Note di Matematica **26,** n. 2, 2006, 1–50.

On Riemann sums

Jean-Jacques Ruch
Laboratoire d'Algorithmique Arithmétique, Université Bordeaux I,
351, cours de la Libération, F-33405 Talence Cedex, France.
ruch@math.u-bordeaux.fr

Michel Weber
Mathématique (IRMA), Université Louis-Pasteur et C.N.R.S.,
7 rue René Descartes, 67084-Strasbourg Cedex, France.
weber@math.u-strasbg.fr

Received: 10/3/2004; accepted: 10/3/2005.

Abstract. The study of almost sure convergence of Riemann sums is a fascinating question which has connections with various problems from Number Theory, among them the Riemann hypothesis through its link with Farey sequences. Moreover, it has been known since the fundamental paper of Rudin, that the convergence almost everywhere of Riemann sums, along a given subsequence of positive integers, definitively relies on the arithmetical properties of the subsequence. The arithmetical characterization of that property is an open and certainly hard question. The study of Riemann sums has for years been an object of constant interest from analysts, ergodicians, and number theorists. It even seems, that its power of attraction has grown even more during this last decade. This is the reason of the present survey. Our motivation in writing it, was to propose a text to the interested reader, giving a direct access to the main results of that theory, as well as an easy understanding, as far as possible each time in each case, of the various methods elaborated by the authors of these results.

Keywords: Riemann sums, chains, Rudin sequences, Farey series, almost sure convergence, entropy criterion, Riemann hypothesis.

MSC 2000 classification: primary 42A24, 40G99, secondary 26A42

1 Introduction

The study of the almost everywhere convergence of the Riemann sums of a measurable function f is a famous and still unsolved problem having deep arithmetical aspects. In the present paper, we will mainly be interested in the study of the almost sure convergence of these sums for Lesbegue integrable functions.

We will state and comment the essential results, discuss their links when necessary and also give indications of proofs. The paper is organized as follows: in Section 2 we introduce Jessen's Theorem on convergence almost everywhere of Riemann sums along chains of integers. A typical example of a chain is the sequence of the powers of two. This is likely to be the first result of the theory. A brief sketch of the proof is indicated, and comments by Marcinkiewicz-Salem about its optimality are also included. We continue with Rudin's Theorem, which is the second fundamental result in the Theory of Riemann sums. This Theorem shows for instance the irregularity of Riemann sums along the sequence of primes. We conclude that section by presenting a striking example obtained by Rudin as a by-product of Jessen's and Rudin's Theorems as well as Dirichlet's Theorem on the distribution of primes in arithmetic progressions, which motivated the study of Riemann sums. This example indeed shows that the study of the convergence almost everywhere of Riemann sums along a given sequence, definitively relies on the arithmetical properties of the sequence. Some extensions of Jessen's Theorem are also included at the end of this Section.

Section 3 is devoted to results of a different nature. They are individual type Theorems. A different approach is considered here. It is indeed possible to obtain sufficient conditions on the function f, sometimes quite sharp, ensuring the convergence almost everywhere of the Riemann sums of f. These conditions are often expressed in terms of the integral modulus of continuity of f, and have a direct translation to properties of the Fourier coefficients of f. The results are mainly due to Marcinkiewicz and Salem.

The next section in some sense combines these points of view by giving new arithmetical characterizations for the convergence of Riemann sums of specific classes of functions.

In Section 5, we discuss a new method introduced and developed by Bourgain in the study of the convergence almost everywhere of sequences of operators like for instance Riemann sums. The method relies on an important tool: the Bourgain's Entropy Criterion. We will see how it allows to prove Rudin's Theorem by different techniques, as well as other new results. Special attention will be given to the study of the convergence of Riemann sums along the sequence of primes.

In Section 6, we are concerned with deep connections between Riemann sums and Number Theory, in particular their link with Riemann Hypothesis through the study of Farey sequences and with the Prime Number Theorem, based on the thorough work of Wintner.

Finally Section 7 is devoted to some parallel results which seemed important to us. They mostly concern operators defined in a slightly different way than the usual Riemann sums.

Throughout the paper we denote the torus by $\mathbb{T} = [0,1[= \mathbb{R}/\mathbb{Z}$ and let m the normalized Lebesgue measure on it. Let f be any measurable function on $\mathbb{T}$. For $n = 1, 2, \dots$, define the Riemann sums of f as follows: for all $n \geqslant 1$

$$R_n(f)(x) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(x + \frac{j}{n}\right) \qquad (\forall x \in \mathbb{T}) \tag{1}$$

When $x = 0$, we simply write

$$R_n(f) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(\frac{j}{n}\right), \tag{2}$$

for the usual Riemann sums. Their role in Number Theory is deep (see Section 6). The results we shall discuss here mostly concern these operators. Before ending this introduction, let us indicate and comment on a fundamental property of Riemann sums. Consider the characters of $\mathbb{T}$: $e_k(x) = \exp(2\pi ik)$, $k \in \mathbb{Z}$. Then, for all $n \geqslant 1$,

$$R_n(e_k(x)) = e_k(x)\frac{1}{n}\sum_{j=0}^{n-1} \exp\left(\frac{2i\pi kj}{n}\right) = e_k(x)\delta_{n|l}. \tag{3}$$

Hence, for $f \in L^2(m)$ with Fourier expansion $f \sim \sum_l a_l e_l$, the Riemann sums of f can be expressed as

$$R_n(f) = \sum_{n|l} a_l e_l. \tag{4}$$

We shall comment this property by means of the infinite Möbius inversion due to Hartman and Wintner [17, page 853]. Consider the following two infinite systems of linear equations (where in (6), $\mu(.)$ denotes the Möbius function, see Section 6)

$$\sum_{m=1}^{\infty} x_{nm} = y_n, \qquad (n = 1, 2, \ldots) \tag{5}$$

$$\sum_{m=1}^{\infty} \mu(m) y_{nm} = x_n, \qquad (n = 1, 2, \ldots) \tag{6}$$

If $x_n = O(n^{-1-\eta})$ for some $\eta > 0$, then (5) has unique solution which is given by (6), namely $x_n = \sum_{m=1}^{\infty} \mu(m) y_{nm}$, $n = 1, 2, \ldots$. Conversely, if $y_n = O(n^{-1-\eta})$ for some $\eta > 0$, then (6) has unique solution which is given by (5).
In our case, this shows that if the Fourier coefficients of f satisfy the condition

$$a_n = O(|n|^{-1-\eta}) \qquad \text{for some} \quad \eta > 0, \tag{7}$$

then, it is possible to obtain f from its Riemann sums. More precisely

$$a_n e_n(x) = \sum_m \mu(m) R_{nm}(f(x)). \tag{8}$$

2 Fundamental results of Jessen and Rudin

We may introduce the problem as follows. When f is Riemann integrable on $\mathbb{T}$, for any real x we have:

$$\lim_{n\to+\infty} R_n(f)(x) = \int_0^1 f(t)dm(t). \tag{9}$$

When f is only Lebesgue integrable ($f \in L^1(\mathbb{T})$), it is a well-known fact that $\{R_n(f), n \geqslant 1\}$ converges to $\int_{\mathbb{T}} f\ dm$ in mean. This is easy to check.

Let us indeed first consider $f \in L^2(\mathbb{T})$ with Fourier expansion $f(x) \sim \sum_l a_l e_l(x)$, where $e_l(x) = \exp(2i\pi l x)$, $l \in \mathbb{Z}$, and $a_0 = \int f dm = 0$. From (3) we deduce

$$R_n f = \sum_{n|l} a_l e_l, \tag{10}$$

so that $||R_n(f)||_2^2 \leqslant \sum_{|l|\geqslant n} a_l^2 \ \to\ 0$ as n tends to infinity. This proves, that for $f \in L^2(\mathbb{T})$ we have $\lim_{n\to+\infty} \big\|R_n(f) - \int f dm\big\|_2 = 0$.

When $f \in L^1(\mathbb{T})$, let $(f_k)_{k\geqslant 1} \subset L^2(\mathbb{T})$ approximating f in $L^1(\mathbb{T})$ so that :

$$\lim_{k\to+\infty} \|f - f_k\|_1 = 0.$$

Let $\epsilon > 0$ be fixed, and choose k large enough such that $\big\|f_k - f\big\|_1 \leqslant \epsilon$. Observe that

$$\begin{aligned} \big\|R_n(f) \ &- \int f dm\big\|_1 \\ \leqslant \big\|R_n(f) \ &- \ R_n(f_k)\big\|_1 + \big\|R_n(f_k) - \int f_k dm\big\|_1 + \big|\int f_k dm - \int f dm\big|. \end{aligned}$$

Since R_n is an $L^1(\mathbb{T})$ contraction, we may write

$$\begin{aligned} \big\|R_n(f) \ &- \int f dm\big\|_1 \\ &\leqslant \ \big\|f - f_k\big\|_1 + \big\|R_n(f_k) - \int f_k dm\big\|_1 + \big|\int f_k dm - \int f dm\big| \\ &\leqslant \ 2\epsilon + \big\|R_n(f_k) - \int f_k dm\big\|_2. \end{aligned}$$

Letting k tend to infinity, we obtain

$$\overline{\lim_{k\to+\infty}} \big\|R_n(f) - \int f dm\big\|_1 \leqslant 2\epsilon.$$

But ϵ is arbitrary, and so we have that

$$\lim_{n\to+\infty}\left\|R_n(f)-\int f\,dm\right\|_1=0.$$

It is natural to inquire about the almost everywhere convergence of these sums. A first study was made in 1914 by Hahn [16] in a paper describing the approximation of Lebesgue integral by Riemann sums. But to our knowledge, it is in an article published by Jessen [19, Theorem A, page 60] in 1934, that one can find the first real result about the almost sure convergence of Riemann sums. Introduce the following definition:

1 Definition. A sequence of positive integers is a chain $\mathcal{S}=(n_k)_{k\geqslant 1}$, if :

$$n_k|n_{k+1} \qquad \text{(for all } k\geqslant 1) \tag{11}$$

By considering such sequences, Jessen was able to prove the following result:

2 Theorem. *Assume that $f\in L^1(\mathbb{T})$. Then*

$$\lim_{k\to+\infty} R_{n_k}f(x)=\int_{\mathbb{T}} f(t)dm(t) \quad a.e.$$

Proof. As noted by Marcinkiewicz and Salem [31], this result is in a certain sense best possible. Indeed, in the particular case when $\mathcal{S}=\{2^n \mid n\geqslant 1\}$, they proved (Theorem 1, page 377) that for every positive and increasing function ω satisfying $\lim_{x\to+\infty}\frac{\omega(x)}{\log x}=0$, it is possible to associate a function f satisfying:

$$\int_{\mathbb{T}}|f|\,\omega(|f|)dm<+\infty \quad \text{and} \quad \int_{\mathbb{T}}\sup_{s\geqslant 0}|R_{2^s}(f)|\,dm=+\infty.$$

Jessen's result is based on the following observation: Since f is 1-periodic, $R_n(f)$ is $\frac{1}{n}$-periodic for any $n\geqslant 1$, and thus $\frac{1}{m}$-periodic if m divides n.

Consequently, since $R_{n_k}f(x)$ is $\frac{1}{n_k}$-periodic for any k, it follows that

$$\Phi(x)=\overline{\lim_{n_k\to+\infty}}R_{n_k}f(x)=C,$$

for almost every x, where C denotes some constant. It suffices here in fact that: For infinitely many p, n_p divides n_m as soon as m is large enough. Let B be some fixed real and put

$$E_k=\{x\mid R_{n_k}(f)(x)>B\}.$$

Then E_k as well as E_k^c are $\frac{1}{n_k}$-periodic. Put

$$E=\{x\mid \sup_{1\leqslant k\leqslant N}R_{n_k}(f)(x)>B\}.$$

We have, $E = E_N + E_N^c \cap E_{N-1} + E_N^c \cap E_{N-1}^c \cap E_{N-2} + \cdots + E_N^c \cap \ldots \cap E_2^c \cap E_1$. Set

$$A_k = E_N^c \cap \ldots \cap E_{k+1}^c \cap E_k.$$

Then A_k is $\frac{1}{n_k}$-periodic. Thus,

$$\int_{A_k} f(x)\ dx = \int_{A_k} f(x + \frac{j}{n_k})\ dx = \int_{A_k} R_{n_k}(f)(x)\ dx \geqslant Bm(A_k).$$

Consequently, by summing over k

$$\int_E f(x)dx \geqslant Bm(E).$$

Letting then N tend to infinity, this leads to

$$\int_{\{x\ \mid\ \sup_{k\geqslant 1} R_{n_k}(f)(x) > B\}} f(x)dx \geqslant Bm(x \mid \sup_{k\geqslant 1} R_{n_k}(f)(x) > B).$$

If $B < C$, $m(x \mid \sup_{k\geqslant 1} R_{n_k}(f)(x) > B) = 1$. The above relation thus shows

$$\int_0^1 f(y)dy \geqslant B \cdot 1 = B.$$

Hence $C \leqslant \int_0^1 f(y)dy$. By replacing f by $-f$, we deduce

$$\int_0^1 f(y)dy \leqslant \liminf_{n_k\to\infty} R_{n_k}(f)(x).$$

Hence the result. $\boxed{QED}$

Later Ursell [60] showed (page 231) that Riemann sums converge almost everywhere along the whole sequence of integers for a special class of square summable functions — namely those, that are monotonic in the periodic interval $[0, 1]$. On the other hand he gave (page 230) the following simple example, which shows that almost everywhere convergence does not take place for Lebesgue integrable functions in general: let $\frac{1}{2} < \delta < 1$ be fixed and take for all $0 < x \leqslant 1$

$$f(x) = |x|^{-\delta}.$$

This result as well as the next one, obtained by Marcinkiewicz and Zygmund [32, Theorem 3 page 157 and Theorem 3' page 158] announce in some sense the much more general result of Rudin [47].

3 Theorem. *There exists an $f \in L^1(\mathbb{T})$ such that*

$$\overline{\lim_{n\to+\infty}} R_{2n+1}(f)(x) = +\infty \qquad a.e.$$

The proof of this Theorem relies upon the fact that it is possible to exhibit a subset $\mathcal{H}$ of $[0,1[$ with positive measure, and possessing the following property: For any $x \in \mathcal{H}$, there exists a sequence of fractions, $\{p_i/q_i\}_{i=1,2,\dots}$, with q_i even, such that:

$$\left| x - \frac{p_i}{q_i} \right| < \frac{4}{q_i^2}, \quad \text{for all } i \geqslant 1.$$

Let $\mathcal{H}^*$ be the set of points $x + \frac{\alpha}{\beta}$, where $x \in \mathcal{H}$ and $\frac{\alpha}{\beta}$ are fractions with even denominators. Since $m(\mathcal{H}) > 0$, $\mathcal{H}^*$ contain almost all reals in $[0,1[$. Moreover, to each $x \in \mathcal{H}^*$ we can associate a sequence of fractions $\{p_i/q_i\}_{i=1,2,\dots}$ satisfying:

$$\left| x - \frac{p_i}{q_i} \right| < \frac{4\beta^2}{q_i^2}, \quad \text{for all } i \geqslant 1,$$

where β depends on x only. Now, it is easy to check that if

$$f(x) = |x|^{-\frac{1}{2}} \log \frac{1}{|x|},$$

for all $|x| \leqslant \frac{1}{2}$, the requested property is satisfies.

In a well-known paper written by Rudin [47], it was shown that there is not necessarily almost sure convergence, even for bounded functions. According to [47, page 322] we have:

4 Theorem. *If $\mathcal{S}$ is an increasing sequence of positive integers satisfying:*

- for any $N \geqslant 1$, there is a set $\mathcal{S}_N$ of N elements of $\mathcal{S}$, none of which divides the least common multiple (l.c.m.) of the others,

then there is a measurable set A of $\mathbb{T}$, such that for $f = \mathbf{1}_A$, $\{R_n(f), n \geqslant 1\}$ does not converge almost everywhere.

For example, one can takes for $\mathcal{S}$ the sequence of primes. We will see in Section 4, that this result can be entirely proved by means of Bourgain's entropy criterion. This Theorem immediately implies, that there is no maximal inequality for the Riemann sums. Indeed, otherwise this would imply, by means of the Banach Principle, that the set of elements of $L^2(m)$ for which $\{R_n(f), n \geqslant 1\}$ converges almost everywhere is closed. Since $\{R_n(f), n \geqslant 1\}$ does converge almost everywhere when f is a finite linear combination of the characters $e_n(x) = \exp(2i\pi nx)$, $n \in \mathbb{Z}$, this set is also everywhere dense in $L^2(m)$ providing a contradiction.

By combining this Theorem with Jessen's result, and using Dirichlet's Theorem on primes in arithmetic progressions, Rudin has built a sequence $\mathcal{S} = \{n_k, k \geqslant 1\}$ possessing some dramatic properties. The construction goes as follows. Let $n_1 = 1$ and assume n_k is defined; there exists an integer $r > 1$ such that $q = 1 + rn_k$ is a prime. Then we set $n_{k+1} = rn_k$. One the one hand, by means of Jessen's Theorem, we know that

a) for any $f \in L^1(m)$,

$$m\{x \mid \lim_{\mathcal{S}\ni n\to+\infty} R_n(f)(x) = \int_{\mathbb{T}} f\ dm\} = 1.$$

And on the other hand, this time by invoking Rudin's Theorem,

b) there exists a bounded Lebesgue measurable f such that

$$m\{x \mid \lim_{\mathcal{S}\ni n\to+\infty} R_{n+1}(f)(x) = \int_{\mathbb{T}} f\ dm\} = 0.$$

This clearly shows that the problem relies on the arithmetical properties of $\mathcal{S}$.

Before ending this section, we give a generalization of Jessen's result. The fact that for $f \sim \sum_l a_l e_l$, the Riemann sums of f can be expressed by $R_n(f) = \sum_{n|l} a_l e_l$ leads to a natural generalization of the problem in L^2-spaces. Assume, that we are given a fixed set of indices $\mathcal{N}$ together with some fixed element (a_l) of ℓ_2. Let μ be a Borel probability measure on $[0,1]$. Let (ψ_n) be an orthonormal sequence of $L^2 = L^2(\mu)$ and define the generalized Riemann sums as follows

$$\mathcal{R}_n = \mathcal{R}_n^{(a)} = \sum_{n|l} a_l \psi_l.$$

The investigation of the problem of knowing whether the convergence almost everywhere of the sums $\mathcal{R}_n$ when n runs along the index $\mathcal{N}$ takes place for *all* orthonormal systems simultaneously generalizes the study of the convergence almost everywhere of Riemann sums as well as the one of orthogonal series. This is quite a fundamental problem in Analysis. It is also quite a hard task, since for instance the periodicity argument used repeatedly by Jessen for proving the convergence of Riemann sums along chains no longer works for arbitrary orthogonal systems. However, in [61, Theorem 11] the following extension of Jessen's Theorem is obtained by means of the theory of stochastic processes.

5 Theorem. *Let $\mathcal{N} = \{n_k, k \geqslant 1\}$ be a chain and put*

$$E_k = \{n :\ n_k|n\}, \qquad F_k = E_k \backslash E_{k+1} \quad \textit{and} \quad \delta_k^2 = \sum_{n\in F_k} a_n^2.$$

Assume, that

(a) $\sum_{n\geqslant 1} \delta_n^2 (\log \frac{1}{\delta_n})^2 (\log\log \frac{1}{\delta_n})^2 (\log\log\log \frac{1}{\delta_n})^{2+\varepsilon} < +\infty, \quad (\varepsilon > 0)$

or,

(b) $\sum_{n\geqslant 1} \delta_n^2 (\log \frac{2}{\delta_n})^{1+h} (\log n)^{1-h} < +\infty, \quad (0 \leqslant h < 1).$

Then, the sequence $(\mathcal{R}_n, n \in \mathcal{N})$ *converges almost surely. In particular, if*

(a') $\sum_{n\geqslant 1} a_n^2(\log \frac{1}{a_n})^2(\log\log \frac{1}{a_n})^2(\log\log\log \frac{1}{a_n})^{2+\varepsilon} < +\infty,\ (\varepsilon > 0)$

or,

(b') $\sum_{n\geqslant 1} a_n^2(\log \frac{2}{a_n})^{1+h}(\log n)^{1-h} < +\infty,\ (0 \leqslant h < 1),$

then the sequence $(\mathcal{R}_n, n \in \mathcal{N})$ *converges almost surely.*

Notice that the latter conditions are of the same type as those in the paper by [31] (see e.g. condition (3.10)).

Extensions of Jessen's Theorem for locally compact groups were also obtained by Ross-Stromberg in 1967 and more recently by Ross-Willis in 1997 (see [49], [50]). A generalization of Jessen's Theorem to one-parameter groups of measure preserving transformations was given by Civin [8] in 1955. Let $T(\epsilon)$ be such a group. If f is an integrable function satisfying $f(s) = f(T(1)s)$, then the result asserts that the sequence of sums $f_n(s) = 2^{-n}\sum_{i=1}^{2^n} f(T(i2^{-n}s))$ converge almost everywhere as $n \to \infty$.

3 Individual Theorems of spectral type

The main contributions are due to Marcinkiewicz and Salem. In [31], various type of results are presented with deep insight. Compared with the preceding section, the approach developed is different. Both authors have examined under which kind of regularity assumptions on f, the associated sequence of Riemann sums converges *a.e.* These conditions are often expressed in terms of the integral modulus of continuity of f, and have a direct translation on the Fourier coefficients of f. For instance (see [31, Theorem 2, page 377])

6 Theorem. *Under the condition*

$$\int_{\mathbb{T}} [f(x+t) - f(x)]^2 \ dx = O(t^{\varepsilon}) \qquad (\varepsilon > 0), \tag{12}$$

the sequence $\{R_n(f), n \geqslant 1\}$ *converges a.e. to* $\int_{\mathbb{T}} f\ dm$.

Let us give a sketch of the proof.

Let $\sum_{\nu=-\infty}^{+\infty} a_\nu e^{2\pi i\nu x}$ be the Fourier series of f. By replacing f by $f - a_0$, if necessary, we may suppose that $a_0 = \int_0^1 f(x)dx = 0$. We have

$$R_n f(x) = \sum_{\nu=-\infty}^{+\infty} a_{n\nu} e^{2\pi i\nu nx},$$

and thus:

$$\begin{aligned}\sum_{n\geqslant 1}\int_0^1 R_n^2 f(x)dx &= \sum_{n\geqslant 1}\sum_{\nu=-\infty}^{+\infty} a_{n\nu}^2 = \sum_{n\geqslant 1}\sum_{\nu\geqslant 1} a_{n\nu}^2\\ &= \sum_{k\geqslant 1} a_k^2 \mathrm{d}(k),\end{aligned}$$

where $d(k)$ is the number of divisors of k. But it is well known (see for example [18, Theorem 315 page 260]) that:

$$d(k) = k^{\delta}, \quad \text{for all positive } \delta.$$

Therefore $\sum_{n\geqslant 1}\int_0^1 R_n^2 f(x)dx < +\infty$, if

$$\sum_{k\geqslant 1} a_k^2 k^{\delta} < +\infty, \quad \text{for some } \delta > 0.$$

Consider now the integral

$$\int_0^1\int_0^1 \frac{\left[f(x+t)-f(x-t)\right]^2}{t^r}\, dtdx.$$

This integral is obviously finite, by condition (12), if we suppose $r < 1+\varepsilon$. Further by Parseval relation we have

$$\int_0^1 \big[f(x+t)-f(x-t)\big]^2 dx = 4\sum_{\nu=-\infty}^{+\infty} a_\nu^2 \sin^2(2\pi\nu t).$$

Therefore

$$\begin{aligned}\int_0^1\int_0^1 \frac{\big[f(x+t)-f(x-t)\big]^2}{t^r}\, dtdx &= 4\sum_{\nu=-\infty}^{\pi f} a_\nu^2 \int_0^1 \frac{\sin^2(2\pi\nu t)}{t^r}\\ &\geqslant C\sum_{\nu=-\infty}^{+\infty} a_\nu^2 \nu^{r-1}.\end{aligned}$$

Thus we get

$$\sum_{n\geqslant 1}\int_0^1 R_n^2 f(x)dx < +\infty,$$

which easily leads to $R_n f(x) \to 0$ for almost all x, and this is exactly the assertion of Theorem 6.

The authors made two interesting comments. First, it is not possible to replace in the integral appearing in (12), $\big[f(x+t)-f(x)\big]^2$ by $\big[f(x+t)-f(x)\big]^p$ with $p<2$. The second comment concerns the following averages:

$$\forall n \geqslant 1, \qquad A_n(f) = \frac{1}{n}\sum_{k=1}^{n} R_k(f). \tag{13}$$

When replacing Riemann sums by their averages $A_n(f)$, assumption (12) can be essentially weakened. Indeed [31, Theorem 3, page 378],

7 Theorem. *Under the condition*

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\big[f(x+t)-f(x)\big]^2}{t|\log\frac{t}{2}|}\, dtdx < \infty \tag{14}$$

the sequence $\{A_n(f), n \geqslant 1\}$ converges a.e. to $\int_{\mathbb{T}} f\ dm$.

Note, that condition (14) is satisfied if, for instance

$$\int_{\mathbb{T}} \big[f(x+t)-f(x)\big]^2\ dx = O\left(\frac{1}{\log^2|\log\ t|}\right), \tag{15}$$

which is essentially less restrictive than (12). This result is generalized for $p<2$. One can find in [31, Theorem 4, page 387]

8 Theorem. *Under the condition*

$$\int_{\mathbb{T}} |f(x+t)-f(x)|\ dx = O\left(\frac{1}{|\log t|^s}\right) \qquad (s>1) \tag{16}$$

the sequence $\{A_n(f), n \geqslant 1\}$ converges a.e. to $\int_{\mathbb{T}} f\ dm$. This is true in particular if f is nondecreasing and

$$\int_{\mathbb{T}} |f(x)|^p\ dx < \infty, \tag{17}$$

for some $p>1$.

Finally, the authors conjecture that the expressions $A_n(f)$ converge *a.e.* for every $f \in L^2(\mathbb{T})$. This is a famous open conjecture. Related to that conjecture a difficult Theorem of Bourgain [5, Theorem 1.10, page 100] asserts

9 Theorem. *Let $f \in L^2(\mathbb{T})$. Then $R_n(f)$ has logarithmic density :*

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}R_n(f) \to \int_{\mathbb{T}} f\ dm \qquad\qquad a.e. \tag{18}$$

Marcinkiewicz and Salem have also observed that some arithmetical properties necessarily intervene in the study of the problem. Let $f = \sum_{p \text{ prime}} c_p e_p$ with $c_p \to 0$ as p tends to infinity. Then, $R_n(f)(x) = 0$ a.e. if n is not a prime number, and $R_n(f)(x) = c_n e_n + c_{-n} e_{-n}$ otherwise. Consequently, $R_n(f)(x) \to 0$ uniformly, outside a measurable set of x's of zero measure. But, we may have f essentially bounded in no interval, which is rather surprising. Next, considering the expressions $R_p(f)$ for p prime only, they proved that (page 384),

$$\sum_p \int_{\mathbb{T}} |R_p f(x)|^2 \, dx \leqslant 2 \sum_{\nu=1}^{\infty} |c_\nu|^2 p(\nu) \tag{19}$$

where

$$f \sim \sum_{\nu \in \mathbb{Z}} c_\nu e_\nu, \qquad c_o = 0, \tag{20}$$

and $p(\nu)$ indicates the number of primes dividing ν. Since $p(\nu) = O\left(\frac{\log \nu}{\log \log \nu}\right)$, it follows that $R_p(f)(x) \to 0$ a.e. whenever

$$\sum_{\nu=3}^{\infty} |c_\nu|^2 \frac{\log \nu}{\log \log \nu} < \infty, \tag{21}$$

The latter condition is satisfied in particular if

$$\int_{\mathbb{T}} [f(x+t) - f(x)]^2 \, dt = O\left(\frac{1}{\log^2 \frac{1}{t}}\right), \tag{22}$$

which is a much weaker condition than (12), as we can easily see. We also mention the following criterion due to Salem [52, page 60] providing a sufficient condition for the convergence a.e. of the Riemann sums $R_{n_i}(f)$ along a given sequence of integers $(n_i)_{i \geqslant 1}$ if the integral modulus of continuity of f is sufficiently smooth.

10 Theorem. *Assume that*

$$\int_{\mathbb{T}} |f(x+t) - f(x)| \, dx = O\left(\frac{1}{|\log t|^{1+\varepsilon}}\right) \qquad (\varepsilon > 0), \tag{23}$$

and let $(n_i)_{i \geqslant 1}$ be an increasing sequence of positive integers. Then $\lim_{k\to\infty} R_{n_k}(f) = \int_{\mathbb{T}} f \, dm$ a.e. whenever

$$\sum_{k \geqslant 1} \left(\frac{1}{\log n_k}\right)^{1+\delta} < \infty, \tag{24}$$

for some $\delta < \varepsilon$.

In [66] Yano has established a similar result.

11 Theorem. *Let $f \in L^1(\mathbb{T})$.*

1) *Assume that $f \in Lip(\alpha, p)$, $0 < \alpha \leqslant 1$, $p \geqslant 1$, and $\sum_k n_k^{-\alpha} < +\infty$. Then,*

$$R_{n_k}(f) \to \int_{\mathbb{T}} f dm \qquad a.e.$$

2) *If $\int_{\mathbb{T}} |f(x+t) - f(x)| dx = O\left(\log^{-s}\left(t^{-1}\right)\right)$ and $\sum_k \log^{-s}(n_k) < +\infty$ for some $s > 1$. Then,*

$$R_{n_k}(f) \to \int_{\mathbb{T}} f dm \qquad a.e.$$

Related to this result, Takahashi [55] proved in the opposite direction, that if $(n_k)_{k\geqslant 1}$ is a sequence of primes, then there exists a function f and a real sequence $(h_k)_{k\geqslant 1}$ such that $\lim_{k\to+\infty} R_{n_k} f(x + h_k)$ fails to exist for almost all x.

Some other authors were more interested in trying to find conditions on the Fourier coefficients. For instance in [54], the author solved the problem of restoration of the harmonics $C_n(f) = a_n(f)\cos(2\pi nx) + b_n(f)\sin(2\pi nx)$ from $R_n(f)$, where $a_n(f)$, $b_n(f)$ are the Fourier coefficients of a continuous function f. Further, in [58, page 230] the following result has been proved :

12 Theorem. *Let f be a function which is integrable in $(0, 2\pi)$, with period 2π and with Fourier series $f \sim \frac{a_0}{2} + \sum_{n\geqslant 1}(a_n \cos nx + b_n \sin nx)$. Assume that*

$$\lim_{n\to+\infty} \sum_{\nu\geqslant 1} \left(|a_{n\nu} - a_{n(\nu+1)}| + |b_{n\nu} - b_{n(\nu+1)}|\right) = 0.$$

Then for almost all x, there exists a sequence of positive integers $(m_k)_{k\geqslant 1}$ (depending on x) such that

$$\lim_{k\to+\infty} R_{m_k}(f)(x) = \int_0^{2\pi} f(t)dt \quad \text{and} \quad \lim_{k\to+\infty} \frac{k}{m_k} = 1.$$

The condition on the Fourier coefficients is satisfied in particular if, for example, $\sum_{\nu\geqslant 1}(|a_n - a_{n+1}| + |b_n - b_{n+1}|) < +\infty$, or if the sequences (a_n) and (b_n) are nonincreasing. This result forms a complement to the Theorems of Marcinkiewicz and Zygmund and Ursell.

In a second paper [59], Tsuchikara has considered functions of the following type:

$$f_t(x) = \sum \varphi_n(t) c_n e_n(x),$$

where $(\varphi_n(t))$ is a sequence of independent Rademacher functions. Assume that the following condition is satisfied:

$$\sum_{n>0} |c_n|^2 \log |n| < +\infty.$$

Then for almost all t, the sequence of Riemann sums associated to f_t converge almost everywhere. Further, if the Fourier coefficients of f are monotonic, the previous condition is enough to ensure the convergence almost everywhere of the Riemann sums associated to f_t.

Also, by considering functions with monotonic Fourier coefficients, Pannikov [35] and [36] has given the following generalization of Theorem 12.

13 Theorem. *Let f be a real-valued, 1-periodic function on $\mathbb{R}$, such that*

$$f(x) = \frac{a_0}{2} + \sum_{n\geqslant 1}(a_n \cos(2\pi nx) + b_n \sin(2\pi nx)).$$

1) If $a_k \downarrow 0$ and $b_k \downarrow 0$ then $\lim_{n\to+\infty} R_n(f) = C$ in measure for some constant C.

2) Moreover if $f \in L^2(0,1)$ then (1) holds almost everywhere and $C = \int_0^1 f dm$.

By using lacunary sequences, the author gives some generalizations of the above fact.

Another important result concerning Fourier series and Riemann sums is given in [34, page 546]. Let $f \in L(0, 2\pi)$ be 2π-periodic and everywhere equal to the sum of its Fourier series, i.e.

$$f(x) = \lim_{N\to+\infty} \sum_{n=-N}^{N} c_n e^{inx}.$$

Define for all $n \geqslant 1$,

$$r_n = \frac{2\pi}{n} \sum_{k=1}^{n} f\left(\frac{2\pi k}{n}\right) - \int_0^{2\pi} f(x)dx.$$

Then the following Theorem is established.

14 Theorem.

1) If $q > 1$ and $c_n + c_{-n} = O(n^{-q})$, then $r_n = O(n^{-q})$.

2) $c_n + c_{-n}$ may be $O(n^{-1})$ without r_n being $O(n^{-1})$.

3) If $q > 1$, $r_n = O(n^{-q})$ and $c_n + c_{-n} = O(n^{-s})$ for some $s > 1$, then $c_n + c_{-n} = O(n^{-q})$.

4) If $r_1 = r_2 = \cdots = 0$ then $c_n + c_{-n} = O(n^{-q})$ may be false for every $q > 1$.

The proofs of 1) and 2) are easy and self-contained. For 3) and 4) they use properties of some arithmetical functions, like the Möbius function.

4 Breadth and dimension

These results are essentially due to Baker [2], Dubins-Pitman [11], Révesz-Ruszla [48] and Bugeaud-Weber [6]. Introduce prealably the following definition

15 Definition. Let $\mathcal{A} \subset L^1(\mathbb{T})$. A sequence $\mathcal{S} = \{n_k,\ k \geqslant 1\}$ of positive integers is called a $\widehat{\mathcal{A}}$-sequence if for every $f \in \mathcal{A}$

$$\lim_{k\to+\infty} R_{n_k}(f) = \int_{\mathbb{T}} f dm \qquad \text{a.e.}$$

Introduce the following notation due to Baker [2]: $\mathcal{L}$ is the class of Lebesgue integrable functions, and $\mathcal{M}$ is the class of bounded measurable functions. Given two arbitrary sequences of positive integers, $\mathcal{S}_1$ and $\mathcal{S}_2$, we denote by $\mathcal{S}_1 \vee \mathcal{S}_2$ the new sequence obtained by ordering the set of positive integers $\{[s_1, s_2],\ s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2\}$, where $[s_1, s_2]$ denotes the least common multiple of s_1 and s_2 according to the natural order.

16 Theorem. *(Theorem 2.1, page 192).*

If $\mathcal{S}_1 = (m_k)$ and $\mathcal{S}_2 = (n_k)$ are two $\widehat{\mathcal{M}}$-sequences, then the sequence $\mathcal{S}_1 \vee \mathcal{S}_2$ is again an $\widehat{\mathcal{M}}$-sequence.

The proof of this result relies upon the following important property for Riemann sums:

$$R_m(R_n(f)) = R_{[m,n]}(f). \tag{25}$$

Recall the notion of Σ-sequences introduced by Cassels [7] in 1950.

17 Definition. Let μ_k be the number of fractions $\frac{j}{m_k}$ $(0 < j < m_k)$ which are not equal to $\frac{l}{m_q}$ (l integer, $q < k$). We say, that (m_k) is a Σ-sequence, if the following condition is satisfied:

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\mu_k}{m_k} > 0.$$

The interest of this notion lies in the fact that if (m_k) is a Σ-sequence, then the system of inequalities

$$\{m_k x\} < \psi(k), \quad (k = 1, 2, \dots),$$

where ψ is a non increasing function, possesses an infinity of solutions for almost all x, whenever the following series $\sum_{k\geqslant 1} \psi(k)$ is divergent. Conversely, there exists an example of decreasing function ψ such that the series $\sum_{k\geqslant 1} \psi(k)$ is convergent, and for which the previous system of inequalities has only finitely many solutions for almost all x. Baker's proof is partially based on this property.

It is interesting to observe that almost all sequences are Σ-sequences, although it is easy to exhibit some which are not. We give a second result due to Baker (Theorem 3.1, page 194).

18 Theorem. *Let (m_k) be a Σ-sequence such that $\liminf\limits_{k\to+\infty} k^{-1}\log m_k = 0$. Then (m_k) is not a $\widehat{\mathcal{L}}$-sequence.*

Baker, however, suggested that the assumption of (m_k) being a Σ-sequence is not likely to be well adapted to this problem, and also established the following remarkable result (Theorem 3.2, page 197):

19 Theorem. *Let $\epsilon > 0$. Assume that (m_k) is a sequence such that:*

$$\forall k \geqslant 1, \quad m_k = O\left(\exp\left(k^{\frac{1}{2}}(\log k)^{-\frac{7}{2}-\epsilon}\right)\right).$$

Then (m_k) is not a $\widehat{\mathcal{L}}$-sequence.

Now, introduce a generalization of the notion of a chain used by Dubins-Pitman [11]:

For sets of positive integers $\mathcal{S}_1,\dots,\mathcal{S}_d$, put

$$[\mathcal{S}_1,\dots,\mathcal{S}_d] = \{[n_1,\dots,n_d] \mid n_i \in \mathcal{S}_i,\ i=1,\dots,d\}, \tag{26}$$

where $[n_1,\dots,n_d]$ denotes the *l.c.m.* of $n_1,\dots,n_d$.

Let $\mathcal{S}$ be a set of positive integers. By the *dimension* of $\mathcal{S}$, we mean the least positive integer d, such that $\mathcal{S}$ is a subset of $[\mathcal{S}_1,\dots,\mathcal{S}_d]$ for some choice of chains $\mathcal{S}_1,\dots,\mathcal{S}_d$. Jessen's Theorem was extended by Dubins and Pitman [11], who proved

20 Theorem. *If $\mathcal{S}$ has dimension d and $f \in L(\log^+ L)^{d-1}$, then*

$$m\left\{ x \mid \lim_{\mathcal{S}\ni n\to\infty} R_n(f)(x) = \int_{\mathbb{T}} f\ dm \right\} = 1. \tag{27}$$

Recall that $L(\log^+ L)^{d-1}$ denotes the set of Lebesgue measurable functions on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} |f|(\log^+|f|)^{d-1}\ dm < \infty,$$

where it is understood, that $\log^+ x = \log_e x$ if $x \geqslant 1$ and equals 0 for $0 < x \leqslant 1$. A partial result ($d = 2$, f bounded) was proved by Baker [2].

The proof of that result consists of associating a converse d-martingale bounded in $\mathcal{L}\log^{d-1}\mathcal{L}$ to the sequence $\mathcal{S}$. Next, one proves the result by using a suitable extension to converse martingales of a maximal inequality for martingales with several parameters.

By considering the example of sequence of dimension two given by Jessen, namely the sequence $\mathcal{S} = \{2^i 3^j,\ i \geqslant 1,\ j \geqslant 1\}$, the authors also showed that it is not possible to improve Theorem 20, replacing $\mathcal{L}\log\mathcal{L}$ by $\mathcal{L}$.

Recently, Nair [33] suggested a more elementary proof avoiding the use of martingale theory. His argumentation is based on dominated estimates [22, page 50], Baker's observation on property (25) for Riemann sums, and an induction argument on the dimension of $\mathcal{S}$, which need a correction. In [6], it is shown that Nair's idea is however tractable. In the same paper, it is also proved that for no $d \geqslant 2$ can $L(\log^+ L)^{d-1}$ in Theorem 20 be replaced by $L(\log^+ L)^{d-2}$, which solves a conjecture by Dubins and Pitman [11]. For $d = 2$, this assertion is due to Baker. The proof of the general case consists of modifications of Baker's arguments, which are based on an elementary but rather technical lemma.

Introduce the following definition, which first appeared in [11]:

21 Definition. We say that a set K of integers has breadth at most d if the least common multiple of every finite subset of K is the least common multiple of at most d elements of that subset. The least such d is called the breadth of K and, if no such d exists, we say that K has infinite breadth.

Rudin's Theorem can be reformulated as follows: If (n_k) is a strictly increasing sequence of integers with infinite breadth, then there exist bounded measurable functions f on $\mathbb{T}$ such that $R_{n_k}(f)$ does not converge almost everywhere. Indeed, as (n_k) has infinite breadth, for every $r \geqslant 2$, there exist $k_1, \ldots, k_r$ such that n_{k_i} does not divide the least common multiple of $n_{k_1}, \ldots, n_{k_{i-1}}, n_{k_{i+1}}, \ldots, n_r$, for $1 \leqslant i \leqslant r$.

There exist sets of integers, which are neither of infinite breath, nor finite dimension, and consequently the a.e. convergence properties of Riemann sums along these sets are not known. Such a sequence has been given explicitely by L. Dubins and J. Pitman [11, Section 3b]. Denote by $p_1 < p_2 < \cdots < p_k < \cdots$ the sequence of primes and consider the set E_1 of all numbers of the type $p_1 \ldots p_{j-1}\, \check{p}_j\, p_{j+1} \ldots p_k$, for $k \geqslant 2$ and $1 \leqslant j \leqslant k$, where the symbol $\check{}$ means that p_j is excluded. In [6, Theorem 2], the authors exhibit a sequence (n_k) for fixed d, with infinite dimension and finite breadth which is not a $L(\log^+ L)^d$-sequence. The sequence is built as follows: let l be a positive integer. With the above notation, consider the set E_l of all integers n ranged in increasing order, such that

$$n = p_1^{a_1} \ldots p_{j-1}^{a_{j-1}} \check{p}_j p_{j+1} \ldots p_k,$$

for $k \geqslant 2,\ 1 \leqslant j \leqslant k$ and $l \geqslant a_1 \geqslant \cdots \geqslant a_{j-1} \geqslant 1$. Then E_l has infinite dimension and breadth not exceeding $l + 1$. The proof of this result relies upon the following extension [6, Lemma 1.5], of a Theorem of Baker.

22 Lemma. *If the sequence (n_k) satisfies the growth condition*

$$n_k = O\left(\exp k^{1/(2d+5)}\right),$$

then (n_k) is not a $L(\log^+ L)^d$-sequence.

In this paper, we also find the following result (proposition 3, page 11) concerning a sequence of finite breath and infinite dimension, namely the sequence E_1.

23 Proposition. *Let $f = \sum_{\nu=0}^{\infty} a_\nu e_\nu$, where $(a_\nu)_{\nu \geqslant 0} \in \ell_2$ satisfies*

$$\sum_{\nu=0}^{\infty} a_\nu^2 \left(\frac{\log l}{\log\log l}\right) < +\infty.$$

Then,

$$m\left\{\lim_{E_1 \ni n \to \infty} R_n(f) = \int_{\mathbb{T}} f \ dm \right\} = 1.$$

Concerning the averages along E_1, writing $E_1 = \{n_k, k \geqslant 1\}$,

$$m\left\{\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} R_{n_k}(f) = \int_{\mathbb{T}} f \ dm \right\} = 1,$$

holds for all $f \in L^2(\mathbb{T})$.

The first part of the proposition is a spectral type result in the sense of Marcinkiewicz-Salem. Since this proposition in some sense constitutes a break with respect to the set of results mentioned above, we briefly indicate the proof. The leading idea will consist in comparing the behavior of the Riemann sums along E_1 with the one of the Riemann sums along some chain.

Proof. Let $t > 0$ and k_0 be fixed. For $f \sim \sum a_l e_l$, we put

$$\Delta_k^j(f) = R_{p_1 \cdots \check{p}_j \cdots p_{k+1}}(f) - R_{p_1 \cdots p_{k+1}}(f).$$

Then

$$m\left\{\sup_{1\leqslant j\leqslant k+1,\ k\geqslant k_0} |\Delta_k^j(f)| > t\right\} \leqslant \frac{1}{t^2} \sum_{\substack{1\leqslant j\leqslant k+1\\ k\geqslant k_0}} \sum_{\substack{p_1\cdots \check{p}_j \cdots p_{k+1}|l\\ (p_j,l)=1}} a_l^2.$$

Given an arbitrary number l, if $k_2 > k_1 \geqslant k_0$ are such that

$$p_1 \cdots \check{p}_{j_1} \cdots p_{k_1+1} | l, \quad p_{j_1} \not| \, l, \qquad p_1 \cdots \check{p}_{j_2} \cdots p_{k_2+1} | l, \quad p_{j_2} \not| \, l,$$

then $j_1 = j_2$. Thus defining $k(l)$ to be the index corresponding to the smallest j such that p_j does not divide l, we get

$$m\left\{\sup_{1\leqslant j\leqslant k+1 \quad k\geqslant k_0} |\Delta_k^j(f)| > t\right\} \leqslant \frac{1}{t^2}\sum_{l\geqslant p_1\cdots p_{k_0}} (k(l)-k_0-1)a_l^2.$$

But as $l \geqslant p_1\cdots p_{k(l)-2}$, we have $k(l) = O\left(\frac{\log l}{\log\log l}\right)$, which allows us to conclude the first half of the proposition. For the second part, observe that if we denote

$$C_{j,\ N}(l) = \mathrm{Card}\left\{j \leqslant k+1,\ k \leqslant N \mid p_j \not| l,\ p_1\cdots \check{p}_j\cdots p_{k+1}|l\right\},$$

then

$$\int\left(\frac{1}{N^2}\sum_{\substack{j\leqslant k+1\\ k\leqslant N}}\Delta_k^j(f)\right)^2 dm = \frac{1}{N^4}\sum_{l=0}^{\infty} a_l^2 C_{j,\ N}(l)^2 \leqslant \frac{(N-k(l))^2}{N^4} \leqslant \frac{1}{N^2}.$$

Therefore,

$$\sum_{N\geqslant 1}\int\left(\frac{1}{N^2}\sum_{\substack{j\leqslant k+1\\ k\leqslant N}}\Delta_k^j(f)\right)^2 dm < +\infty,$$

which combined with Jessen's Theorem implies:

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{\substack{j\leqslant k+1\\ k\leqslant N}} R_{p_1\cdots\check{p}_j\cdots p_{k+1}}(f) = \int f\ dm,$$

and this easily allows us to get the second half of the proposition. $\boxed{QED}$

In [48], Révész and Ruzsa have considered this problem in a wider arithmetical setting, apparently independent from the papers published by Baker and Dubins-Pitman. To this end, they introduced a new notion:

24 Definition. A sequence $\mathcal{S}$ of positive integers has Rudin-dimension d when the following property holds: There exists sets $\mathcal{S}_l = \{n_{k_1},\ldots,n_{k_l}\} \subset \mathcal{S}$ such that,

$$\forall i \in [1,l], \quad n_{k_i} \not| \ [n_{k_1},\ldots,n_{k_{i-1}},n_{k_{i+1}},\ldots,n_{k_l}],$$

if and only if $l \leqslant d$.

Then, a sequence of Rudin-dimension 1 is a chain, whereas a sequence of infinite Rudin-dimension is simply a Rudin sequence. That notion is in fact equivalent to the notion of breadth, since a sequence $\mathcal{S}$ is of finite Rudin-dimension d, if and only if, it has a breadth equal to d. Using this definition they have given another proof of Theorem 7 , but also the following interesting result:

25 Theorem. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ have Rudin-dimensions α and β respectively, then the Rudin-dimension γ of the sequence $\mathcal{S}_1 \vee \mathcal{S}_2$ satisfies*

$$\gamma \leqslant \alpha + \beta.$$

Since one can find sequences for which the latter inequality is in fact an equality, this result is also optimal. Observe that a sequence of integers which is built from a given set of d primes, is of Rudin-dimension d. One could believe, in view of this result, that any sequence with large dimension can be built by means of sequences of smaller dimension. This is in turn wrong. The authors showed in [48] the existence of a sequence of dimension 3 which cannot be represented by means of a finite number of chains. The proof is based on Van der Waerden's Theorem. In addition, they have established an important link between the density of a sequence and its Rudin-dimension. This is the object of the next statement.

26 Theorem. *Let $\mathcal{S}$ be a sequence of integers with Rudin-dimension equal to d. If $S(n) = Card([1,n] \cap \mathcal{S})$, then there exists a positive constant C such that for all $n \geqslant 1$,*

$$S(n) < C(\log n)^d.$$

5 The method of Bourgain

This section is devoted to the results based on a powerful method discovered by Bourgain in [4] for disproving the almost everywhere convergence of sequences of operators. This method is often referred to as Bourgain's entropy condition, and gives a unified way of solving different problems as:

(1) Does there exist a bounded function f on $[0,1[$ such that the Riemann sums diverges?

(2) Khinchine's problem: Does there exist a bounded function f on $\mathbb{T}$ such that the averages $\frac{1}{n}\sum_{j=1}^{n} f(jx)$ diverge? This was first solved [27] by Marstrand by different techniques.

(3) The Bellow problem: Let (a_k) be a sequence of real numbers which is converging to zero. Does there exist a bounded function f on the real line such that the following average $\frac{1}{n}\sum_{k=1}^{n} f(x+a_k)$ diverges a.e.?

Recall first the entropy criteria of Bourgain. For this, we need to introduce the following useful notion. Let H be an arbitrary Hilbert space, the canonical Gaussian process $\{Z_a, a \in H\}$ is defined as follows :

$$\forall a, b \in H, \qquad \mathbb{E}Z_a = 0, \qquad \mathbb{E} < Z_a, Z_b >=< a, b > .$$

A countable subset A of H is said to be Gaussian bounded (GB) if the supremum $\mathbb{E}\left(\sup_{a\in A} Z_a\right)$ is finite. This is a fine notion of compactness in a Hilbert space, and it is well approximated by using metric entropy. Now, we state a first criterion ([4, Proposition 1], see also [62, Theorem 4.1.1])

27 Theorem. *Let $(X, \mathcal{A}, \mu)$ be a probability space with a complete σ-algebra $\mathcal{A}$. Let $\{S_n, n \geqslant 1\}$ be a sequence of $L^2(\mu)$-contractions. Assume that there exists a sequence of positive isometries $\{T_j,\ j \geqslant 1\}$ of $L^2(\mu)$, preserving* 1*, commuting with the sequence $\{S_n, n \geqslant 1\}$ and satisfying the mean ergodic Theorem in $L^1(\mu)$:*

$$\forall f \in L^\infty(\mu), \qquad \lim_{J\to\infty} J^{-1} \sum_{j\leqslant J} T_j f = \int f d\mu. \tag{28}$$

Moreover, assume for some $p \in [2, \infty)$

$$\forall f \in L^p(\mu), \quad \sup_{n\geqslant 1} |S_n(f)| < \infty, \qquad \mu - \text{a.e.}$$

Then for any $f \in L^p(\mu)$, the set $C_f = \{S_n(f), n \geqslant 1\}$ is a GB subset of $L^2(\mu)$. In particular, there exists a constant C depending on the sequence $\{S_n, n \geqslant 1\}$ such that

$$\forall f \in L^p(\mu), \qquad \sup\left\{\varepsilon\sqrt{\log N_f(\varepsilon)}, \varepsilon > 0\right\} \leqslant C\|f\|_2, \tag{29}$$

where $N_f(\varepsilon)$ denotes the minimal number of hilbertian balls of radius δ centered in C_f enough to cover C_f.

It is worth noticing here, that the entropy numbers estimate is actually optimal. The next criterion is the one most used by ergodicians. It concerns the case $L^\infty(\mu)$ (see [4, Proposition 2], and also [62, Theorem 4.2.1] for a detailed account)

28 Theorem. *Let $\{S_n, n \geqslant 1\}$ be a sequence of $L^2(\mu)$-$L^\infty(\mu)$ contractions. Assume there exists a sequence $\{T_j\ , j \geqslant 1\}$ of positive $L^2(\mu)$-isometries preserving* 1*, commuting with the sequence $\{S_n, n \geqslant 1\}$, and satisfying* (28)*. Moreover, assume*

$$\forall f \in L^\infty(\mu), \quad \mu\left\{(S_n(f))_{n\geqslant 1}\right\} \text{ converges } = 1.$$

Then for any $\delta > 0$,

$$C(\delta) = \sup_{f \in L^\infty(\mu),\ ||f||_2 = 1} N_f(\delta) < \infty. \tag{30}$$

With the help of this criterion Bourgain has recovered Rudin's Theorem. Indeed, for every $r \geqslant 2$, there exist $k_1, \dots, k_r$ such that for $1 \leqslant i \leqslant r$, n_{k_i} does not divide the least common multiple of $n_{k_1}, \dots, n_{k_{i-1}}, n_{k_{i+1}}, \dots, n_r$. Hence, there are $p_1, \dots, p_r$ distinct primes such that

$$v_{p_i}(n_{k_i}) > v_{p_i}(n_{k_j}), \quad \text{whenever} \quad i \neq j,$$

where v_p denotes the p-adic valuation.

Put $N = \operatorname{lcm}(n_{k_1}, \dots, n_{k_r})/(p_1 \dots p_r)$ and notice that n_{k_i} does not divide N for $1 \leqslant i \leqslant r$. Consider the set of integers

$$E = \{N p_1^{\alpha_1} \dots p_r^{\alpha_r} \mid \alpha_i \in \{0, 1\}\}$$

and the function

$$f = 2^{-r/2} \sum_{n \in E} e^{2i\pi nx}.$$

Then

$$R_{n_{k_s}}(f) = 2^{-r/2} \sum_{n \in (E \cap N p_s \mathbb{Z})} e^{2i\pi nx},$$

and, for $1 \leqslant s \neq t \leqslant r$,

$$||R_{n_{k_s}}(f) - R_{n_{k_t}}(f)||_2 = 1/\sqrt{2}.$$

This shows that $C(1/\sqrt{2}) = \infty$ and so achieves the proof.

A slight improvement of this result using the following notion, was given in [1]. Assume $(X, \mathcal{A}, \mu)$ is a non-atomic Lebesgue probability space.

29 Definition.

1) A sequence $\{S_n, n \geqslant 1\}$ of linear operators is said to be strong sweeping out if given $\epsilon > 0$ there is a set B with $m(B) < \epsilon$ such that :

$$-\quad \overline{\lim_{n \to +\infty}}\, S_n 1_B(x) = 1 \quad \text{and} \quad \liminf_{n \to +\infty} S_n 1_B(x) = 0 \qquad \text{a.e.}$$

2) Let $0 < \delta \leq 1$. A sequence $\{S_n, n \geqslant 1\}$ of linear operators is said to be $\delta -$ sweeping out if given $\epsilon > 0$ there is a set B with $m(B) < \epsilon$ such that:

$$-\quad \overline{\lim_{n \to +\infty}}\, S_n 1_B(x) \geqslant \delta \text{ a.e.}$$

Notice that the first point of this definition is stronger than the second one. The proof of Rudin, as well as the previous one, shows that the Riemann sums are at least $\frac{1}{2}$-sweeping out. But in [1] the authors proved (Theorem A.2, page 62) by using several ideas which were given by Rudin in his construction, that the Riemann sums even have the strong sweeping out property.

Following the method introduced by Bourgain, some other authors (see [44], [45], [46], [51]) have given new results on particular averages of Riemann sums, namely:

For $\mathcal{U} = (n_k)_{k\geqslant 1}$ a sequence of positive integers put

$$A_N^{\mathcal{U}}(f) = \frac{1}{N}\sum_{k=1}^{N} R_{n_k}(f), \tag{31}$$

and

$$B_N^{\mathcal{U}}(f) = \frac{1}{\sigma_N}\sum_{k=1}^{N} \lambda_k R_{n_k}(f), \quad \sigma_N = \sum_{k=1}^{N}\lambda_k, \tag{32}$$

where $(\lambda_k)_{k\geqslant 1}$ is a sequence of positive reals such that $\sum_{k\geqslant 1}\lambda_k = +\infty$.

It is obvious, that for the study of these means, $\mathcal{U}$ has been chosen in the sequences for which we know, that the Riemann sums are not convergent. For example, $\mathcal{U}$ may be the whole sequence of integers or a Rudin sequence. In the particular case when $\mathcal{U}$ is the sequence $\mathcal{P} = (p_i)_{i\geqslant 1}$ of prime numbers, Ruch and Weber have proved the following statement (Theorem 1.3 in [51]). Put,

$$\forall N \geqslant 1, \quad A_N^{\mathcal{P}}(f) = \frac{1}{N}\sum_{k=1}^{N} R_{p_k}(f) \tag{33}$$

and introduce for all $N \geqslant 1,\ \alpha \geqslant 1$ the Césarò averages

$$C_N^{\alpha}(f) = \frac{1}{\Gamma_N^{\alpha}}\sum_{k=1}^{N} \Gamma_{N+1-k}^{\alpha-1}\ R_{p_k}(f), \tag{34}$$

where $\Gamma_N^{\alpha} = \dfrac{(\alpha+1)\cdots(\alpha+N-1)}{(N-1)!}$.

30 Theorem.

a) For each $2 \leqslant p < \infty$, there exists an $f \in L^p(\mathbb{T})$ such that for every $\alpha \geqslant 1$, the sequence $\{C_N^{\alpha}(f), N \geqslant 1\}$ does not converge almost everywhere.

b) There exists a bounded measurable f such that the sequence $\{A_N^{\mathcal{P}}(f), N \geqslant 1\}$ does not converge almost everywhere.

For proving this proposition, we shall need the following Lemma.

31 Lemma. *For any measurable f on $\mathbb{T}$ we have:*

$$\{A_N^{\mathcal{P}}(f),\ N \geqslant 1\} \quad \textit{converges a.e.} \quad \Rightarrow \quad \{M_N^{\mathcal{P}}(f),\ N \geqslant 1\} \quad \textit{converges a.e.},$$

where

$$M_N^{\mathcal{P}}(f) = \frac{1}{N} \sum_{i=N+1}^{2N} R_{p_i}(f).$$

Although it is very immediate, this Lemma is useful in the sense, that it seems to be much easier to prove the divergence of the moving averages $\big(M_N^{\mathcal{P}}(f)\big)_{N\geqslant 1}$. Let us now give some ideas of the proof of the part b), which among other things uses Theorem 28.

Let u be a fixed integer and $s,\ t$ two other integers such that

$$2 \leqslant s < t \leqslant u+1.$$

For all $j \geqslant 1$, put $N_{2^j} = P_{2^j+1} \cdots P_{2^{j+1}}$ and define:

$$\begin{aligned} E &= E_u = \{N_1^{\alpha_1} \cdots N_u^{\alpha_u}, \alpha_i \in \{0,1\}\} \\ f(x) &= f_u(x) = \frac{1}{|E_u|^{\frac{1}{2}}} \sum_{n\in E_u} e_n(x). \end{aligned}$$

By observing that

$$\forall n \in E,\ \forall j \in [1, s-1],\ \forall i \in [2^j+1, 2^{j+1}], \quad p_i|n \iff N_j|n$$

we may write:

$$A_{2^s}(f) = \frac{1}{|E|^{\frac{1}{2}}} \sum_{n\in E} \frac{1}{2^s}\Big(\sum_{i=1}^{2^s} \delta_{p_i|n}\Big) e_n = \frac{1}{|E|^{\frac{1}{2}}} \sum_{n\in E} \frac{1}{2^s}\Big(\sum_{j=1}^{s-1} 2^j \delta_{N_j|n}\Big) e_n.$$

And so,

$$\begin{aligned} \|A_{2^s} - A_{2^t}\|_2^2 &= \frac{1}{|E|} \sum_{n\in E} \Big[\frac{1}{2^s}(\sum_{j=1}^{s-1} 2^j \delta_{N_j|n}) \ - \ \frac{1}{2^t}(\sum_{j=1}^{t-1} 2^j \delta_{N_j|n})\Big]^2 \\ &= \frac{1}{|E|}\Big[(\frac{1}{2^s} - \frac{1}{2^t})^2 \sum_{n\in E} (\sum_{j=1}^{s-1} 2^j \delta_{N_j|n})^2 \ + \ (\frac{1}{2^t})^2 \sum_{n\in E} (\sum_{j=s}^{t-1} 2^j \delta_{N_j|n})^2 \\ &\qquad -2\frac{1}{2^t}(\frac{1}{2^s} - \frac{1}{2^t}) \sum_{n\in E} (\sum_{j=1}^{s-1} 2^j \delta_{N_j|n})(\sum_{j=s}^{t-1} 2^j \delta_{N_j|n})\Big] \end{aligned}$$

Estimating the three sums appearing in the above expression leads to

$$\|A_{2^s} - A_{2^t}\|_2^2 \geqslant \left(\frac{1}{2\sqrt{3}}\right)^2.$$

So, for all $\delta \leqslant \frac{1}{2\sqrt{3}}$ we get,

$$N(\{A_s(f), s \geqslant 1\}, \delta) = +\infty.$$

and this completes the proof. In [45], Ruch extends these results to general weighted averages. Using nearly the same methods, he has shown

32 Proposition. *Let $\mathcal{P} = (p_i)_{i\geqslant 1}$ be the sequence of primes. There exists a bounded measurable function f on the torus such that the sequence of averages*

$$B_N^{\mathcal{P}}(f) = \frac{1}{\sigma_N}\sum_{i=1}^{N} \lambda_k R_{p_i}(f), \quad \sigma_N = \sum_{k=1}^{N} \lambda_k \to +\infty,$$

does not converge almost everywhere.

Theorem 9 in section 3 clearly shows, that it is not possible to generalize this result when replacing the sequence $\mathcal{P}$ by any Rudin sequence. Further in [44], the author gives the following example of a sequence of integers for which the usual averages of Riemann sums converge. Let $(k_l)_{l\geqslant 1}$ be a sequence of integers which has density zero, i.e.

$$\frac{1}{N}\sum_{1\leqslant k\leqslant N,\ k\in(k_l)} 1 \ \to 0 \quad \text{when} \quad N \to +\infty,$$

and define the sequence $\mathcal{U} = (n_k)_{k\geqslant 1}$ by

$$n_k = \begin{cases} 2^k & \text{if } k \neq k_l \\ p_{k_l} & \text{if } k = k_l \end{cases}$$

where p_{k_l} is a prime number in $[2^{k_l}, 2^{k_l+1}]$. It is easy to see, that $\mathcal{U}$ is a Rudin sequence. Further, by comparing the averages of Riemann sums along the sequence $\mathcal{U}$ and the sequence $(2^k)_{k\geqslant 1}$ and using Jessen's Theorem, we can prove that the sequence $\{A_N^{\mathcal{U}}(f),\ N \geqslant 1\}$ converges for all bounded function on the torus. So Rudin's result is no longer true, when we take averages of Riemann sums.

In [46], The following result concerning Marcinkiewicz-Zygmund's conjecture is also proved. The proof relies upon a strategy worked out by Bourgain.

33 Theorem. *Let $0 < \epsilon_1 < 1$, $\epsilon_2 \geqslant \epsilon_1$ and consider a sequence $(N_l)_{l\geqslant 1}$ of positive integers satisfying*

$$\forall l \geqslant 1, \quad N_l^{1+\epsilon_1} < N_{l+1} < N_l^{1+\epsilon_2}. \tag{35}$$

Then,

$$\forall f \in L^2(\mathbb{T}), \quad A_{N_l}(f) = \frac{1}{N_l}\sum_{n=1}^{N_l} R_n(f) \to \int_{\mathbb{T}} f dm \qquad a.e.$$

The interest of this theorem lies in the fact, that it shows the validity of the conjecture when considering hypergeometric subsequences, like for instance $(2^{2^l})_{l\geqslant 1}$. The techniques involved in the proof, however, does not seem to give any more. We give a brief sketch of the proof. A standard argumentation reduces the problem to showing the existence of a maximal inequality, namely:

$$|| \sup_{l\geqslant 1} A_{N_l}(f)||_2 \leqslant C||f||_2 \quad (\forall f \in L^2(\mathbb{T})).$$

Observe that if $f \sim \sum \hat{f}(k)e_k$, then

$$A_{N_l}(f) = \sum \frac{d(k, N_l)}{N_l}\hat{f}(k)e_k,$$

with $d(k, N_l) = \mathrm{Card}\{1 \leqslant n \leqslant N_l,\ n|k\}$. As the estimation of these quantities makes the proof of the maximal inequality difficult, this sequence of multipliers is replaced by another one which is noted by $(\mu_k^{(N_l)})$. Let χ_z denote the indicator function of $z\mathbb{Z}$, and denote by $\mathcal{P}$ the set of primes and $\mathcal{P}^* = \cup_{j\geqslant 1}\mathcal{P}^j$. Clearly

$$d(k, N) = \sum_{n=1}^{N} \prod_{\substack{z\in\mathcal{P}^* \\ z \not| n}} (1 - \chi_z(n)).$$

Consider the multipliers

$$\mu_k^N = \prod_{\substack{z\in\mathcal{P}^*,\ z\leqslant N \\ z \not| n}} (1 - \frac{1}{z}) = \prod_{z\in\mathcal{P}^*,\ z\leqslant N} |\hat{\mu}_z(k)|^2,$$

where

$$|\hat{\mu}_z(k)|^2 = (1 - \frac{1}{z}) + \frac{1}{z}\chi_z(k),$$

and μ_z is the probability measure on $\mathbb{T}$ defined by

$$\mu_z = \left(1-\frac{1}{z}\right)^{\frac{1}{2}}\delta_0 + \frac{1}{z}\left(1-\left(1-\frac{1}{z}\right)^{\frac{1}{2}}\right)\left(\delta_0+\delta_{\frac{1}{z}}+\cdots+\delta_{\frac{z-1}{z}}\right).$$

Define now

$$A^*f = \sup_{l\geqslant 1}\left|\sum \frac{d(k,N_l)}{N_l}\hat{f}(k)e_k\right|, \quad \text{and} \quad A_1^*f = \sup_{l\geqslant 1}\left|\sum \mu_k^{(N_l)}\right|.$$

The choice of multipliers and a Theorem by Rota [43] ensure that

$$||A_1^*f||_2 \leqslant C_1||f||_2,$$

so that,

$$||A^*f||_2 \leqslant C_1||f||_2 \ + \ \sup_{k\in\mathbb{Z}}\left(\sum_{l\geqslant 1}\left|\frac{d(k,N_l)}{N_l} - \mu_k^{(N_l)}\right|^2\right)^{\frac{1}{2}}||f||_2.$$

A very precise evaluation of the quantity $\left|\frac{d(k,N_l)}{N_l} - \mu_k^{(N_l)}\right|$ allows us to show, that the previous series converges uniformly in k, and so to obtain the Theorem.

6 Connection with Number Theory

It seemed necessary for us to include a section in this presentation, devoted to the important work of Mikolás, [28], [29], [30] regarding on the one hand, the convergence of averages associated to Farey sequences of a periodic function f with those of the Riemann sums of f, and on the other hand, the error of approximation made in this convergence (for a class of functions with bounded derivative) with Riemann Hypothesis (RH). This work is still motivating number theorists. One can refer for instance to the recent paper of Yoshimoto [67],(see also [23]). We shall start, by discussing the link between Farey sequences and Riemann sums, and we will recall some estimates concerning classical arithmetic functions: the Euler function, the Möbius function, useful for our purpose. For the clarity of the exposition, we will display the arguments concerning the building of this link. At the end of this section we will give some other results, which are connecting Riemann sums with Number Theory, and especially with the Prime Number Theorem.

Let $x \geqslant 1$, we denote by $F_x = \{\frac{k}{n},\ 0 < k \leqslant n \leqslant x,\ (k,n) = 1\}$ the Farey sequence of order x. The ν-th term is denoted ρ_ν^x or ρ_ν, when there is no confusion. The number of these fractions is

$$\Phi(x) = \sum_{n=1}^{[x]} \varphi(n) \qquad (n > 1),$$

where $\varphi(n)$ is the Euler function defined by:

$$\varphi(n) = \text{ Card } \{m \leqslant n,\ (m,n) = 1\}, n > 1.$$

Also, let μ be the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 | n \\ (-1)^k & \text{if } n = p_1 \dots p_k. \end{cases}$$

From the formula $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$, $s > 1$, we can classically estimate Φ (see for example [18, page 287]):

$$\Phi(x) = \frac{3}{\pi^2} x^2 + O(x \log x). \tag{36}$$

Recall also, that [18, page 270]

$$M(x) = \sum_{n=1}^{x} \mu(n) = o(x), \quad \text{and} \quad \sum_{n=1}^{x} |\mu(n)| = \frac{6}{\pi^2} + O(\sqrt{x}). \tag{37}$$

For h an arbitrary real valued function defined on $[0,1]$, we note the associated Riemann sums by

$$R_n(h) = \frac{1}{n} \sum_{k=1}^{n} h(\frac{k}{n}). \tag{38}$$

The link between Farey sequences and Riemann sums established by Mikolás is deduced via the Möbius inversion formula [18, page 234]. The following proof of this link, shorter than the original one, has been given by Gerald Tenenbaum in a private communication. By the Möbius inversion formula

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \tag{39}$$

we get

$$\begin{aligned}
\sum_{1\leqslant \nu\leqslant \Phi(x)} h(\rho_\nu) &= \sum_{1\leqslant a\leqslant b\leqslant [x],\ (a,b)=1} h(a/b) \\
&= \sum_{1\leqslant a\leqslant b\leqslant [x]} h(a/b) \sum_{d|(a,b)} \mu(d) \\
&= \sum_{1\leqslant m\leqslant n\leqslant [x]} h(m/n) \sum_{1\leqslant d\leqslant [x]/n} \mu(d) \\
&= \sum_{1\leqslant n\leqslant [x]} \sum_{1\leqslant m\leqslant n} h(m/n) M([x]/n) \\
&= \sum_{1\leqslant n\leqslant [x]} nR_n(h) M([x]/n).
\end{aligned}$$

Thus for any real A it is easy to deduce

$$\frac{1}{\Phi(x)} \sum_{1\leqslant \nu\leqslant \Phi(x)} h(\rho_\nu) - A = \frac{1}{\Phi(x)} \sum_{1\leqslant n\leqslant [x]} n(R_n(h) - A) M([x]/n). \tag{40}$$

So we see that if $R_u(h) \to A$ as $u \to \infty$, then

$$\frac{1}{\Phi(x)} \sum_{\nu=1}^{\Phi(x)} h(\rho_\nu) \ \to \ A, \tag{41}$$

as $x \to \infty$, provided that we have an suitable criterion for matrix summation method. The relevant criterion is the following result due to Toeplitz (see [21, page 75])

34 Lemma. *Let $t_1, t_2, \ldots, t_n$ be a sequence of reals converging to 0, and let $(a_{k,l},\ k,l = 1,2,\ldots$ be an array of reals satisfying the following conditions*

$$\begin{aligned}
&(1) \qquad \forall l, \quad \lim_{k\to\infty} a_{k,l} = 0, \\
&(2) \qquad S(k) = |a_{k,1}| + |a_{k,2}| + \ldots + |a_{k,k}| = O(1).
\end{aligned}$$

Then, the sequence $t_k^{'} = (a_{k,1}t_1 + a_{k,2}t_2 + \ldots + a_{k,k}t_k)_{k\geqslant 1}$ converges to zero.

We show that conditions (1) and (2) are indeed satisfied:

$$(1) \qquad \text{for all fixed } n \qquad \frac{\left|M(\frac{x}{n})\right|}{\Phi(x)} \leqslant \frac{x}{\Phi(x)} \simeq \frac{\pi^2}{3x} \ \to 0, \qquad (x \to +\infty),$$

$$(2) \qquad \frac{x}{\Phi(x)} \sum_{n=1}^{[x]} n \left|M(\frac{x}{n})\right| \leqslant \frac{x^2}{\Phi(x)} \simeq \frac{\pi^2}{3}, \qquad (x \to +\infty).$$

We therefore deduce the following Theorem.

35 Theorem. *Let h be such that the Riemann sums $R_n(h)$ converge to a (finite) real A as n tends to infinity. Then, the associated Farey averages converge to A:*

$$\mathcal{F}_n h = \frac{1}{\Phi(x)} \sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) \quad \to \quad A.$$

And so, if $\lim_{n\to\infty} R_n(h) = \int_0^1 h(t)dt$, then

$$\lim_{n\to\infty} \mathcal{F}_n h = \int_0^1 h(t)dt.$$

What will interest us in the sequel, is the study of the error of approximation

$$\frac{1}{\Phi(x)} \sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) - \int_0^1 h(t)dt, \tag{42}$$

and its connection with (RH). For this, we will suppose that h is a function which has bounded derivate in $[0,1]$. In this case (this immediately implies that $d\left[R_d(h) - \int_0^1 h(t)dt\right] = O(1)$), Mikolás showed that

$$\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) - \Phi(x) \int_0^1 h(t)dt = O(x \log x). \tag{43}$$

This result may easily be improved. By using the following relation (see [24] or [25] II page 176)

$$\sum_{\nu=1}^{\Phi(x)} \left(\rho_\nu^x - \frac{\nu}{\Phi(x)}\right)^2 = O(1) \tag{44}$$

and writing that

$$\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) = \sum_{\nu=1}^{\Phi(x)} \left(h(\rho_\nu^x) - h(\frac{\nu}{\Phi(x)})\right) + \sum_{\nu=1}^{\Phi(x)} h(\frac{\nu}{\Phi(x)})$$

we have

$$\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) - \Phi(x) \int_0^1 h(t)dt = O(1) \sum_{\nu=1}^{\Phi(x)} \left|\rho_\nu^x - \frac{\nu}{\Phi(x)}\right| + O(1)$$

and so by applying the estimate $\Phi(x) \asymp x^2$ and Cauchy-Schwarz's inequality, we obtain

$$\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) - \Phi(x) \int_0^1 h(t)dt = O\left(x \left[\sum_{\nu=1}^{\Phi(x)} \left(\rho_\nu^x - \frac{\nu}{\Phi(x)}\right)^2\right]^{\frac{1}{2}}\right). \tag{45}$$

But, by means of Franel's identity (see [15] or [25] II page 173)

$$\sum_{\nu=1}^{\Phi(x)} \left(\rho_\nu^x - \frac{\nu}{\Phi(x)}\right)^2 = \frac{1}{12\Phi(x)} \left\{ \sum_{a=1}^{[x]} \sum_{b=1}^{[x]} M(\frac{x}{a}) M(\frac{x}{b}) \frac{(a,b)^2}{ab} - 1 \right\}, \tag{46}$$

and a result by Tchudakov (see [56]) on the error of approximation in the Prime Number Theorem

$$\pi(x) - \int_2^x \frac{du}{\log u} = O\left(x e^{-c_1 (\log x)^\gamma}\right), \tag{47}$$

more precisely by using its analogue for the Möbius function (see [12])

$$M(x) = O\left(x e^{-c_2 (\log x)^\gamma}\right), \tag{48}$$

where $\gamma \in]\frac{1}{2}, \frac{11}{21}[$, while $c_1 = c_1(\gamma)$, $c_2 = c_2(\gamma)$ are constants, one gets the following estimation, much better than (44).

$$\sum_{\nu=1}^{\Phi(x)} \left(\rho_\nu^x - \frac{\nu}{\Phi(x)}\right)^2 = O\left(x e^{-c_3 (\log x)^\gamma}\right). \tag{49}$$

On the other hand under (RH) we have the well-known relations, the first implying the second

$$M(x) = O\left(x^{\frac{1}{2} + c_4 \frac{\log\log\log x}{\log\log x}}\right), \tag{50}$$

$$\sum_{\nu=1}^{\Phi(x)} \left(\rho_\nu^x - \frac{\nu}{\Phi(x)}\right)^2 = O\left(x^{1 + c_5 \frac{\log\log\log x}{\log\log x}}\right). \tag{51}$$

We can therefore deduce the following Theorem

36 Theorem. *Assume that h has a bounded derivative. Then*

$$\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) = \Phi(x) \int_0^1 h(t)dt + O\left(x e^{-c (\log x)^\gamma}\right),$$

where $\gamma \in]\frac{1}{2}, \frac{11}{21}[$, and $c = c(\gamma)$ is a constant.

And if (RH) is true, then for every $\epsilon > 0$

$$\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) = \Phi(x) \int_0^1 h(t)dt + O\left(x^{\frac{1}{2} + \epsilon}\right).$$

Conversely, if h has a bounded derivative and

(1) $\sum_{\nu=1}^{\Phi(x)} h(\rho_\nu^x) = \Phi(x) \int_0^1 h(t)dt + O\left(x^{\frac{1}{2}+\epsilon}\right)$

(2) $F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\left(nR_n(h) - n\int_0^1 h(t)dt\right)$ *is regular and has no zero in the strip* $\Re(s) > \frac{1}{2}$*,*

then (HR) is true.

As an application, and by means of the theory of Dirichlet series, Mikolás showed

37 Theorem. *Let* $f \in \mathcal{C}^3([0,1])$ *such that* $f^{'''}(t)$ *is not identically 0, and*

$$\frac{\left|f^{'}(1) - f^{'}(0)\right|}{\int_0^1 |f^{'''}(t)|\, dt} > \frac{3\zeta(3)}{2\pi} \approx 0,574\ldots,$$

then (HR) is equivalent to : $\displaystyle\sum_{\nu=1}^{\Phi(x)} f(\rho_\nu^x) = \Phi(x)\int_0^1 f(t)dt + O\left(x^{\frac{1}{2}+\epsilon}\right).$

Examples are given by $f(t) = \exp(\lambda t)$ $\left(\lambda \neq 0,\ |\lambda| < \dfrac{2\pi}{3\zeta(3)}\right)$, or by the function $f(t) = \cos\lambda t$ $\left(0 < \lambda \leqslant \dfrac{\pi}{2}\right)$. The proof consists in establishing condition (2) of Theorem 36, under the assumptions made. For proving that F has no zero in the strip $\Re(s) > \frac{1}{2}$, Mikolás use Euler-Maclaurin sum-formula to develop $nR_n(h) - n\displaystyle\int_0^1 h(t)dt$. This allows to represent F as a difference of two Dirichlet series, and reduces the study of the zeroes of F to finding good bounds for these two Dirichlet series.

Notice, that Yoshimoto [67] recently showed that the constant $\frac{3\zeta(3)}{2\pi}$ can be slightly improved by

$$\frac{\sqrt{3}}{6}\left[\pi^2 + \frac{2}{3}\log 2 - \frac{2}{3}\right].$$

Gerald Tenenbaum has established the following improvement of the first part of Theorem 36. Denote

$$B_1(u) = u - [u] - \frac{1}{2} \tag{52}$$

the first Bernoulli function.

38 Lemma. *Let* h *an absolutely continuous function on* $[0,1]$*. Then we have* *for all* $N \geqslant 1$

$$\sum_{1\leqslant\nu\leqslant\Phi(N)} h(\rho_\nu) = \Phi(N)\int_0^1 h(t)dt + \frac{1}{2}\{h(1)-h(0)\} + \sum_{1\leqslant n\leqslant N} M(N/n)\int_0^1 B_1(nt)h'(t)dt. \tag{53}$$

Indeed by (40) we have

$$\begin{aligned} R_n(h) &= \frac{1}{n}\int_0^1 h(t)d[nt] = \frac{1}{n}\int_0^1 h(t)d(nt-\frac{1}{2}-B_1(nt)) \\ &= \int_0^1 h(t) - \frac{1}{n}\int_0^1 h(t)dB_1(nt) \\ &= \int_0^1 h(t) + \frac{h(1)-h(0)}{2n} + \frac{1}{n}\int_0^1 B_1(nt)h'(t)dt. \end{aligned}$$

Now applying (40) for h such that, $h(1)=1$ and $h(t)=0$ for $t\in[0,1[$ we get

$$\sum_{1\leqslant n\leqslant N} M(N/n) = 1 \quad (N\geqslant 1).$$

From these two results we obtain (53).

Let $c \in]0,\frac{1}{2}[$ and $R_c : [1,+\infty[\to [1,+\infty[$ a growing function such that

(i) $x^c/R_c(x)$ is asymptotically growing;

(ii) $M(x)\leqslant x/R_c(x)$ for $x\geqslant 1$.

By the Vinogradov-Korobov estimation of the error term in the Prime Number Theorem, for all $c\in]0,\frac{1}{2}[$ we may choose

$$R_c(x) = e^{a(\log x)^{3/5}/(\log_2 3x)^{1/5}}$$

with a a positive constant. Moreover under (RH), we may choose

$$R_c(x) = x^c.$$

Then, see for example [57], we have

$$M(x)\leqslant x^{\frac{1}{2}+b/\log_2 x} \quad (x\geqslant 3),$$

where b is a positive constant. With this notation we get the following result.

39 Theorem. *Let h an absolutly continuous function on $[0,1]$ such that $h' \in L^2[0,1]$. We have uniformly for all $N \geqslant 1$*

$$\frac{1}{\Phi(N)} \sum_{1\leqslant \nu \leqslant \Phi(N)} h(\rho_\nu) = \int_0^1 h(t)dt + O\left(\frac{1}{NR_c(N)}\right). \tag{54}$$

Let $e(u) = e^{2\pi i u}$. It is well-known that

$$B_1(t) = \frac{i}{\pi} \sum_{k\in\mathbb{Z}^*} \frac{e(kt)}{k}.$$

Introducing the Fourier coefficients,

$$c_m = \int_0^1 e(-mt)h'(t)dt,$$

by Parseval identity we get

$$\int_0^1 B_1(nt)h'(t)dt = \frac{1}{i\pi} \sum_{k\in\mathbb{Z}^*} \frac{c_{kn}}{k}.$$

Denoting now $c_m^* = |c_m| + |c_{-m}|$ and $\tau(m)$ the number of divisors of m in $[1,N]$ we obtain

$$\begin{aligned}
\sum_{1\leqslant n\leqslant N} M(N/n) \int_0^1 B_1(nt)h'(t)dt &\leqslant N \sum_{1\leqslant n\leqslant N} \sum_{k\geqslant 1} \frac{c_{kn}^*}{knR_c(N/n)} \\
&\leqslant N \sum_{m\geqslant 1} \frac{c_m^*}{m} \sum_{n|m,\ n\leqslant N} \frac{1}{R_c(N/n)} \\
&\leqslant N \sum_{m\geqslant 1} \frac{c_m^*}{m} \sum_{n|m,\ n\leqslant N} \left(\frac{m}{N}\right)^c \frac{(N/n)^c}{R_c(N/n)} \\
&\leqslant N \sum_{m\geqslant 1} \frac{c_m^*}{m^{1-c}N^c} \frac{\tau(m)N^c}{R_c(N)} \\
&\leqslant \frac{N}{R_c(N)} \sum_{m\geqslant 1} \frac{c_m^*\tau(m)}{m^{1-c}}.
\end{aligned}$$

The Cauchy-Schwarz inequality implies

$$\left(\sum_{m\geqslant 1} \frac{c_m^*\tau(m)}{m^{1-c}}\right)^2 \leqslant \sum_{m\geqslant 1} (c_m^*)^2 \sum_{m\geqslant 1} \frac{\tau(m)^2}{m^{2-2c}} < +\infty.$$

Thus, by (53) we get the result.

It is possible to give the same result by supposing that for some $q \in [1, 2[$ h' satisfies $\sum_{m\in\mathbb{Z}} |c_m(h')|^q < +\infty$.

Before ending this section, we indicate some other results established by several other authors, linking Riemann sums with the Prime Number Theorem. At first, we quote the thorough paper of Wintner [65]. Let f be an everywhere finite function on $(-\infty, +\infty)$ with period one and having a finite Lebesgue integral on $[0, 1]$. Write the Fourier series $f \sim \sum_1^\infty g_m(x)$, where $g_m(x) = c_m e(mx) + c_{-m} e(-mx)$ and $c_0 = 0$. With this notation, we have $R_n(f) \sim \sum_1^\infty g_{nm}(x)$. So, if the Fourier series of f converges to f everywhere, then that of $R_n(f)$ will converge to $R_n(f)$ everywhere. Hence, formally one has $g_n(x) = \sum_1^\infty \mu(m) f_{nm}(x)$ where μ is the Möbius function. The author investigates the convergence of this last formula, which represents the term of the Fourier series of f in terms of the equidistant Riemann sums. It is shown that the series converges for every x if f satisfies a Lipschitz condition of order greater than $\frac{1}{2}$, and that it need not always converge, even if the Fourier series converges absolutely. Moreover, Wintner obtains the following interesting, necessary and sufficient condition for the analyticity of f in $[0, 1]$: For every integer n and every real x and for some constants K and $q < 1$

$$\left| R_n(f)(x) - \int_0^1 f(t)dt \right| \leqslant K q^n.$$

In [3, page 181], the authors deduce the following special case from the above result.

40 Theorem. *Let f be a real step function on* $]0, 1]$,

$$f(x) = a_n \quad \textit{throughout } (\frac{1}{n+1}, \frac{1}{n}], \quad n = 1, 2, \cdots$$

Suppose the special sequence of Riemann sums

$$\frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}), \quad n = 1, 2, \cdots$$

converges. Then, so does the improper Riemann integral $\int_{0^+}^1 f(t)dt$, and to the same limit.

From this Theorem, the authors deduce the Prime Number Theorem in a rather simple fashion. For this, let for every real $x \geqslant 1$,

$$\Psi(x) = \sum \log p,$$

where the sum is taken over all ordered pairs (p, m) for which p is a prime and m a natural number satisfying $p^m \leqslant x$. Define

$$f(x) = \Psi(x^{-1}) - [x^{-1}].$$

Denoting the number of positive divisors of k by $d(k)$, we have for $n \geqslant 1$,

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} \sum_{j|k,\ j\geqslant 1} 1 = \sum_{k=1}^{n} \sum_{j=1}^{\left[\frac{n}{k}\right]} 1 = \sum_{k=1}^{n} \left[\frac{n}{k}\right] \cdot \tag{55}$$

A classical result of Dirichlet therefore yields, for $n \geqslant 1$ (γ being Euler's constant),

$$\sum_{k=1}^{n} \left[\frac{n}{k}\right] = n \log n + (2\gamma - 1)n + O(\sqrt{n}). \tag{56}$$

Moreover, we also have

$$\sum_{k=1}^{n} \Psi\left(\left[\frac{n}{k}\right]\right) = n \log n - n + O(1 + \log n), \quad n = 1, 2, \cdots \tag{57}$$

In view of the two equalities (56) and (57), we obtain function f that $R_n(f) \to 2\gamma$. Applying Theorem 38, we now get that $\int_{0+}^{1} f(t)dt$ converges. Then it is easy to deduce, that so does the integral $\int_{0+}^{1} (\Psi(t^{-1}) - t^{-1})dt$. Therefore,

$$\lim_{x\to+\infty} \frac{\Psi(x)}{x} = 1,$$

from which the Prime Number Theorem follows in an elementary way.

In [53], Selvaraj has given a much easier proof of the preceding result by using a Theorem of Landau [26, page 568]. Let us give the ideas of that proof. Let $g(x) = f(\frac{1}{x})$, where $f(x) = a_{[\frac{1}{x}]}$ throughout $(0, 1]$ is exactly the function f which is defined in Theorem 38. Put also

$$G(x) = \sum_{k\leqslant x} g(\frac{x}{k}) = \sum_{k\leqslant x} f(\frac{k}{x})\cdot$$

For $0 < \varepsilon < 1$,

$$\begin{aligned} \int_{\varepsilon}^{1} f(x)dx &= \int_{\varepsilon}^{1} g(\frac{1}{x})dx = \int_{1}^{\frac{1}{\varepsilon}} \frac{1}{t^2} g(t)dt \\ &= \int_{1}^{\frac{1}{\varepsilon}} \frac{1}{t^2} \sum_{k\leqslant t} \mu(k) G(\frac{t}{k})dt = \sum_{k\leqslant \frac{1}{\varepsilon}} \int_{k}^{\frac{1}{\varepsilon}} \frac{\mu(k)}{t^2} G(\frac{t}{k})dt, \end{aligned}$$

where μ denotes the Möbius function. Since $G(x) = G([x])$,

$$\frac{G(x)}{x} = \frac{1}{[x]} \sum_{k \leqslant [x]} f(\frac{k}{[x]}) \cdot \frac{[x]}{x},$$

and thus

$$\lim_{x \to +\infty} \frac{G(x)}{x} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) = L.$$

Hence, $G(x) = Lx + o(x)$ as $x \to +\infty$. Now, using the fact that $\int_k^{+\infty} \frac{1}{t} dt$ diverges, we have by applying a result of Landau [26, page 568]

$$\int_\varepsilon^1 f(x)dx = \sum_{k \leqslant \frac{1}{\varepsilon}} \frac{\mu(k)}{k} \int_k^{\frac{1}{\varepsilon}} \left(\frac{L}{t} + o\left(\frac{1}{t}\right) \right) dt = L \cdot S(\frac{1}{\varepsilon}) + o\left(S(\frac{1}{\varepsilon}) \right),$$

where

$$\begin{aligned} S(\frac{1}{\varepsilon}) &= \sum_{k \leqslant \frac{1}{\varepsilon}} \frac{\mu(k)}{k} \int_k^{\frac{1}{\varepsilon}} \frac{1}{t} dt = \sum_{k \leqslant \frac{1}{\varepsilon}} \frac{\mu(k)}{k} \log \frac{1}{\varepsilon} - \sum_{k \leqslant \frac{1}{\varepsilon}} \frac{\mu(k)}{k} \log k \\ &= \log \frac{1}{\varepsilon} \cdot \left(\frac{1}{\log \frac{1}{\varepsilon}} \right) - \sum_{k \leqslant \frac{1}{\varepsilon}} \frac{\mu(k)}{k} \log k. \end{aligned}$$

Therefore, as $\varepsilon \to 0^+$,

$$\int_\varepsilon^1 f(x)dx = L\left(o(1) - (-1)\right) + o(1),$$

and

$$\int_{0^+}^1 f(x)dx = L$$

which is exactly the wanted result.

7 Other type of Riemann sums

In this section, we are concerned with Riemann sums defined in a slightly different way than in (1), namely

$$M_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k+x}{n} \right), \quad n = 1, 2, \cdots \tag{58}$$

or

$$\widetilde{M}_n f(x) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k-x}{n}\right), \quad n = 1, 2, \cdots \tag{59}$$

It is easy to see, that we have the following properties: For all $n \geqslant 1$ and $x \in [0,1]$,

$$M_n f(x) = R_n f\left(\frac{x}{n}\right), \tag{60}$$

and

$$\widetilde{M}_n f(x) = R_n f\left(\frac{-x}{n}\right). \tag{61}$$

Even though the difference with the usual Riemann sums does not seem so important, it brings dramatic effects on the results. Let us show these differences. Notice, that if f a Riemann integrable function, the averages $(M_n f)_{n\geqslant 1}$ and $(\widetilde{M}_n f)_{n\geqslant 1}$ converge to the integral of f. This is easy to check. To our knowledge, the first important results on these sums were given by Chui [9], who considered Riemann integrable functions. In [9], he proved the following result:

41 Theorem. *Let f be a Riemann integrable function on $[0,1]$. If f is absolutely continuous on $[0,1]$, then $\widetilde{M}_n f(\frac{1}{2}) - \int_0^1 f(t)dt = o\left(\frac{1}{n}\right)$. Moreover, if $f(0) = f(1)$, then $\widetilde{M}_n f(x) - \int_0^1 f(t)dt = o\left(\frac{1}{n}\right)$, for all x in $[0,1]$.*

In [10], he generalized this by proving some new Theorems. In a first Theorem (Theorem 1 page 279), the author gave some results which were already known.

42 Theorem.

a) If f is a Riemann integrable function on $[0,1]$, then

$$\widetilde{M}_n f(x) - \int_0^1 f(t)dt = o(1), \quad \textit{for each } x \in [0,1].$$

b) If f is a function of bounded variation on $[0,1]$, then

$$\widetilde{M}_n f(x) - \int_0^1 f(t)dt = O\left(\frac{1}{n}\right), \quad \textit{for each } x \in [0,1].$$

c) If f is is absolutely continuous on $[0,1]$, then

$$\widetilde{M}_n f\left(\frac{1}{2}\right) - \int_0^1 f(t)dt = o\left(\frac{1}{n}\right).$$

d) *If f is differentiable on $[0,1]$ and its derivative f' is of bounded variation on $[0,1]$, then*

$$\left|\widetilde{M}_n f\left(\frac{1}{2}\right) - \int_0^1 f(t)dt\right| \leqslant \frac{T(f')}{8n^2}, \quad \textit{for all } n,$$

where $T(f')$ is the total variation of f' on $[0,1]$.

Result a) is an easy consequence of the convergence for the Riemann integrable function. The proofs of b), and of the weaker and somewhat different forms of c) and d) are also given in [42]. Moreover, b) is also proved in [9], see Theorem 41. But, in [10] Chui gives a unified way of proving b), c) and d). The interest in this proof is the use of the saw-tooth functions which are defined by:

$$v_n(t) = \sum_{k=1}^{n} \chi_{\frac{k-\frac{1}{2}}{n}}(t) - nt, \tag{62}$$

where for $0 \leqslant s \leqslant 1$, χ_s denotes the characteristic function of the closed interval $[s,1]$. For each n, these function verify, $v_n(0) = v_n(1) = 0$, and v_n lies between $-\frac{1}{2}$ and $\frac{1}{2}$ and is linear with the exception of n unit jumps at the points $\frac{k-\frac{1}{2}}{n}$, $k = 1, \cdots, n$. It is noted by the author, that the saw-tooth functions $v_n(t)$ plays a role in the study of the convergence of the operators defined in (58) and (59) similar to the role of the functions $e^{2\pi int}$ in the study of Fourrier series. The difficulty, in working with the functions $v_n(t)$ is, that they are not orthogonal. However, they still satisfy some interesting properties as it is established in the following lemma.

43 Lemma. *Let m and n positive integers such that the quotient $\frac{m}{n}$ is an odd integer. Then*

$$\int_0^1 v_n(t)v_m(t)dt = \frac{n}{12m}\cdot$$

In particular

$$\int_0^1 v_n(t)^2 dt = \frac{1}{12},$$

for all positive integers n.

Using these properties, Chui established some other results (Theorem 2,3,4 and 5 page 280) that we summarize in the following Theorem.

44 Theorem.

a') *Let $(\varepsilon_n)_{n\geqslant 1}$ be a sequence of positive reals which converges monotonically to zero. Then there exists a Riemann integrable function f on $[0,1]$ such that*

$$\widetilde{M}_n f(0) - \int_0^1 f(t)dt \leqslant \varepsilon_n, \quad \textit{for all } n.$$

b') There exist a positive number ε_0, an increasing function f of total variation less than one, and a sequence of positive integers $(n_k)_{k\geqslant 1}$ such that:

$$n_k\left(\int_0^1 f(t)dt - \widetilde{M}_{n_k} f\left(\frac{1}{2}\right)\right) \leqslant \varepsilon_0 \quad \textit{for all } k.$$

c') Let $(\varepsilon_n)_{n\geqslant 1}$ be any sequence of positive reals converging to zero. Then, there is an absolutely continuous f on $[0,1]$ such that

$$n_k\left(\int_0^1 f(t)dt - \widetilde{M}_{n_k} f\left(\frac{1}{2}\right)\right) \leqslant \varepsilon_0 \quad \textit{for all } k,$$

where $(n_k)_{k\geqslant 1}$ is some sequence of positive integers tending to infinity.

d') If f is twice differentiable and f'' is bounded and almost everywhere continuous on $[0,1]$, then

$$\lim_{n\to+\infty} n\left(\int_0^1 f(t)dt - \widetilde{M}_n f\left(\frac{1}{2}\right)\right) = \frac{1}{24}\int_0^1 f''(t)dt = \frac{f'(0)-f'(1)}{24}.$$

Moreover if $f'(0)=f'(1)$ then $\widetilde{M}_n f(\frac{1}{2}) - \int_0^1 f(t)dt = o(\frac{1}{n^2})$.

Notice that a'), b') and c') (in Theorem 44) implies that a), b) and c) (in Theorem 42) cannot be improved.

Following the same methods, Petrovič has studied in [40]

$$\widetilde{M}'_n\varphi(x) = \frac{2\pi}{n}\sum_{k=0}^{n-1}\varphi\left(\frac{2\pi(k-x)}{n}\right), \quad n=1,2,\cdots \tag{63}$$

for $0\leqslant x\leqslant 1$ and φ a 2π-periodic function. He proved that

45 Theorem.

1) If $\varphi = \sum_{m\geqslant 1} b_m \sin(mx)$ with $b_m \downarrow 0$, then there exists a constant $C(x)$ such that

$$\left|\widetilde{M}'_n\varphi(x)\right| \leqslant C(x)b_n, \quad \textit{for } x\in[0,1].$$

2) If $\varphi = \frac{1}{2}a_0 + \sum_{m\geqslant 1} a_m \cos(mx)$ with $a_m\downarrow 0$, and $x\neq 0 \mod 2\pi$, then there exists a constant $C(x)$ such that

$$\left|\widetilde{M}'_n\varphi(x)_\pi a_0\right| \leqslant C(x)a_n, \quad \textit{for } 0<x<1.$$

3) Consequently, if $\varphi = \frac{1}{2}a_0 + \sum_{m\geqslant 1} a_m\cos(mx) + \sum_{m\geqslant 1} b_m \sin(mx)$ with $a_m\downarrow 0$ and $b_m\downarrow 0$, then

$$\widetilde{M}'_n\varphi(x) \to \pi a_0, \quad \textit{for } 0<x<1.$$

In [41], Petrovič proved nearly the same results by considering the averages

$$M_n'\varphi(x) = \frac{2\pi}{n}\sum_{k=0}^{n-1}\varphi\left(\frac{2\pi(k+x)}{n}\right), \quad n = 1, 2, \cdots \tag{64}$$

for $0 \leqslant x \leqslant 1$ and φ a 2π-periodic function.

But, in some way he gave the main result in [39], where he studied averages defined as in (58).

46 Definition. Let f be a measurable and periodic real-valued function on the real line with period one. If, for some increasing sequence $(n_k)_{k\geqslant 1}$ of integers and almost all x, the limit $\lim_{k\to+\infty} M_{n_k}f(x)$ exists and is equal to c, then c is called a partial integral of f.

Using this definition, the author established

47 Theorem. *There is a non decreasing function f in $(0,1)$ with two different partial integrals.*

This result announces in a certain way those of Fominykh and Pannikov.

In the study of averages as defined in (58),(59), (63) or (64), it is natural to ask about the almost everywhere convergence, as we did in the case of the usual Riemann sums. More precisely, do Jessen's and Rudin's Theorems carry over to these new averages, and is there any relation between their convergence and the convergence of the usual Riemann sums? Fominykh was the first author to study such questions. In [13], he proved the following results (Theorem 1 page 86, Theorem 2 page 87, Theorem 3 page 88, Theorem 4 and 5 page 90 and Theorem 6 page 92).

48 Theorem. *Suppose $f \in L^p(0,1)$, where $1 \leqslant p \leqslant +\infty$. Then*

$$\left\|M_n(f) - \int_0^1 f(x)dx\right\|_p \leqslant 2\omega_p(\frac{1}{n}, f),$$

where $\omega_p(\frac{1}{n}, f)$ is the modulus of continuity in $L^p(0,1)$.

The author noticed, that it is impossible to improve the order of the estimate. Moreover, by relation (60) this result remains true for the usual Riemann sums. He also proved, that in the case $p = 1$, 2 is the best constant possible. As a simple consequence of this Theorem, he obtained (this result is also true for the sequence $(R_n f)_{n\geqslant 1}$):

49 Theorem. *Suppose $f \in L(0,1)$. If the increasing sequence $(n_k)_{k\geqslant 1}$ satisfies the condition $\sum_{k\geqslant 1}\omega_1(\frac{1}{n_k}, f) < +\infty$, then for almost all $x \in [0,1]$*

$$\lim_{k\to+\infty} M_{n_k}f(x) = \int_0^1 f(t)dt.$$

Moreover, he stated:

50 Theorem.

1) Suppose $f \in L^2(0,1)$*. If the modulus of continuity* $\omega_2(\delta, f) = O(\delta^\varepsilon)$ *for some* $\varepsilon > 0$*. Then for almost all* $x \in [0,1]$

$$\lim_{n \to +\infty} M_n f(x) = \int_0^1 f(t)dt.$$

2) Suppose $f \sim \frac{a_0}{2} + \sum_{k \geqslant 1} a_k \cos(2\pi kx) + \sum_{k \geqslant 1} b_k \sin(2\pi kx)$*, such that:*

$$\begin{gathered} \lim_{n \to +\infty}(|a_n| + |b_n|) = 0, \\ \sum_{n \geqslant 1}(|a_{n+1} - a_n| + |b_{n+1} - b_n|) < +\infty. \end{gathered}$$

Then

$$\lim_{n \to +\infty} M_n f(x) = \frac{a_0}{2}, \quad \textit{for any } x \in [0,1]$$

In [14], the author established the following important Theorem (Theorem 1, page 69).

51 Theorem. *For any sequence* $(n_k)_{k \geqslant 1}$ *and any* $\varepsilon > 0$*, there exists a function* f *which is the characteristic function of an open set* A *in* $[0,1]$ *for which* $m(A) \leqslant \varepsilon$ *but* $\overline{\lim}_{k \to +\infty} M_{n_k} f(x) = 1$ *for any* $x \in [0,1]$*.*

Notice, that this result clearly shows that Jessen's result is no longer true when considering the averages $(M_n f)_{n \geqslant 1}$. Let us give a sketch of the proof. This theorem is a consequence of the following lemma:

52 Lemma. *Suppose a sequence* $(n_k)_{k \geqslant 1}$ *and a number* δ *are given. There exists then a function* g*, which is the characteristic function of an open set* A*, a natural number* q*, and a set* $E \subset [0,1]$*,* $m(E) > 1 - \delta$ *that satisfy the conditions:*

$$\int_0^1 g(x)dx = \delta,$$

and for any $x \in E$ *there exists* k *(*$1 \leqslant k \leqslant q$*) for which:*

$$M_{n_k} f(x) = 1.$$

By using this lemma the author defines increasing numbers k_i, functions f_i nonnegative on $[0,1]$ and sets $B_i \subset [0,1]$ with Lebesgue measure greater than $1 - \delta_i$, where $\delta_i = \frac{\epsilon}{2^{i+1}}$, such that the following conditions are fulfilled:

$$\int_0^1 f_i(x)dx = \delta_i, \tag{65}$$

and for any $x \in B_i$ there exists k ($k_i \leqslant k \leqslant k_{i+1}$) for which:

$$M_{n_k} f_i(x) = 1. \tag{66}$$

Suppose now, that $f(x) = \sup_k f_k(x)$. Using (65), it is easy to check that $\int_0^1 f(x)dx < \varepsilon$. Moreover, for the points of the set $B = \overline{\lim}_k B_k$ we have by (66)

$$\overline{\lim_{k\to+\infty}} M_{n_k} f(x) = 1. \tag{67}$$

Denote $A = [0,1] \setminus B$. Thus, since for all $i \geqslant 1$, $m(B_i) > 1 - \delta_i$, we have $m(A) = 0$. Therefore, if we set $f(x) = 1$ on all the sets

$$A_{n_i} = \left\{ x \in [0,1],\ x = \frac{y+k}{n},\ y \in A,\ 0 \leqslant k < n_i \right\},$$

relation (67) will be fulfilled on the entire interval $[0,1]$.

In the same paper, Fominykh also established the two following results. Let $(n_k)_{k\geqslant 1}$ be a sequence of positive integers. Let $\alpha(k)$ denote the number of term in $(n_k)_{k\geqslant 1}$ which are divisors of k and put

$$\begin{aligned}
\alpha_m &= \max_{2^{m-2}\leqslant k<2^{m-1}} \alpha(k),\\
\Delta_t f &= f(x+t) - f(x),\\
\Omega(t) &= \int_0^1 (\Delta_t f)^2 dx.
\end{aligned}$$

53 Theorem. *If*

$$\sum_{m\geqslant 2} \alpha_m \Omega\left(\frac{1}{2^m}\right) < +\infty,$$

then

$$\lim_{k\to+\infty} M_{n_k} f(x) = \int_0^1 f(t)dt, \quad \textit{for almost all } x \in [0,1].$$

As noticed by the author, this result remains true for the usual Riemann sums, and thus implies a Marcinkiewicz-Salem result (see for example Theorem 6).

Nearly at the same time, Pannikov gave also a proof of Theorem 51. In fact, he proved something more (see [37, page 504])

54 Theorem. *For any sequence $(n_k)_{k\geqslant 1}$ and any $\varepsilon > 0$, there exists a function f which is the characteristic function of an open set A on $[0,1]$ for which $m(A) \leqslant \varepsilon$ and*

$$\overline{\lim_{k\to+\infty}} \frac{1}{n_k} \sum_{j=1}^{n_k} f\Big(x + \frac{y+j}{n_k}\Big) = 1,$$

for almost all pairs (x,y).

It is easy to check that Theorem 51 is a consequence of this result. Let us give a quick sketch of the proof.

Let

$$f_\varepsilon(x+1) = f_\varepsilon(x) = \begin{cases} 1, & \text{if } 0 < x < \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \tag{68}$$

and

$$f(x) = \sup_s f_{\frac{\varepsilon}{2^s}}\left(\Big(\prod_{l=2^s+1}^{2^{s+1}} n_l\Big)x\right) \tag{69}$$

Then f is the characteristic function of an open set in $[0,1]$ which satisfies the condition :

$$\int_0^1 f(x)dx \leqslant \sum_{s=1}^{+\infty} \int_0^1 f_{\frac{\varepsilon}{2^s}}(x)dx = \sum_{s=1}^{+\infty} \frac{\varepsilon}{2^s} = \varepsilon.$$

Moreover, it is easy to check that for each natural number m such that $2^s+1 \leqslant m \leqslant 2^{s+1}$, $s = 1, 2, \cdots$, we have

$$\frac{1}{n_m}\sum_{j=1}^{n_m} f_{\frac{\varepsilon}{2^s}}\left(\Big(\prod_{l=2^s+1}^{2^{s+1}} n_l\Big)\Big(\frac{y+j}{n_m}\Big)\right) = f_{\frac{\varepsilon}{2^s}}\left(\Big(\prod_{\substack{2^s+1\leqslant l\leqslant 2^{s+1}\\ l\neq m}} n_l\Big)y\right). \tag{70}$$

Indeed, if g is periodic with period $\frac{1}{n}$, $n \in \mathbb{N}^*$, then

$$\frac{1}{n}\sum_{j=1}^{n} g\left(\frac{y+j}{n}\right) = g\left(\frac{y}{n}\right)\cdot$$

It follows from equation (69) that

$$f(x) \geqslant f_{\frac{\varepsilon}{2^s}}\left(\Big(\prod_{l=2^s+1}^{2^{s+1}} n_l\Big)x\right), \quad s = 1, 2, \cdots \tag{71}$$

Then, by using (68), (70) and (71), we get for arbitrary $s \in \mathbb{N}*$ and $2^s+1 \leqslant m \leqslant 2^{s+1}$,

$$\begin{aligned} &\mu\left\{y \in (0,1),\ \frac{1}{n_m}\sum_{j=1}^{n_m} f\Big(\frac{y+j}{n_m}\Big) = 1\right\} \\ \geqslant\ & \mu\left\{y \in (0,1),\ \frac{1}{n_m}\sum_{j=1}^{n_m} f_{\frac{\varepsilon}{2^s}}\left(\left(\prod_{l=2^s+1}^{2^{s+1}} n_l\right)\Big(\frac{y+j}{n_m}\Big)\right) = 1\right\} \\ =\ & \mu\left\{y \in (0,1),\ f_{\frac{\varepsilon}{2^s}}\left(\Big(\prod_{\substack{2^s+1\leqslant l\leqslant 2^{s+1}\\ l\neq m}} n_l\Big)y\right) = 1\right\} = \frac{\varepsilon}{2^s}, \end{aligned}$$

where μ denotes the Lebesgue measure on $[0,1]$.
Therefore, by summing over m

$$\sum_{m\geqslant 3}\mu\Big\{y\in(0,1),\ \frac{1}{n_m}\sum_{j=1}^{n_m} f\Big(\frac{y+j}{n_m}\Big)=1\Big\}\geqslant +\infty.$$

Notice that this leads directly to the Theorem 51.

Now, for arbitrary x

$$\sum_{m\geqslant 1}\mu\Big\{y\in(0,1),\ \frac{1}{n_m}\sum_{j=1}^{n_m} f\Big(x+\frac{y+j}{n_m}\Big)=1\Big\}=+\infty,$$

and thus

$$\sum_{m\geqslant 1}\mu\otimes\mu\Big\{(x,y)\in(0,1)^2,\ \frac{1}{n_m}\sum_{j=1}^{n_m} f\Big(x+\frac{y+j}{n_m}\Big)=1\Big\}=+\infty.$$

Since the sets involved in the above sum are independent, then by virtue of Borel-Cantelli lemma the assertion of Theorem 54 follows.

In [38], Pannikov also established the following result.

55 Theorem. *For every $\alpha>0$ there is an open set $A\subset[0,1]$, with $m(A)<\alpha$ and for which its characteristic function*

$$f_A(x)=\frac{a_0}{2}+\sum_{k\geqslant 1}\left(a_k\cos(2\pi kx)+b_k\sin(2\pi kx)\right),$$

satisfies the following conditions:

$$\overline{\lim_{n\to+\infty}}\, n^{-1}\sum_{j=0}^{n-1} f_A\big(x+\frac{y+j}{n}\big)=1,$$

for almost all pairs (x,y);

$$\overline{\lim_{n\to+\infty}}\, n^{-1}\sum_{j=0}^{n-1} f_A\big(\frac{y+j}{n}\big)=1,$$

for almost all y, and

$$\sum_{k\geqslant 2}(a_k^2+b_k^2)k^{\frac{(1-\varepsilon)\log 2}{\log\log k}}<+\infty,\quad \textit{for all } \varepsilon>0.$$

Let us finally indicate another line of results concerning numerical analysis. The idea is to of approach the integral $\int_0^1 f(x)dx$ by Riemann sums defined relative to random subdivisions. As a typical result, we may mention the following one due to Kahane

56 Theorem. *[20, page 1073]*

Assume that $f \in L^p[0,1]$ with $p > 2$ and suppose that the points $x_k^{(n)}$ are chosen independently and at random from the intervals $I_k^{(n)} = \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1, \ldots n$. Then,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k^{(n)}) \quad \longrightarrow \quad \int_0^1 f(x)dx,$$

almost everywhere.

8 Concluding remarks

Riemann sums are classical objects for most of the mathematicians. But their connections with many fields of the mathematics does not always seem so obvious. In the above discussion we were mainly interested in the convergence almost everywhere of these sums. Through this study, we could establish their impact on analysis, ergodic theory and number theory. Even if there are many interesting and important areas that are related to Riemann sums, but that we have mentioned only briefly, or in some cases, not at all, the point of view we chose allowed us to find some interesting and still open problems. Most of them are associated with deep arithmetical properties of subsequences. For example the conjecture of Marcinkiewicz-Salem, which is a still unsolved and probably very difficult question. It is clear, that there are many interesting questions that remain in this area, and it seems that the more we discover, the more questions arise.

Acknowledgements. The authors thanks G. Tenenbaum for useful remarks concerning Section 6.

Final Note: Recent advances concerning Marcinkiewicz-Salem conjecture have been obtained since the writing of this survey, by the second named author. In [63], the validity of Marcinkiewicz-Salem conjecture for the randomly sampled trigonometric system is proved. In [64], a very sharp sufficient condition for the convergence almost everywhere of averages of Riemann sums is obtained. Results concerning the Littlewood-Paley square function associated to these averages are also established. These results are conforting the validity of Marcinkiewicz-Salem conjecture.

References

[1] M. Akcoglu, A. Bellow, R. Jones, V. Losert, K. Reinhold-Larsson, M. Wierdl: *The strong sweeping out property for lacunary sequences, Riemann sums, convolution powers, and related matters*, Erg. Theo. Dynam. Syst., **16**, 1996, p. 207–253.

[2] R. C. Baker: *Riemann sums and Lebesgue integrals*, Quart. J. Math. Oxford Ser., **27**, 1976, p. 191–198.

[3] J. S. Byrnes, A. Giroux, O. Shisha: *Riemann sums and improper integrals of step functions related to the prime number theorem*, J. Approx. Theory, **40**, 1984, p. 180–192.

[4] J. Bourgain: *Almost sure convergence and bounded entropy*, Israël J. Math., **63**, 1988, p. 79–95.

[5] J. Bourgain: *Problems of almost everywhere convergence related to harmonic analysis and number theory*, Israël J. Math., **71**, 1990, p. 97–127.

[6] Y. Bugeaud, M. Weber: *Examples and counterexamples for Riemann sums*, Indag. Math., **9**, 1998, p. 1–13.

[7] J. W. S. Cassels: *Some metrical theorems in diophantine approximation*, Proc. Cambridge Philo. Soc., **46**, 1950, p. 209–218.

[8] P. Civin: *Abstract Riemann sums*, Pacific J. Math., **5**, 1955, p. 861–868.

[9] C. K. Chui: *A convergence theorem for certain Riemann sums,* Canad. Math. Bull., **12**, 1969, p. 523–525.

[10] C. K. Chui: *Concerning rates of convergence of Riemann sums,* J. Approximation Theory, **4**, 1971, p. 279–287.

[11] L. E. Dubins, J. Pitman: *A pointwise ergodic theorem for the group of rational rotations*, Trans. Amer. Math. Soc., **251**, 1979, p. 299–308.

[12] E. Fogels: *On the average value of arithmetical functions*, Publ. du seminaire math. de l'Univ. de Lettonie à Riga, 1940, p. 15–16.

[13] M. Yu. Fominykh: *Properties of Riemann sums (Russian)*, Izv. Vysš. Učebn. Zaved Matematika, **88**, 1985, p. 65–73.; translation in Soviet Math. (Iz. VUZ), **29**, 1985, p. 83–93.

[14] M. Yu. Fominykh: *Convergence almost everywhere of Riemann sums (Russian)*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., **6**, 1987, p. 67–70.; translation in Moscow Univ. Math. Bull.

[15] J. Franel: *Les suites de Farey et le problème des nombres premiers*, Göttinger Nachrichten, 1924, p. 198–201.

[16] H. Hahn: *Über Annäherung an Lebesgue Integrale durch Riemannsche Summen*, Wiener Sitzungsberichte Oxford Ser.,**123**, 1914, p. 713–743.

[17] P. Hartman, A. Wintner: *On Möbius' inversion* , Amer. J. Math. **69**, 1947, p. 853–858.

[18] G. H. Hardy, E. M. Wright: *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press Oxford, 1960.

[19] B. Jessen: *On the approximation of Lebesgue integrals by Riemann sums*, Annals of Math., **35**, 1934, p. 248–251.

[20] C. S. Kahane: *Evaluating Lebesgue integrals as limits of Riemann sums*, Math. Japon. 38 (1993), no. 6, 1073–1076.

[21] K. Knopp: *Theorie der Anwendung der unendlichen Reihen*, 8-eme Edition, Berlin 1931.

[22] U. Krengel: *Ergodic theorems*, W. de Gruyter, Berlin-New York (1985).

[23] S. Kanemitsu, M. Yoshimoto: *Farey series and the Riemann hypothesis*, Acta Arithmetica, **LXXV**, no4 1996, p. 351–374.

[24] E. Landau: *Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel*, Göttinger Nachrichten, 1924, p. 202–206.

[25] E. Landau: *Vorlesungen über Zahlentheorie*, vol. I-II, Leipzig, 1927.

[26] E. Landau: *Handbuch der Lehre von der Verteilung der Primzahlen*, Chelsea, New-York, 1953.

[27] J. M. Marstrand: *On Khinchin's conjecture about strong uniform distribution*, Proc. London Math. Soc., **21**, 1970, p. 540–556.

[28] M. Mikolás: *Farey series and their connection with the prime problem (I)*, Acta Sci. Math. Szeged, **13**, 1949, p. 93–117.

[29] M. Mikolás: *Farey series and their connection with the prime problem (II)*, Acta Sci. Math. Szeged, **14**, 1951, p. 5–21.

[30] M. Mikolás: *Sur l'hypothèse de Riemann*, C. R. Acad. Sci Paris, **228**, 1949, p. 633–636.

[31] J. Marcinkiewicz, R. Salem: *Sur les sommes riemanniennes*, Comp. Math., **7**, 1940, p. 376–389.

[32] J. Marcinkiewicz, A. Zygmund: *Mean values of trigonometrical polynomials*, Fund. Math., **28**, 1937, p. 131–166.

[33] R. Nair: *On Riemann sums and Lebesgue Integrals*, Monatsh. Math., **120**, 1995, p. 49–54.

[34] D. J. Newman, O. Shisha: *Magnitude of Fourier coefficients and degree of approximation by Riemann sums*, Numer. Funct. Anal. Optim., **12**, 1991, p. 545–550.

[35] B. V. Pannikov: *The convergence of Riemann sums of functions that can be represented by trigonometric series with monotone coefficients (Russian)*, Mat. Zametki, **8**, 1970, p. 607–618.

[36] B. V. Pannikov: *The convergence of Riemann sums of divergent functions (Russian)*, Dokl. Akad.Nauk SSSR, **201**, 1971, p. 542–544.; translation in Soviet Math. Dokl., **12** , 1971, p. 1687–1690.

[37] B. V. Pannikov: *An example of the divergence of Riemann sums (Russian)*, Sibirsk. Math. Zh., **29**, 1988, p. 208–210.; translation in Siberian Math. J., **29**, 1988, p. 503–506.

[38] B. V. Pannikov: *On a Weyl multiplier for the convergence of Riemann sums (Russian)*, Anal. Math., **19**, 1993, p. 243–253.

[39] A. J. Petrovič: *The convergence of subsequences of Riemann sums (Russian)*, Mat. Zametki, **16**, 1974, p. 645–656.; translation in Math. Notes, **16**, 1974, p. 975–982.

[40] A. J. Petrovič: *Riemann sums for functions that are expanded in trigonometric series (Russian)*, Izv. Vysš. Učebn. Zaved Matematika, **12 (151)**, 1974, p. 70–72.; translation in Soviet Math. (Iz. VUZ), **18**, 1974, p. 61–64.

[41] A. J. Petrovič: *Properties of Riemann sums for functions that are representable by a trigonometric series with monotone coefficients (Russian)*, Mat. Sb., **97**, 1975, p. 360 -378;

[42] G. Pólya, G. Szegö: *Aufgaben und Lehrsätze aus der Analysis, Band I: Reihen, Integralrechnung, Funktionentheorie*, p.37, Springer, Berlin, 1925; second edition, 1954.

[43] G. C. Rota: *An "Alternierende Verfahren" for general positive operations*, Bull. Am. Math. Soc., **58**, 1962, p. 85–102.

[44] J. J. Ruch: *Contribution a l'étude des sommes de Riemann* , Prépublication de l'I.R.M.A, thèse, 1997.

[45] J. J. Ruch: *Étude de moyennes pondérées de sommes de Riemann*, Expo. Math., **16**, 1998, p. 277–285.

[46] J. J. Ruch: *Convergence presque sûre de moyennes de sommes de Riemann*, Acta Arith., **87**, 1998, p. 1–12.

[47] W. Rudin: *An arithmetic property of Riemann sums*, Proc. Amer. Math. Soc., **15**, 1964, p. 321–324.

[48] S. G. Revesz, I. Z. Ruzsa: *On approximating Lebesgue integrals by Riemann sums*, Glasgow Math. Jour., **33**, 1991, p. 129–134.

[49] K. Ross, K. Stromberg: *Jessens's theorem on Riemann sums for locally compact groups*, Pacific J. Math., **20**, 1967, p. 135–147.

[50] K. Ross, G. Willis: *Riemann sums and modular functions on locally compact groups*, Pacific J. Math., **180**, 1967, p. 325–331.

[51] J. J. Ruch, M. Weber: *Quelques résultats dans la théorie des sommes riemanniennes*, Expo. Math., **15**, 1997, p. 279–288.

[52] R. Salem: *Sur les sommes riemanniennes des fonctions sommables* , Mat. Tidsskr. B, **6**, 1948, p. 60–62.

[53] S. Selvaraj: *A note on Riemann sums and improper integrals related to the prime number theorem*, J. Approx. Theory, **66**, 1991, p. 106–108.

[54] L. I. Shloma: *Harmonic components of a function and Riemann sums (Russian)*, Dokl. Akad. Nauk. BSSR 31, **92**, 1991, p. 13–16.

[55] S. Takahashi: *A remark on the Riemann sum*, Sci. Rep. Kanazawa Univ, **6**, 1959, p. 57–59.

[56] N. Tcudakov: *On zeros of Dirichlet's L-functions*, Mat. Sbornik, **1**, 1936, p.591- 602.

[57] E. C. Titchmarsh: *The theory of the Riemann zeta function*, revised by D. R. Heath-Brown, Oxford ,1986.

[58] T. Tsuchikara: *Notes on Fourier Analysis 13: A theorem on Riemann sums*, J. Math. Soc. Japan, **1**, 1949, p. 232–234.

[59] T. Tsuchikara: *Some remarks on the Riemann sums*, Tôhoku Math. J., **3**, 1951, p. 197–202.

[60] H. D. Ursell: *On the behaviour of a certain sequence of functions derived from a given one*, J. London Math. Soc., **12**, 1937, p. 229–232.

[61] M. Weber: *Some theorems related to almost sure convergence of orthogonal series*, Indag. Math. (N.S.) 11 (2000), no. 2, 293–311.

[62] M. Weber: *Entropie métrique et convergence presque partout*, Coll. "Travaux en Cours" **58**, Hermann, Paris.

[63] M. Weber: *A theorem related to Marcinkiewicz-Salem Conjecture*, Results in Math. **45**, 2004, 169–184.

[64] M. Weber: *Almost sure convergence and square functions of averages of Riemann sums.* Results Math. **47**, 2005, 340–354.

[65] A. Wintner: *Fourier constants and equidistant Riemann sums*, J. Math. Pures Appl., **36**, 1957, p. 251–261.

[66] S. Yano: *Notes on Fourier Analysis 19: A remark on Riemann sums*,Tùhoku Math. J., **2**, 1950, p. 1–3.

[67] M. Yoshimoto: *Farey series and the Riemann hypothesis*, Acta Math. Hungar., **78**, no4 1998, p. 287–304.