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An analogue of Rionero's functional for reaction-diffusion equations and an application thereof

James N. Flavin
Department of Mathematical Physics, National University of Ireland, Galway, Ireland
james.flavin@nuigalway.ie

Salvatore Rionero
Department of Mathematics and Applications "R. Caccioppoli", University of Naples Federico II, Monte Sant'Angelo, Naples, Italy
rionero@unina.it

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Abstract. In the context of stability properties of a reaction-diffusion system of p.d.e.s, the paper discusses an analogue of Rionero's Lyapunov functional, which has analogous properties. By way of application, it is shown how to derive L^2 stability estimates for the gradient of the solution of a Lotka-Volterra system.

Keywords: reaction-diffusion, Lotka-Volterra system, stability, pointwise estimates, Lyapunov functional

MSC 2000 classification: primary 34D20, secondary 35B35, 35B45, 35K57

1 Introduction

In connection with the stability and instability properties of the zero solution of a pair of nonlinear, reaction-diffusion p.d.e.s, subject to appropriate boundary conditions, Rionero has, in a sequence of papers (see [1–3]), used a peculiar Lyapunov functional. The pivotal, or central, theorem expresses the time-derivative of the functional, along the solutions, in a particularly revealing manner: the time derivative is linked to the eigenvalues of a pair of o.d.e.s in an interesting manner.

Both the functional and its time-derivative in the aforesaid pivotal theorem involve (space) integrals of the dependent variables. The pivotal theorem of the present paper exhibits an analogous functional with analogous properties; both the functional and its time-derivative involve integrals of the *gradients* of the dependent variables. The pivotal theorem of this paper is deduced from that of Rionero by means of a lemma which is dealt with in Section 2.

Section 3 gives the two pivotal theorems as aforesaid, and discusses important similarities between the two results, in so far as they bear upon stability considerations for the reaction-diffusion system in question.

By way of application, an equilibrium solution of the Lotka-Volterra reaction diffusion system is considered—for simplicity, in one spatial dimension—and an L^2 stability estimate is obtained for the gradient of the perturbation. As a by-product, one may deduce a *pointwise* stability estimate.

2 Basic equations and a useful lemma

Consider smooth solutions of the reaction-diffusion system, in the fixed spatial domain Ω (where indicial notation is used, including the summation convention, the indices taking the values 1,2 in the context of the present paper)

$$\frac{\partial u_i}{\partial t} = a_{ij}u_j + \gamma_{ij}\nabla^2 u_j + F_i(u_1, u_2) \tag{1}$$

subject to

$$u_i = 0 \quad \text{on } \partial\Omega\,, \tag{2}$$

where $\partial\Omega$ is the smooth boundary of Ω. Moreover a_{ij},γ_{ij} are constants and F_i are smooth functions of u_i such that

$$F_i(0,0) = 0\,. \tag{3}$$

The latter condition ensures that $u_i = 0$ is a solution to the system.

The lemma that follows exhibits a useful connection between functionals defined along the solutions of (1), (2). The functionals are defined as follows:

$$\begin{cases} V(t) = \dfrac{1}{2}\displaystyle\int_\Omega R_{ij}\, u_i\, u_j\, d\Omega\,, \\ \overline{V}(t) = \dfrac{1}{2}\displaystyle\int_\Omega R_{ij}\, \nabla u_i \cdot \nabla u_j\, d\Omega\,, \\ \overline{\overline{V}}(t) = \dfrac{1}{2}\displaystyle\int_\Omega R_{ij} \nabla^2 u_i\, \nabla^2 u_j\, d\Omega\,, \end{cases} \tag{4}$$

where R_{ij} are constants (such that $R_{ij} = R_{ji}$ w.l.o.g.). Differentiating $V(t)$, $\overline{V}(t)$, $\overline{\overline{V}}(t)$ with respect to t (the time variable), using the divergence theorem together with (1)–(3), gives the following lemma:

1 Lemma. *The time derivatives of the functionals defined in (4) along the* *solutions of the system defined by (1)–(3), are given by:*

$$\begin{cases} (a)\, V'(t) = \displaystyle\int_\Omega R_{ij}\left[a_{ik}u_ju_k - \gamma_{ik}\nabla u_j\cdot\nabla u_k + F_iu_j\right]d\Omega\,, \\ (b)\, \overline{V}'(t) = \displaystyle\int_\Omega R_{ij}\left[a_{ik}\nabla u_j\cdot\nabla u_k - \gamma_{ik}\nabla^2u_j\nabla^2u_k + \nabla F_i\cdot\nabla u_j\right]d\Omega\,, \\ (c)\, \overline{\overline{V}}'(t) = \displaystyle\int_\Omega R_{ij}\left[a_{ik}\nabla^2u_j\nabla^2u_k - \gamma_{ik}\nabla(\nabla^2u_j)\cdot\nabla(\nabla^2u_k) + \nabla^2F_i\nabla^2u_j\right]d\Omega \end{cases} \tag{5}$$

provided that, in the case (c), the determinant of coefficients γ_{ij} is non-zero, e.g. the quantity $\overline{V}'(t)$ is formally the same as $V(t)$, with the following changes: u_i, ∇u_i, F_i are replaced by ∇u_i, $\nabla^2 u_i$, ∇F_i respectively, and appropriate products replace those arising in $V'(t)$.

Whereas the derivation of *(c)* is similar to that of *(a)*, *(b)*, it also entails establishing that the Laplacian of u_i vanishes on $\partial\Omega$.

3 The fundamental theorem and a discussion thereof

In the remainder of the article we confine attention to the case of no cross-diffusion

$$\gamma_{12} = \gamma_{21} = 0\,, \tag{6}$$

and for convenience write

$$\gamma_{11} = \gamma_1, \quad \gamma_{22} = \gamma_2\,, \tag{7}$$

both assumed positive. It proves convenient to introduce the positive scaling constants α, β, which may be chosen subsequently, and new dependent variables u,v such that

$$u_1 = \alpha u\,, \quad u_2 = \beta v\,. \tag{8}$$

Further write

$$f(u,v) = \alpha^{-1}F_1(u,v), \quad g(u,v) = \beta^{-1}F_2(u,v) \tag{9}$$

where the foregoing $F_1(u,v)$, $F_2(u,v)$ mean $F_1(u_1,u_2)$, $F_2(u_1,u_2)$ expressed in terms of the new variables u and v. Define the constants

$$\begin{aligned} b_1 &= a_{11} - \overline{\alpha}\gamma_1, \quad b_4 = a_{22} - \overline{\alpha}\gamma_2, \\ b_2 &= (\beta/\alpha)a_{12}, \quad b_3 = (\alpha/\beta)a_{21}, \end{aligned} \tag{10}$$

where $\overline{\alpha}$ is a positive constant, yet to be chosen.

We shall denote by $< \cdot,\cdot >$ the L^2 scalar product, $||\cdot||$ the L^2 norm, for scalar and vector functions as appropriate.

Rionero (see [1–3]) proved (though using a slightly different notation) that the functional, defined along the solutions,

$$V(t) = \frac{1}{2}\left[A\left(\|u\|^2 + \|v\|^2\right) + \|b_1 v - b_3 u\|^2 + \|b_2 v - b_4 u\|^2\right], \tag{11}$$

wherein

$$A = b_1 b_4 - b_2 b_3 = b_1 b_4 - a_{12} a_{21}, \quad I = b_1 + b_4, \tag{12}$$

satisfies

$$\frac{dV}{dt} = AI\left(\|u\|^2 + \|v\|^2\right) + \Psi^* + \Psi, \tag{13}$$

where

$$\Psi^* = \gamma_1 \alpha_1 \left[- \|\nabla u\|^2 + \overline{\alpha}\,\|u\|^2\right] + \gamma_2 \alpha_2 \left[- \|\nabla v\|^2 + \overline{\alpha}\,\|v\|^2\right] \\ + (\gamma_1 + \gamma_2)\alpha_3 \left[\langle \nabla u, \nabla v\rangle - \overline{\alpha}\,\langle u, v\rangle\right], \tag{14}$$

$$\Psi = \langle \alpha_1 u - \alpha_3 v, f\rangle + \langle \alpha_2 v - \alpha_3 u, g\rangle, \tag{15}$$

wherein

$$\alpha_1 = A + b_3^2 + b_4^2, \quad \alpha_2 = A + b_1^2 + b_2^2, \quad \alpha_3 = b_1 b_3 + b_2 b_4. \tag{16}$$

2 Remark. As pointed out by Rionero (op. cit.), the eigenvalues λ_1,λ_2 of the binary system of o.d.e.s

$$\begin{aligned} \frac{d\xi}{dt} &= b_1 \xi + a_2 \eta, \\ \frac{d\eta}{dt} &= a_3 \xi + b_4 \eta \end{aligned} \tag{17}$$

are given by

$$\lambda_{1,2} = \frac{I \pm \sqrt{I^2 - 4A}}{2}, \tag{18}$$

where I,A, defined by (12), are such that

$$\begin{aligned} I &= \lambda_1 + \lambda_2, \\ A &= \lambda_1 \lambda_2. \end{aligned} \tag{19}$$

Moreover, Rionero discussed interesting connections between the stability and instability of the zero solution of the system of o.d.e.s (17) and the stability and instability of the reaction-diffusion system considered in this section. To this end, he uses the result (13) etc.

We now cite the fundamental theorem of the present paper: it follows from Lemma 1 and the result (11)–(16), or it can, of course, be proved directly (although the proof is lengthy).

3 Theorem. *Defining*

$$\overline{V}(t) = \frac{1}{2}\left[A\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \|b_1\nabla v - b_3\nabla u\|^2 + \|b_2\nabla v - b_4\nabla u\|^2\right], \quad (20)$$

along the solutions, one has

$$\frac{d\overline{V}}{dt} = AI\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \overline{\Psi}^* + \overline{\Psi}, \quad (21)$$

where

$$\overline{\Psi}^* = \gamma_1\alpha_1\left[-\left\|\nabla^2 u\right\|^2 + \overline{\alpha}\,\|\nabla u\|^2\right] + \gamma_2\alpha_2\left[-\left\|\nabla^2 v\right\|^2 + \overline{\alpha}\,\|\nabla v\|^2\right] + (\gamma_1+\gamma_2)\alpha_3\left[\left\langle\nabla^2 u, \nabla^2 v\right\rangle - \overline{\alpha}\,\langle\nabla u, \nabla v\rangle\right], \quad (22)$$

$$\begin{aligned}\overline{\Psi} &= \langle\alpha_1\nabla u - \alpha_3\nabla v, \nabla f\rangle + \langle\alpha_2\nabla v - \alpha_3\nabla u, \nabla g\rangle \\ &= \langle\alpha_1\nabla u - \alpha_3\nabla v, f_u\nabla u + f_v\nabla v\rangle \\ &\quad + \langle\alpha_2\nabla v - \alpha_3\nabla u,\ g_u\nabla u + g_v\nabla v\rangle \end{aligned} \quad (23)$$

where A, I, α are defined by (12), (16) and where the subscripts u and v denote the partial differentiation with respect to these variables.

The main application of Theorem 3, dealt with in this paper, is to the derivation of stability estimates for $\overline{V}(t)$ in the context of a reaction-diffusion system.Rionero (op.cit.) uses the results (11)–(16) to obtain, *inter alia*, stability criteria for the reaction-diffusion system defined by (1)–(3), (6): stability is established (for the zero solution) in the measure $V(t)$. We now make some remarks concerning the means by which stability may be established in the measure $\overline{V}(t)$ analogous to those obtained by Rionero.

4 Remark. In the context of (11)–(16), Rionero chooses the constant $\overline{\alpha}$ to be the best (least, positive) number for which the inequality

$$\|\nabla\Phi\|^2 \geq \overline{\alpha}\,\|\Phi\|^2 \quad (24)$$

is valid (essentially) for arbitrary smooth functions Φ such that

$$\Phi = 0 \quad \text{on } \partial V; \quad (25)$$

i.e. the lowest eigenvalue of the eigenvalue problem defined by

$$\nabla^2\Phi + \lambda\Phi = 0, \text{ in } V, \ \ \Phi = 0 \text{ on } \partial V \quad (26)$$

etc. Similarly the most appropriate value for $\overline{\alpha}$ in the context of Theorem 3 is adjudged to be the best (least, positive) number $\overline{\alpha}$ for which the inequality

$$\left\|\nabla^2\Phi\right\|^2 \geq \overline{\alpha}\left\|\nabla\Phi\right\|^2 \tag{27}$$

for arbitrary smooth functions such that (25) holds. The best number $\overline{\alpha}$ in this case is the same eigenvalue as before (i.e. the lowest eigenvalue of (26) e.g. [4]). *This value of* $\overline{\alpha}$ *is assumed henceforward.*

5 Remark. When considering stability, Rionero (e.g. [3]) essentially requires that V be positive-definite in u, v. Similarly when Theorem 1 is used for stability $\overline{V}$, defined by (20), is required to be positive-definite in u, v. A sufficient condition for positive-definiteness in both cases is

$$A > 0; \text{ i.e. } b_1 b_4 - a_{12}a_{21} > 0 \tag{28}$$

The condition (28) is assumed henceforward.

Again for stability (in the measures V, $\overline{V}$ respectively) one requires

$$\frac{dV}{dt} \leq 0 \text{ or } \frac{d\overline{V}}{dt} \leq 0\,, \tag{29}$$

as appropriate. In the absence of nonlinear source/forcing terms F_1, F_2 (or f, g), sufficient conditions, in addition to (28), are given by

$$I < 0; \text{ i.e. } b_1 + b_4 < 0 \tag{30}$$

together with

$$\Psi^* \leq 0 \text{ or } \overline{\Psi}^* \leq 0 \tag{31}$$

as appropriate.

6 Remark. Rionero [3] obtains conditions on data in order that $(31)_1$ be valid, and it is the purpose of this remark to point out that the aforementioned conditions are also sufficient for the validity of $(31)_2$.

Rionero [3] requires $\Psi^* \leq 0$ for all kinematically admissible values of u,v [(essentially) all smooth functions vanishing on the boundary]: such conditions on data ($b.$, $a.$, $\gamma.$) are derived using the variational characterization (24)–(25) together with algebraic considerations. In view of the variational characterization (27),(25) it follows that Rionero's conditions on data, as aforementioned, sufficient for the validity of $(31)_1$, are also sufficient for the validity of $(31)_2$.

An example of such conditions (e.g. [3]) is:

$$b_1 b_4 b_2 b_3 < 0 \tag{32}$$

in which case the scaling constants α, β (defined by (8)) are chosen such that

$$\frac{\alpha}{\beta} = \left|\frac{b_2 b_4}{b_1 b_3}\right|^{\frac{1}{2}} \tag{33}$$

ensuring that

$$\alpha_3 = 0 \tag{34}$$

i.e. the conditions (32),(33) imply $(31)_2$.

This may, of course, easily be proved directly.

7 Remark. Rionero [3] obtains conditions on the forcing terms f, g such that

$$|\Psi| \le \delta V^{1+k} \tag{35}$$

δ, k being positive constants, leading to a differential inequality for V (on using (11)–(16), $(31)_1$ etc.) that implies conditional, exponential asymptotic stability, in the measure V, of the reaction-diffusion system considered here. A similar approach is used hereunder, in the context of a Lotka-Volterra reaction-diffusion system, with Dirichlet boundary conditions, in one spatial dimension: a stability estimate is established in the measure $\overline{V}$ from which a pointwise estimate may be deduced.

4 A Lotka-Volterra system: stability estimates for the solution gradient

Here we consider a Lotka-Volterra system of reaction-diffusion equations in one spatial dimension, with Dirichlet boundary conditions: Theorem 3 is used to obtain a stability estimate in the measure $\overline{V}$, for an equilibrium configuration, from which a pointwise stability estimate may be deduced.

We discuss the Lotka-Volterra system (discussed in [5], for example).

$$\begin{aligned} \frac{\partial S_1}{\partial t} &= \gamma_1 S_{1,xx} + a_1 S_1 - c_1 S_1 S_2\,, \\ \frac{\partial S_2}{\partial t} &= \gamma_2 S_{2,xx} - a_2 S_2 + c_2 S_1 S_2, \end{aligned} \tag{36}$$

where $\gamma.$, $a.$, $c.$ are all positive constants, in the interval $0 < x < 1$ (the symbol x is used, instead of x_1, as the spatial variable, and subscripts in x denote partial differentiation with respect to x). We consider the equilibrium configuration of (36), in the presence of constant boundary conditions:

$$S_1 = \frac{a_2}{c_2}, \quad S_2 = \frac{a_1}{c_1}\,. \tag{37}$$

We write

$$S_1 = \left(\frac{a_2}{c_2}\right) + u_1, \quad S_2 = \left(\frac{a_1}{c_1}\right) + u_2\,, \tag{38}$$

where the perturbations $u_1(x,t)$, $u_2(x,t)$ satisfy (in the interval $0 < x < 1$)

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \gamma_1 u_{1,xx} - c_1\left(\frac{a_2}{c_2}\right) u_2 - c_1 u_1 u_2\,, \\ \frac{\partial u_2}{\partial t} &= \gamma_2 u_{2,xx} + c_2\left(\frac{a_1}{c_1}\right) u_1 + c_2 u_1 u_2, \end{aligned} \tag{39}$$

subject to

$$u = 0 \quad \text{on } \; x = 0, 1. \tag{40}$$

The equations (39) are of the type (1) with

$$F_1 = -c_1 u_1 u_2\,, \quad F_2 = c_2 u_1 u_2\,. \tag{41}$$

We use scaled variables (see (8)) and in the context of these we have

$$f = -c_1 \beta u v\,, \quad g = c_2 \alpha u v\,, \tag{42}$$

and

$$b_1 = -\pi^2\gamma_1,\; b_2 = \left(\frac{\beta}{\alpha}\right)\left(-\frac{c_1 a_2}{c_2}\right),\; b_3 = \left(\frac{\alpha}{\beta}\right)\left(\frac{a_1 c_2}{c_1}\right),\; b_4 = -\pi^2\gamma_2\,, \tag{43}$$

on noting that the relevant eigenvalue in this case is π^2.

Prior to using Theorem 3 we note that the condition (32) is automatically satisfied here. Thus we choose $\frac{\alpha}{\beta}$ in accordance with (33) in which case $\alpha_3 = 0$. Using Theorem 3 in the context described above, we obtain the following: the measure of the perturbation u,v

$$\overline{V}(t) = \frac{1}{2}\left[A\left(\|u_x\|^2 + \|v_x\|^2\right) + \|b_1 v_x - b_3 u_x\|^2 + \|b_2 v_x - b_4 u_x\|^2\right], \tag{44}$$

satisfies, in view of Remark 6,

$$\frac{d\overline{V}}{dt} \le AI\left(\|u_x\|^2 + \|v_x\|^2\right) + \overline{\Psi}, \tag{45}$$

where $\overline{\Psi}$ is given by

$$\begin{aligned} \overline{\Psi} &= \int_0^1 \left[-\alpha_1 c_1 \beta v u_x^2 + \alpha_2 c_2 \alpha u v_x^2 + \left(-\alpha_1 c_1 \beta u + \alpha_2 c_2 \alpha v\right) u_x v_x\right] dx \\ &\le M \int_0^1 \left(|u| + |v|\right)\left(u_x^2 + v_x^2 + |u_x v_x|\right) dx \\ &\le \frac{3}{2} M \int_0^1 \left(|u| + |v|\right)\left(u_x^2 + v_x^2\right) dx \end{aligned} \tag{46}$$

where

$$M = \max\left[c_1\alpha_1\beta, c_2\alpha_2\alpha\right] . \tag{47}$$

The following fundamental inequality (e.g. [6]) is used hereunder:

$$|\Phi(x)|^2 \leq x(1-x)\,\|\Phi_x\|^2 , \tag{48}$$

or the weaker version thereof,

$$|\Phi(x)| \leq \frac{1}{2}\,\|\Phi_x\| , \tag{49}$$

where Φ is any smooth function vanishing at $x = 0, 1$. Using (49) we obtain

$$\begin{aligned}\int_0^1 (|u| + |v|)\left(u_x^2 + v_x^2\right) dx \\ &\leq \frac{1}{2}\left(\|u_x\| + \|v_x\|\right)\left(\|u_x\|^2 + \|v_x\|^2\right) \\ &\qquad \leq 2^{-\frac{1}{2}}\left(\|u_x\|^2 + \|v_x\|^2\right)^{\frac{3}{2}} .\end{aligned} \tag{50}$$

This together with (46) gives

$$\overline{\Psi} \leq \delta_1\left(\|u_x\|^2 + \|v_x\|^2\right)^{\frac{3}{2}} \tag{51}$$

where

$$\delta_1 = 3 \cdot 2^{-\frac{3}{2}} M . \tag{52}$$

It follows from (44) etc. (e.g. [3]) that

$$k_1\left(\|u_x\|^2 + \|v_x\|^2\right) \leq \overline{V} \leq k_2\left(\|u_x\|^2 + \|v_x\|^2\right) \tag{53}$$

where

$$k_1 = \frac{A}{2}; \quad k_2 = \frac{A}{2} + \sum_{i=1}^{4} b_i^2 . \tag{54}$$

Thus, using (44)–(45),(51),(53)–(54), we obtain the differential inequality

$$\frac{d\overline{V}}{dt} \leq -dV + d_1 V^{\frac{3}{2}} \tag{55}$$

where

$$d = \frac{A|I|}{k_2}; \quad d_1 = \frac{\delta_1}{k_1^{(3/2)}} . \tag{56}$$

It may be noted, *en passant*, that

$$A|I| = (\pi^4\gamma_1\gamma_2 + a_1a_2)\pi^4\gamma_1\gamma_2 \tag{57}$$

using (12) etc.

From (55) we obtain the following (e.g. [7]). Supposing that the initial perturbation (assumed known) is such that

$$\left\{\overline{V}(0)\right\}^{\frac{1}{2}} \leq \frac{d}{d_1}\,, \tag{58}$$

then

$$\frac{d\overline{V}}{dt} \leq -\eta\overline{V} \tag{59}$$

where

$$\eta = \alpha\left[1 - d_1d^{-1}\left\{\overline{V}(0)\right\}^{\frac{1}{2}}\right]\,, \tag{60}$$

whence

$$\overline{V}(t) \leq \overline{V}(0)e^{-\eta t} \tag{61}$$

A pointwise estimate follows from this on using (48), (61):

$$\{u(x,t)\}^2 + \{v(x,t)\}^2 \leq x(1-x)k_1^{-1}\overline{V}(0)e^{-\eta t}\,. \tag{62}$$

8 Theorem. *The equilibrium configuration (37) of the Lotka-Volterra system (36) etc. is conditionally exponentially stable*

(a) in the measure $\overline{V}$, as conveyed by (58), (60), (61);

(b) pointwise, as conveyed by (62) etc.

9 Remark. It should be emphasized that pointwise stability estimates (in one dimension) are also obtainable by these methods for a large class of reaction-diffusion systems of the type (1)–(2), (6)–(7).

10 Remark. It will be noted that the system (36) also admits an equilibrium configuration $S_1 = S_2 = 0$ (in the presence of constant boundary conditions). Stability estimates, analogous to those of Theorem 8, can be obtained in a similar manner, subject to the restriction, essentially following from (30),

$$\pi^2\gamma_1 - a_1 > 0\,. \tag{63}$$

This condition is consistent with the fact that the equilibrium in question is known to be unstable in the absence of diffusion.

11 Remark. A similar approach may be used to obtain similar stability estimates for $\overline{\overline{V}}(t)$ in any number of spatial dimensions. Pointwise stability estimates may be deduced using a Sobolev inequality of the type

$$|\Phi(\mathbf{x})| \leq K \left\| \nabla^2 \Phi \right\|, \tag{64}$$

where K is a constant, e.g. [6].

12 Remark. Many issues cognate to those discussed in this paper may be found in [8] and in the many references therein.

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References

[1] S. Rionero: *A rigorous reduction of the L^2-stability of solutions to a nonlinear binary reaction-diffusion system of P.D.E.s to the stability of the solutions to a binary linear system of O.D.E.s*, J. Math. Anal. Appl., **319** (2006), 372–392.

[2] S. Rionero: *A nonlinear* L^2 *stability analysis for two species dynamics with dispersal*, Math. Biosci. Engineering, **3** (2006), n. 1, 189–204.

[3] S. Rionero: *L^2 stability of solutions to a nonlinear binary reaction-diffusion system of P.D.E.s*, Rend. Mat. Acc. Lincei, s.9, **16** (2005), 227–238.

[4] J. N. Flavin, S. Rionero: *Asymptotic and other properties for a nonlinear diffusion model*,J. Math. Anal. Appl., **228** (1998), 119–140.

[5] A. Okubo, S. A. Levin: Diffusion and Ecological Problems: Modern Perspectives. Interdisciplinary Mathematics, vol. 14., Springer-Verlag, Berlin, Heidelberg, New York 2001.

[6] J. N. Flavin, S. Rionero: Qualitative Estimates for P.D.E.s. An introduction, CRC Press, Boca Raton, FL. 1995.

[7] B. Straughan: The Energy Method, Stability and Nonlinear Convection, Appl. Math. Sci. Ser., vol. 91., Springer-Verlag, 2nd edition, 2004.

[8] R. S. Cantrell, C. Cosner: Spatial ecology via reaction-diffusion equations, John Wiley and Sons Ltd., Chichester (U.K.) 2004.