Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal. Vol. 3, Issue 2 (2010), 134 – 149
ISSN 2070-5948, DOI 10.1285/i20705948v3n2p134

LINEAR MOMENTS STUDY UNDER RANKED SET SAMPLING

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Received 5 June 2009; Accepted 1 September 2009
Available online 14 October 2010

Abstract*: A ranked set sample consists of independently distributed order statistics and can occur naturally in many experimental settings. The weighted least squares method is used to find linear estimators of unknown parameters from location-scale family of distributions which required the full matrix of all variances and covariances of order statistics of the sample size n which is difficult to obtain in large samples, finding the inverse of this matrix and the estimators can be computed numerically only for small sample sizes. Also, the weighted least squares can not be used when we have distribution which has more than two parameters, for example, generalized Pareto distribution. In this article, we are looking for method in the class of linear estimation which can be applied for any distribution under ranked set sample regardless of the number of the parameters and easy to use. The linear moment-L-moments- method does not require the full matrix of order statistics and easy to use. Also, we derive unbiased estimators of population L-moments using sample linear moments based on k independent ranked set sample. We obtain distribution-free estimate for the sample mean from any distribution under ranked set sample in terms of sample variance and sample L-moments. We illustrate our method on the generalized Pareto distribution.*

Keywords*: Order statistics, Sampling; Linear estimation, Pareto distribution, Estimation.*

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1. Introduction

One of the keys to any statistical inference is to collect data via some sampling techniques. These data enable the experimenter to make valid judgments on the questions of interest. One of the most common sampling techniques for obtaining such data is that of a simple random sampling (SRS). Other more structured sampling designs, such as stratified sampling or cluster sampling are also possible to help data collector in finding a sample that has a good representation of the population of interest. Any such additional structure of this type revolves around how the sample data themselves should be collected in order to provide an informative image of the larger population. With any of these approaches, once the sample items have been chosen the desired measurements are collected from each of the selected items.

The concept of ranked set sampling (RSS) is a recent development that enables one to provide more structure to the collected sample items. This approach to data collection was first introduced by [16] for situations where taking the actual measurements for sample observations is difficult (e.g. costly, time-consuming, destructive), but mechanisms for either informally or formally ranking a set of sample units is relatively easy and reliable. In particular, McIntyre was interested in improving the precision of RSS was first in estimation of average yield from large plots of arable crops without a substantial increase in the number of fields from which detailed expensive and tedious measurements need to collect. For discussions of some of the settings on ranked set sampling technique, see; [17], [3] and [2].

The method of generalized least squares, based on the Gauss-Markoff least-squares theorem, was developed by [15] and was used in ranked set sampling by [4], [14] and [1], among others. Using this method, best linear unbiased estimators (BLUE) of location and scale parameters; μ and σ from distributions of the type $(1/\sigma)f[(x-\mu)/\sigma]$, which employ order statistics in a systematic manner and have minimum variance in the class of linear unbiased estimators, can be obtained. The method of least squares required the full matrix of all variances and covariances of order statistics of the sample size n which is difficult to obtain, finding the inverse of this matrix and the estimators can be computed numerically only for small sample sizes. Also, this method can be applied when we have location-scale family of distribution. This make this method is limited by such condition.

Therefore, we are looking for method in the class of linear estimation which can be applied for any distribution regardless of the number of the parameters and easy to use. This method is the L-moments which is introduced recently by [13] and found many applications in such fields of applied research as civil engineering, meteorology and hydrology; see, for example, [10], [12], [9] and [19]. We study this method under ranked set sampling. Also, we define sample linear moments and show that they are unbiased estimators of the corresponding population quantities. The method is not intended to replace existing methods but rather to complement them especially in situations where we find difficult to find least squares estimators or the random variable does not belong to location-scale family.

2. Ranked Set Sampling

When we select a simple random sample X_1, $X_2,\ldots,X_n$ from a fixed population of interest what makes resulting statistical inference procedures appropriate is not the fact that each individual measurement in the sample is likely to be representative of the population characteristics, such as mean or median, of interest. Rather it is through the concept of sampling distributions of the relevant statistics that we should obtain a set of sample observations that are representative of the entire population. However, in practice we obtain only a single random sample and the concept does not help much if the particular population items selected for our sample are, in fact, not really very representative of the entire population. We are simply bound by the statistical inferences for this particular sample that go with the concept unless we are willing to increase our sample size and expand the number of sample observations.

There are a number of ways to address the problems associated with obtaining a representative sample from a population. One method for dealing with this issue is to involve a more structured sampling scheme than simple random sampling. Such approaches include stratified sampling schemes, cluster sampling, proportional sampling and multistage sampling, among others; see, for example [5]. Note that this additional structure about which items to collect and measure is imposed on our data collection process prior to the actual decision, and, as such, is correctly viewed as sampling technique.

On the other hand, the ranked set sampling utilizes the basic intuitive properties associated with simple random sampling but it is also takes advantage of additional information available in the population to provide an "artificially stratified" sample with more structure that enables us to direct our attention toward the actual measurement of more representative units in the population. The net result is a collection of measurements that are more likely to span the range of values in the population than can be guaranteed by virtue of a simple random sample.

We describe how this additional structure is captured in a single ranked set sample of k measured observations. First, an initial simple random sample of k units from the population is selected and subjected to ordering on the attribute of interest via some ranking process. This judgment ranking can result from a variety of mechanisms, including expert opinion, visual comparisons, or the use of easy-to-obtain auxiliary variables, but it can not involve actual measurements of the attribute of interest on the sample units. Once this judgment ranking of the k units in our initial random sample has been accomplished, the item judged to be the smallest is included as the first item in our ranked set sample and the attribute of interest will be formally measured on this unit. The remaining (k-1) unmeasured units in the first random sample are not considered further. We denote this measurement by $Y_{[1]}$, where a square bracket [1] is used instead of the usual round bracket (1) for the smallest order statistics because $Y_{[1]}$ is only the smallest judgment ordered item. It may or may not actually have the smallest attribute measurement among our k sampled units. Note that the remaining (other than $Y_{[1]}$) units in our first random sample is not considered further in the selection of our ranked set sample or eventual inference about the population. The sole purpose of these other (k-1) units is to help select an item for measurement that represents the smaller attribute values in the population.

Following selection of $Y_{[1]}$, a second independent random sample of size k is selected from the population and judgment ranked without formal measurement on the attribute of interest. This time we selected the items judged to be the second smallest of the k units in this second random sample and include it in our ranked set sample for measurement of the attribute of the interest, this second measured observation is denoted by $Y_{[2]}$. we select the unit judgment ranked to be the third smallest, $Y_{[3]}$, for measurement and inclusion in the ranked set sample. This process is continued until we have selected the unit judgment ranked to be the largest of the k units in the k^{th} random sample, denoted by $Y_{[k]}$, for measurement and inclusion in our ranked set sample. This entire process is referred to as a cycle and the number of observations in each random sample, k in our example, is called the set size. Thus to complete a single ranked set cycle, we need to judgment rank k independent random samples of size k involving a total of k^2 sample units in order to obtain k measured observations $Y_{[1]}, Y_{[2]}, \ldots, Y_{[k]}$. These k observations represent a balanced ranked set sample with set size k, where the word balance denote to the fact that we have collected one judgment order statistics for each of the ranks $i = 1,2,...k$. In order to obtain a ranked set sample with desired total number of measured observations km, we repeat the entire cycle process m independent time, yielding the data $Y_{[1]j}, Y_{[2]j}, ..., Y_{[k]j}$ for $j = 1,2,...,m$; see, for example, [22].

3. Distribution of perfect balanced ranked set sample

To understand what makes the ranked set sample (RSS) different from a simple random sample (SRS) of the same size, we consider the simple case of a single cycle ($m = 1$) with set size k and perfect judgment ranking. In this case, the ranked set sample observations are also the respective order statistics. Let $X_1, X_2, ..., X_n$ denote a simple random sample of size k from a continuous population with probability density function $f(x)$, cumulative distribution function $F(x)$ and quantile function $x(F)$, $0 \le F \le 1$ and let $Y_1, Y_2, ..., Y_k$ be a perfect ranked set sample of size k obtained as in Section 2.

In the case of a SRS the k observations are independent and each of them is viewed as representing a typical value of the population. However, there is no additional structure imposed on their relationship to one another. Letting $X_{(1)} \le X_{(2)} \le .... \le X_{(k)}$ be the order statistics associated with these SRS observations, we note that they are dependent random variables with joint probability density function (p.d.f) given by:

$$g_{SRS}\left(x_{(1)}, ..., x_{(k)}\right) = k! \prod_{i=1}^{k} f\left(x_{(i)}\right) \qquad -\infty < x_{(1)} < ... < x_{(k)} < \infty \tag{1}$$

For the RSS setting, additional information and structure has been provided through the judgment ranking process involving a total of k^2 sample units. The k measurements $Y_{(1)}, ..., Y_{(k)}$ are also order statistics but in this case they are independent but not identically distributed where each of them provides information about a different aspect of the population. The joint p.d.f. for RSS is given by:

$$g_{RSS}\left(y_{(1)}, ..., y_{(k)}\right) = \prod_{i=1}^{k} f\left(y_{(i)}\right) \qquad -\infty < y_{(1)} < .... < y_{(k)} < \infty \tag{2}$$

Where:

$$f_{y_{(i)}}(y) = \frac{k!}{(i-1)!(k-i)!} F^{i-1}(y)\left[1-F(y)\right]^{k-i} f(y) \tag{3}$$

is the probability density function for the ith order statistics for a SRS of size k. It is this extra structure provided by the judgment ranking and the independence of the resulting order statistics that enables procedures based on RSS data to be more efficient than comparable procedures based on SRS with the same number of measured observations. On the other hand, these same features also make the theoretical development of properties for RSS procedures more difficult than for their SRS counterparts. In the next section, we introduce the L-moments under ranked set sample.

4. Linear Moments under Ranked Set Sample

Linear moments are linear combinations of ranked observations that do not requiring squaring or cubing of the observations, as do product-moment estimators. As a result they work in the case of order data and therefore can be used in perfect ranked set sample.

4.1 Population Linear Moments

Let $Y_{1:k}, Y_{2k}, ..., Y_{k:k}$ be a perfect ranked set sample of size k from a continuous distribution with cumulative distribution function $F_Y(.)$, density function $f_Y(.)$ and quantile function $y(F)$, $0 \le F \le 1$. [13] defined the rth linear moments λ_r as:

$$\lambda_r = r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E\left(Y_{r-j:r}\right) \tag{4}$$

as the expectation of $Y_{i:r}$ can be written as:

$$E(Y_{i:r}) = \frac{\Gamma(r+1)}{\Gamma(i)\Gamma(r-i+1)} \int_0^1 y(F) F^{i-1} (1-F)^{r-i} dF$$

We may re-express (4) as:

$$\lambda_r = \int_0^1 y(F) \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} \binom{r-1+k}{j} F^j dF \tag{5}$$

It is straightforward to establish from (4) and (5) the following expressions for the first four L-moments:

$$\lambda_1 = E(Y_{1:1}) = \int_0^1 y(F)dF$$
$$\lambda_2 = \frac{1}{2}E(Y_{2:2} - Y_{1:2}) = \int_0^1 y(F)(2F-1)dF$$
$$\lambda_3 = \frac{1}{3}E(Y_{3:3} - 2Y_{2:3} + Y_{1:3}) = \int_0^1 Y(F)(6F^2 - 6F + 1)dF$$
$$\lambda_4 = \frac{1}{4}E(Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}) = \int_0^1 y(F)(20F^3 - 30F^2)dF$$

Where λ_1 is a measure of location (population mean) and λ_2 is a measure of scale (population scale). The scale-free quantities $\tau_3 = \lambda_3/\lambda_2$ and $\tau_4 = \lambda_4/\lambda_2$ are measure of skewness and kurtosis which are less sensitive to the extreme tails of the distribution than β_1 and β_2, the usual measures of skweness and kurtosis. For more details; see, for example, [21].

4.2 Sample Linear Moments

[13] defined the sample linear moments l_r, corresponding to the population linear moments λ_r given in (4) as follows:

$$l_r = \binom{k}{r}^{-1} \sum_{1 \le i_1} \sum_{< i_2} \cdots \sum_{< i_r \le k} \frac{1}{r} \sum_{j=0}^{r-1} (-1)^k \binom{r-1}{j} Y_{i_{r-j:k}}, \qquad r = 1, 2, \dots, j \tag{6}$$

For example, from (6) the first two sample moments corresponding to λ_1 and λ_2 are:

$$l_1 = \frac{1}{k}\sum_{i=1}^{k} Y_{i:k} \quad \text{and} \quad l_2 = \frac{1}{2\binom{k}{2}} \sum_{i<j}\sum (Y_{j:k} - Y_{i:k})$$

as [7] have pointed out, it is not necessary to iterate over all subsamples of size r when calculating l_r, as it can be written as linear combination of order statistics as:

$$l_r = \frac{1}{r\binom{k}{r}} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \sum_{i=1}^{k} \binom{i-1}{r-j-1} \binom{k-i}{j} Y_{i:k} \tag{7}$$

Where we find from [8] that:

$$\sum_{i=1}^{k} (i-1)^{(v)} (k-i)^{(u)} E(Y_{i:k}) = v!u!\binom{k}{v+u+1} E(Y_{v+1:v+u+1})$$

note that $n^{(r)}=n(n-1)....(n-r+1)$. Putting $v+1=r-j$ and $v+u+1=r$ we obtain:

$$\sum_{i=1}^{n}(i-1)^{(r-j-1)}(k-i)^{(j)}E(Y_{i:k})=(r-j-1)!\,j!\binom{k}{r}E(Y_{r-j:j}) \tag{8}$$

This equation gives us:

$$\hat{E}(Y_{r-j:j})=\frac{\sum_{i=1}^{k}\binom{i-1}{r-j-1}\binom{k-i}{j}Y_{i:k}}{\binom{k}{r}}$$

Substituting by this equation in (4) give us equation (7). Equation (7) allows us to re-express the first four sample linear moments in the readily computable forms:

$$l_1=\frac{1}{k}\sum_{i=1}^{k}Y_{i:k}$$
$$l_2=\frac{1}{k(k-1)}\sum_{i=1}^{k}(2i-1-k)Y_{i:k}$$
$$l_3=\frac{1}{k(k-1)(k-2)}\sum_{i=1}^{k}\left[6(i-1)(i-2)-6(k-2)(i-1)+(k-1)(k-2)\right]Y_{i:n}$$
$$l_4=\frac{1}{k(k-1)(k-2)(k-3)}\sum_{i=1}^{k}\left[20(i-1)(i-2)(i-3)-30(k-3)(i-1)(i-2)\right.$$
$$\left.+12(k-2)(k-3)(i-1)-(k-1)(k-2)(k-3)\right]Y_{i:n}$$

Standardized unit-free versions of the symmetry and kurtosis measures are $t_3=l_3/l_2$ and $t_4=l_4/l_2$ corresponding to the population versions $\tau_3=\lambda_3/\lambda_2$ and $\tau_4=\lambda_4/\lambda_2$.

Theorem 1. l_r is an unbiased estimator of λ_r under ranked set sample. Hence:

$$E(l_r)=\lambda_r$$

Proof:
From equation (7) we find that:

$$E(l_r)=\frac{1}{r\binom{k}{r}}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}E\left[\sum_{i=1}^{k}\binom{i-1}{r-j-1}\binom{k-i}{j}Y_{i:k}\right]$$

From equation (8) we obtain:

$$E\left(Y_{r-j:j}\right)=\frac{\sum_{i=1}^{k}\binom{i-1}{r-j-1}\binom{k-i}{j}E\left(Y_{i:k}\right)}{\binom{k}{r}}$$

This gives us:

$$E\left(l_r\right)=\frac{1}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}E\left[Y_{r-j:j}\right]=\lambda_r$$

Thus, l_r is nonparametric an unbiased estimator of λ_r under ranked set sample whatever the underlying distribution.

5. Applications

In this section, we study the properties of the sample mean under ranked set sampling using some of the properties of L-moments given by [6], [13] and [8]. Also, we estimate the parameters of generalized Pareto distribution under ranked set sample and give comparison with the same estimators under simple random sample.

5.1 The sample mean in terms of sample L-moments

Let $\bar{X}=\frac{1}{k}\sum_{i=1}^{k}X_i$ and $\bar{Y}=l_1=\frac{1}{k}\sum_{i=1}^{k}Y_{(i)}$ be the SRS and RSS sample mean, respectively. It is well known that $\bar{X}$ and $\bar{Y}$ are unbiased estimators of the population mean $\mu=\lambda_1$ and it has variance $\frac{\sigma^2}{k}$, where σ^2 is the population variance. How does $\bar{Y}$ compare with this estimator? First we note that the mutual independence of the $Y_{(i)}$, $i=1,2,...,k$. This enables us to write:

$$E\left(\bar{Y}\right)=\frac{1}{k}\sum_{i=1}^{k}E\left(Y_{(i)}\right) \tag{9}$$

And

$$\operatorname{var}\left(\bar{Y}\right)=\frac{1}{k^2}\sum_{i=1}^{k}\operatorname{var}\left(Y_{(i)}\right) \tag{10}$$

Since we have assumed perfect ranking, $Y_{(i)}$ is distributed as ith order statistics from a continuous distribution. Hence, from [8] we find that:

$$\sum_{i=1}^{k}(i-1)^{(v)}(k-i)^{(u)}E\left(Y_{(i)}\right)=v!u!\binom{k}{v+u+1}E\left(Y_{v+1:v+u+1}\right) \tag{11}$$

When $v=u=0$ we find that:

$$\sum_{i=1}^{k}E\left(Y_{(i)}\right)=kE\left(Y_{1:1}\right) \tag{12}$$

Substituting by (12) in (9) we obtain:

$$E(\overline{Y})=\frac{1}{k}kE(Y_{1:1})=E(Y_{1:1})=\mu$$

Thus, $\overline{Y}$ is an unbiased estimator of μ.
Certainly, there is a difference between these unbiased estimator and unbiased estimator under SRS. The k components of the SRS are mutually independent and identically distributed and each is it self an unbiased estimator for μ. While the k components of the RSS average $\overline{Y}$ are also mutually independent, they are not identically distributed and none of them are individually unbiased for μ except for the middle order statistics when the distribution is symmetric about μ.

Theorem 2. If $E(Y_{1:1})$ exists, we have:

$$Var\left(\overline{Y}\right)=\frac{1}{k}\left[E\left(Y_{1:1}^{2}\right)-\sum_{r=0}^{k-1}\frac{k!(k-1)!(2r+1)}{(k+r)!(k-1-r)!}\lambda_{r+1}^{2}\right] \tag{13}$$

***Proof*:**
Where $Y_{(i)}$ are independent, we have:

$$Var\left(\overline{Y}\right)=\frac{\sum_{i=1}^{k}\mathrm{var}\left(Y_{(i)}\right)}{k^{2}}$$

This can be written as:

$$Var\left(\overline{Y}\right)=\frac{1}{k^{2}}\left[\sum_{i=1}^{k}E\left(Y_{(i)}^{2}\right)-\sum_{i=1}^{k}E^{2}\left(Y_{(i)}\right)\right] \tag{14}$$

From [20] and [6] we find that:

$$\frac{1}{k}\sum_{i=1}^{k} E^2\left(Y_{(i)}\right) = \sum_{r=0}^{k-1} \frac{k!(k-1)!(2k+1)}{(k+r)!(k-1-r)!} \lambda^2{}_{r+1} \tag{15}$$

and

$$\frac{1}{k}\sum_{i=1}^{k} E\left(Y_{(i)}^2\right) = E\left(Y_{1:1}^2\right) \tag{16}$$

Substituting by (15) and (16) in (14) we find that:

$$Var\left(\bar{Y}\right) = \frac{1}{k}\left[E\left(Y_{1:1}^2\right) - \sum_{r=0}^{k-1} \frac{k!(k-1)!(2r+1)}{(k+r)!(k-1-r)!}\lambda_{r+1}^2\right]$$

Where λ_r are the population L-moments. This completes the proof. Not that, we may write the variance of $\bar{X}$ in terms of order statistics as:

$$Var\left(\bar{X}\right) = \frac{1}{k}\left[E\left(Y_{1:1}^2\right) - E^2\left(Y_{1:1}\right)\right]$$

Therefore, we can re-write equation (13) as:

$$\mathrm{var}\left(\bar{Y}\right) = \mathrm{var}\left(\bar{X}\right) - \sum_{r=1}^{k-1} \frac{\left[(k-1)!\right]^2 (2r+1)}{(k+r)!(k-1-r)!}\lambda_{r+1}^2 \tag{17}$$

Since $\lambda_{r+1}^2 \geq 0$, we find that: $\mathrm{var}\left(\bar{Y}\right) \leq \mathrm{var}\left(\bar{X}\right)$

Hence the variance of the sample mean under ranked set sample is always less than the variance of the sample mean under simple random sample.
We can find distribution-free estimator of $\mathrm{var}\left(\bar{Y}\right)$ from the data given regardless of the underlying distribution for the data. From equation (17) we obtain:

$$\hat{\mathrm{v}}\mathrm{ar}\left(\bar{Y}\right) = \frac{s^2}{k} - \sum_{r=1}^{k-1} \frac{\left[(k-1)!\right]^2 (2r+1)}{(k+r)!(k-1-r)!} l_{r+1}^2$$

where s^2 is the usual sample variance, l_r is the sample L-moments given in (7). The relative efficiency of the sample mean under simple random sample relative to the sample mean under ranked set sample is:

$$R.eff.\left(\bar{X},\bar{Y}\right)=\frac{\operatorname{var}\left(\bar{Y}\right)}{\operatorname{var}\left(\bar{X}\right)}=1-\frac{\sum_{r=1}^{k-1}\frac{\left[(k-1)!\right]^2(2r+1)}{(k+r)!(k-1-r)!}\lambda_{r+1}^2}{\sigma^2/k}\le 1$$

This equation has advantage that it could be estimated from the data as:

$$\hat{R}.eff.\left(\bar{X},\bar{Y}\right)=1-\frac{\sum_{r=1}^{k-1}\frac{\left[(k-1)!\right]^2(2r+1)}{(k+r)!(k-1-r)!}l_{r+1}^2}{s^2/k}$$

s^2 and l_r as before.

5.2 Generalized Pareto Distribution

If we have the generalized Pareto distribution with density:

$$f\left(x\right)=\alpha^{-1}e^{-(1-k)z}$$

Where:

$$z=-\beta^{-1}\log\left[1-\beta(x-\xi)/\alpha\right]\text{for } \beta\ne 0 \text{ and } z=(x-\xi)/\alpha \text{ for } \beta=0$$

The L-moments for the generalized Pareto distribution can be obtain as: $\lambda_1=\xi+\alpha/\left(1+\beta\right)$ and

$$\lambda_r=\frac{\alpha\Gamma\left(\beta+1\right)\Gamma\left(r-1-\beta\right)}{\Gamma\left(1-\beta\right)\Gamma\left(r+1+\beta\right)} \text{ for } \beta>-1$$

Then the variance and relative efficiency is given by:

$$\operatorname{var}\left(\bar{Y}\right)=\frac{\alpha^2}{k\left(\beta+1\right)^2\left(2\beta+1\right)}-\sum_{r=1}^{k-1}\frac{\left[(k-1)!\right]^2(2r+1)\alpha^2\Gamma^2\left(\beta+1\right)\Gamma^2\left(r-\beta\right)}{(k+r)!(k-1-r)!\Gamma^2\left(1-\beta\right)\Gamma^2\left(r+\beta+2\right)}$$

and

$$R.eff.\left(\bar{X},\bar{Y}\right)=1-\sum_{r=1}^{k-1}\frac{k\left(\beta+1\right)^2\left(2\beta+1\right)\left[(k-1)!\right]^2(2r+1)\Gamma^2\left(\beta+1\right)\Gamma^2\left(r-\beta\right)}{(k+r)!(k-1-r)!\Gamma^2\left(1-\beta\right)\Gamma^2\left(r+\beta+2\right)}$$

Table 1 below gives the relative efficiency for sample mean under RSS with respect to sample mean under SRS from uniform, exponential and generalized Pareto with different choices of the shape parameter β .

Table 1. The exact relative efficiency of the sample mean under SRS relative to the sample mean under RSS from uniform distribution (Unif.), exponential distribution (Expo.) and generalized Pareto (G.Pareto) distribution using different values of β.

				β					
Set				$-.4\dot{9}$	$-.45$	$-.25$	0.001	0.25	0.5
Size		Unif.	Expo.				G.pareto		
	2	0.6667	0.7500	1	0.9583	0.8367	0.7497	0.7037	0.6800
	3	0.5000	0.6111	1	0.9308	0.7382	0.6107	0.5476	0.5167
k	4	0.4000	0.5208	1	0.9103	0.6699	0.5204	0.4503	0.4173
	5	0.3333	0.4566	1	0.8940	0.6188	0.4562	0.3834	0.3502
	6	0.2857	0.4083	1	0.8805	0.5784	0.4079	0.3345	0.3019
	7	0.2500	0.3704	1	0.8691	0.5456	0.3699	0.2970	0.2654
	8	0.2222	0.3397	1	0.8591	0.5181	0.3392	0.2673	0.2369
	9	0.2000	0.3143	1	0.8503	0.4945	0.3139	0.2432	0.2139
	10	0.1818	0.2928	1	0.8424	0.4741	0.2924	0.2232	0.1950

The density of the generalized Pareto distribution from some ranges from $\xi = 0, \alpha = 1$ and $\beta = -.49$ to $\xi = 0, \alpha = 1$ and $\beta = 1$ is shown in the graph 1 below.

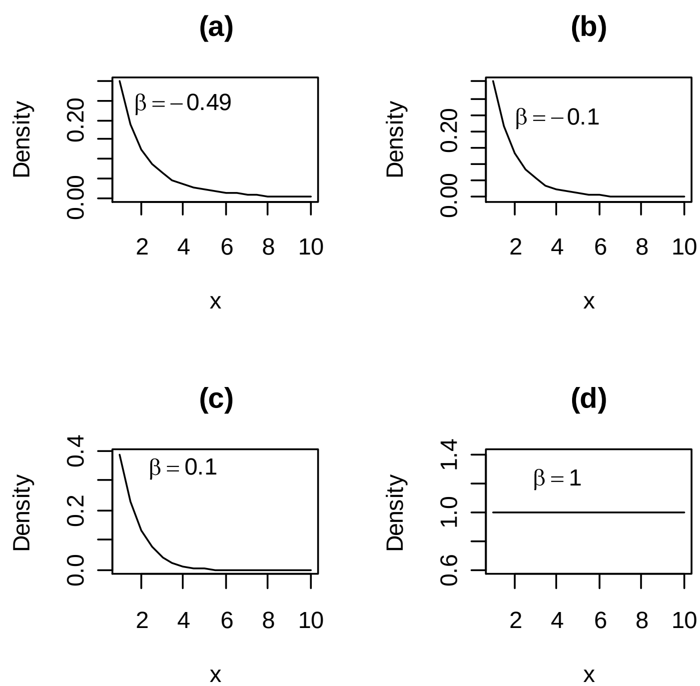

Figure 1. Density function of the generalized Pareto distribution with ξ=0, α=1 and (a) β=-0.49, (b) β=-0.1, (c) β=0.1 and (d) β=1

Like the exponential distribution, the generalized Pareto distribution is often used to model the tails of another distribution; see, for example [12]. For example, you might have washers from a manufacturing process. If random influences in the process lead to differences in the sizes of the washers, a standard probability distribution, such as the normal, could be used to model those sizes. However, while the normal distribution might be a good model near its mode, it might not be a good fit to real data in the tails and a more complex model might be needed to describe the full range of the data. On the other hand, only recording the sizes of washers larger (or smaller) than a certain threshold means you can fit a separate model to those tail data, which are known as exceedences. You can use the generalized Pareto distribution in this way, to provide a good fit to extremes of complicated data. The generalized Pareto distribution allows a continuous range of possible shapes that includes both the exponential and Pareto distributions as special cases. The generalized Pareto distribution has three basic forms, each corresponding to a limiting distribution of exceedence data from a different class of underlying distributions.

- Distributions whose tails decrease exponentially, such as the normal, lead to a generalized Pareto shape parameter of zero.
- Distributions whose tails decrease as a polynomial, such as Student's t, lead to a positive shape parameter.
- Distributions whose tails are finite, such as the beta, lead to a negative shape parameter.

The most realistic case of the generalized Pareto distribution which used in practice when ξ is known. Without loss of generality we assume that $\xi = 0$. Where the population linear moments is defined in terms of the quantile function $y(F)$, we find the quantile function for Pareto distribution as:

$$y(F) = \xi + \frac{\alpha}{\beta}\left[1-(1-F)^{\beta}\right] \text{ for } \beta \neq 0 \text{ and } y(F) = \xi - \alpha \log(1-F) \text{ for } \beta = 0$$

Then:

$$\beta = \lambda_1 / \lambda_2 - 2 \text{ and } \alpha = (1+\beta)\lambda_1$$

That shows that the parameters β and α are functions of population linear moments. This can be estimated as:

$$\hat{\beta} = l_1 / l_2 - 2 \quad \text{and} \quad \hat{\alpha} = (1+\hat{\beta})l_1$$

Now we investigate the properties of these estimators under SRS with respect to RSS in Table 2. This table shows that the estimation under ranked set sampling is more efficient than simple random sampling. For example, the relative efficiency for $\hat{\beta}$ is 0.43 at the sample size 6, also the relative efficiency is 0.33 for $\hat{\alpha}$ at the same sample size using RSS with respect to SRS.

Table 2. Variance and relative efficiency of the parameters of the generalized Pareto distribution using SRS with respect to RSS.

	$\beta=-.0001$			$\alpha=1$		
k	var. RSS	var. SRS	R.eff	var. RSS	var SRS	R.eff
3	1.209	6.233	0.19	1.660	15.1	0.11
4	0.629	1.972	0.32	0.841	2.48	0.33
5	0.391	0.905	0.43	0.518	1.17	0.44
6	0.270	0.634	0.43	0.356	1.07	0.33
	var. RSS	var. SRS	R.eff	var RSS	var SRS	R.eff
3	1.137	16.34	0.07	2.057	7.59	0.27
4	0.614	1.763	0.31	1.069	2.46	0.43
5	0.394	0.730	0.53	0.674	1.30	0.51
6	0.278	0.500	0.57	0.472	.942	0.50

We obtain these values by noting that the order statistics from generalized Pareto distribution are:

$$E\left(Y_{i:k}\right)=\frac{k!\alpha}{(i-1)!(k-i)!\beta}\left[Beta\left(i,k-i+1\right)-Beta\left(i,k-i+\beta+1\right)\right]$$

and

$$E\left(Y^2\right)=\frac{k!\alpha^2}{(i-1)!(k-i)!\beta^2}\left[Beta\left(i,k-i+1\right)-2Beta\left(i,k+\beta-i+1\right)+Beta\left(i,k-i+2\beta+1\right)\right]$$

The variance of $\hat{\alpha}$ and $\hat{\beta}$ can be obtained using the variance of the ratio:

$$\operatorname{var}\left(\frac{l_1}{l_2}\right)\cong\frac{\operatorname{var}(l_2)E^2(l_1)}{E^4(l_2)}+\frac{\operatorname{var}(l_1)}{E^2(l_2)}-\frac{2\operatorname{cov}(l_1,l_2)E(l_1)}{E^3(l_2)}$$

and

$$\operatorname{var}\left(\hat{\alpha}\right)=\operatorname{var}\left[E\left(\hat{\alpha}/\hat{\beta}\right)\right]+E\left[\operatorname{var}\left(\hat{\alpha}/\hat{\beta}\right)\right]$$

See [18] and also the variances of $\hat{\alpha}$ and $\hat{\beta}$ in the case of simple random sample are given in [11].

5. Conclusions

In this article, we have study the L-moments under ranked set sampling in the perfect case and shows that the sample L-moments is an unbiased estimator for corresponding population L-moments. We re-expressed the sample mean in terms of L-moments and obtained the distribution-free estimator for the sample mean in terms of sample variance and sample L-moments. Moreover, we have used L-moments to estimate the parameters from generalized Pareto distribution where the generalized least square is not applicable. We showed that the estimate of the parameter under ranked set sampling is more efficient than the simple random sampling. The question which will be investigated in the future, what is the performance of L-moments under imperfect order in the case of ranked set sampling?

Acknowledgments We would like to thank an anonymous referee whose comments have helped to improve the paper.

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