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On the geometry of variational calculus on some functional bundles

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Abstract. We first generalize the operation of formal exterior differential in the case of finite dimensional fibered manifolds and then we extend it to certain bundles of smooth maps. In order to characterize the operator order of some morphisms between our bundles of smooth maps, we introduce the concept of fiberwise (k, r)-jet. The relations to the Euler-Lagrange morphism of the variational calculus are described.

Keywords: Formal exterior differential, bundle of smooth maps, operator order of a morphism, fiberwise (k, r)-jet, Euler-Lagrange morphism.

MSC 2000 classification: 58A20, 58E30

1 Introduction

Our geometrical research was inspired by the paper on the Schrödinger operator by the second author and M. Modugno, [5], as well as by their previous joint paper with A. Jadczyk, [4]. We are interested mainly in certain geometric objects and operations related with the functional bundle $S(E, Q)$ of all sections $E_x \to Q_x$ of a 2-fibered manifold $Q \to E \to M$, $x \in M$. In [2], the first and the third authors established the theory of connections in a somewhat more general situation of the bundle $\mathcal{F}(Y_1, Y_2) \to M$ of all smooth maps between the fibers

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over the same base point of two fibered manifolds $Y_1 \to M$ and $Y_2 \to M$ with the same base M. The main purpose of the present paper is to introduce some geometric concepts and to study some geometric operations that could be useful for the variational calculus on these functional bundles.

Our approach to the variational calculus is based on the formal exterior differential on finite dimensional fibered manifolds introduced by A. Trautman, [12], and further developed by the third author, [6, 7]. In Section 1 of the present paper we study a slight finite dimensional generalization of this concept in a form suitable for our next purposes. Section 2 is devoted to some geometric properties of the bundles $\mathcal{F}(Y_1, Y_2)$ and $S(E, Q)$ in the framework of the Frölicher's theory of smooth structures, [3]. The morphisms between our functional bundles represent a kind of differential operators. As pointed out already in [2], one can distinguish an important class of them that have finite order in the operator sense. In Section 3 we modify this idea to the morphisms defined on the r-th jet prolongation $J^r\mathcal{F}(Y_1, Y_2)$. This leads us to an original concept of fiberwise (k, r)-jet of a base preserving morphism of finite dimensional fibered manifolds. Section 4 deals with the formal exterior differentiation over the functional bundle $\mathcal{F}(Y_1, Y_2)$. In Section 5 we study its restriction to the bundle $S(E, Q)$ of sections. In Proposition 11 we characterize an important situation in which the finite dimensional formal exterior differential and the analogous operation over $S(E, Q)$ are naturally related. Finally, Section 6 is devoted to the Euler-Lagrange morphism on $S(E, Q)$ from the viewpoint of our previous operations.

If we deal with finite dimensional manifolds and maps between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. On the other hand, the smooth spaces and maps in the sense of A. Frölicher are said to be F-smooth. Unless otherwise specified, all morphisms are assumed to be base preserving. In all standard situations we use the terminology and notation from the monograph [8].

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2 The formal exterior differential in finite dimension

We recall that $Q \xrightarrow{q} E \xrightarrow{p} M$ is said to be a 2-fibered manifold, if both q and p are surjective submersions. Consider two 2-fibered manifolds $Z \to Y \to M$, $W \to Y \to M$, a fibered manifold $N \to M$ and a morphism

$$\psi : Z \underset{Y}{\times} W \to N \,,$$

where $Z \underset{Y}{\times} W$ is interpreted as a fibered manifold over M. Then the rule

$$J^k\psi(j^k_x s, j^k_x \sigma) = j^k_x \psi(s,\sigma) \tag{1}$$

defines a map

$$J^k\psi : J^k Z \underset{J^kY}{\times} J^k W \to J^k N\,. \tag{2}$$

In the case of

$$\psi : J^r Z \underset{J^tY}{\times} J^s W \to N\,, \quad r \geq t \leq s\,,$$

we obtain

$$J^k\psi : J^k J^r Z \underset{J^kJ^tY}{\times} J^k J^s W \to J^k N\,.$$

Then we introduce

$$J^k_{\mathrm{hol}}\psi : J^{k+r} Z \underset{J^{k+t}Y}{\times} J^{k+s} W \to J^k N \tag{3}$$

by means of the canonical inclusions of the holonomic jet prolongations into the iterated jet prolongations.

In particular, consider

$$\varphi : J^r Y \underset{J^sY}{\times} V J^s Y \to Z\,, \quad s \leq r\,, \tag{4}$$

where $Z \to M$ is a fibered manifold. By using the well known identification $\varkappa_s : VJ^sY \to J^sVY$, we construct

$$\varphi \circ (\mathrm{id}_{J^rY} \underset{J^sY}{\times} \varkappa_s^{-1}) : J^r Y \underset{J^sY}{\times} J^s VY \to Z$$

and

$$J^k_{\mathrm{hol}}(\varphi \circ (\mathrm{id}_{J^rY} \underset{J^sY}{\times} \varkappa_s^{-1})) : J^{k+r} Y \underset{J^{k+s}Y}{\times} J^{k+s} VY \to J^k Z\,.$$

Then we define

$$\begin{aligned} \mathcal{J}^k_{\mathrm{hol}}\varphi := J^k_{\mathrm{hol}}(\varphi \circ (\mathrm{id}_{J^rY} \underset{J^sY}{\times} \varkappa_s^{-1})) \circ (\mathrm{id}_{J^{k+r}Y} \underset{J^{k+s}Y}{\times} \varkappa_{s+k}) : \\ : J^{k+r} Y \underset{J^{k+s}Y}{\times} V J^{k+s} Y \to J^k Z\,. \end{aligned} \tag{5}$$

Let $\eta : Y \to VY$ be a vertical vector field on Y and $\mathcal{J}^s\eta : J^sY \to VJ^sY$ be its flow prolongation. Write

$$\varphi(\mathcal{J}^s\eta) = \varphi \circ (\mathrm{id}_{J^rY} \underset{J^sY}{\times} \mathcal{J}^s\eta) : J^r Y \to Z\,.$$

Then

$$J^k_{\mathrm{hol}}(\varphi(\mathcal{J}^s\eta)) : J^{k+r}Y \to J^kZ\,.$$

On the other hand,

$$\mathcal{J}^k_{\mathrm{hol}}\varphi : J^{k+r}Y \underset{J^{k+s}Y}{\times} VJ^{k+s}Y \to J^kZ\,,$$

so that

$$(\mathcal{J}^k_{\mathrm{hol}}\varphi)(\mathcal{J}^{k+s}\eta) : J^{k+r}Y \to J^kZ\,.$$

1 Proposition. *For every φ and η, we have*

$$(\mathcal{J}^k_{\mathrm{hol}}\varphi)(\mathcal{J}^{k+s}\eta) = J^k_{\mathrm{hol}}(\varphi(\mathcal{J}^s\eta))\,. \tag{6}$$

Proof. This follows from the well known fact $\mathcal{J}^s\eta = \varkappa_s^{-1} \circ J^s\eta$, where $J^s\eta : J^sY \to J^sVY$ is the functorial prolongation of η. QED

Consider the case $Z = \bigwedge^l T^*M$ in (4). The exterior differential d on M is a first order operator, so that d determines the associated map $\delta : J^1 \bigwedge^l T^*M \to \bigwedge^{l+1} T^*M$ satisfying $d\omega = \delta \circ (J^1\omega)$ for every l-form $\omega : M \to \bigwedge^l T^*M$.

2 Definition. For every morphism $\varphi : J^rY \underset{J^sY}{\times} VJ^sY \to \bigwedge^l T^*M$, we define its *formal exterior differential* by

$$D\varphi := \delta \circ (\mathcal{J}^1_{\mathrm{hol}}\varphi) : J^{r+1}Y \underset{J^{s+1}Y}{\times} VJ^{s+1}Y \to \bigwedge^{l+1} T^*M\,. \tag{7}$$

Proposition 1 implies that this concept represents a generalization of that one introduced by the third author in [6]. In fact, φ is assumed to be linear in VJ^sY in [6], while in (7) φ is quite arbitrary.

Consider some local fiber coordinates x^i, x^p on Y, $i = 1, \dots, m = \dim M$, $p = m+1, \dots, m+n = \dim Y$. Let α and σ be multiindices of the range m. Write

$$x^p_\alpha\,, \quad 0 \leq \|\alpha\| \leq r$$

for the induced coordinates on J^rY and

$$x^p_\sigma\,,\ X^p_\sigma = dx^p_\sigma, \quad 0 \leq ||\sigma|| \leq s$$

for the induced coordinates on VJ^sY. If

$$a_{i_1\dots i_l}(x^i, x^p_\alpha, X^p_\sigma)\, dx^{i_1} \wedge \dots \wedge dx^{i_l} \tag{8}$$

is the coordinate expression of φ, then the coordinate form of $D\varphi$ is

$$\left(\frac{\partial a_{i_1\dots i_l}}{\partial x^i} + \frac{\partial a_{i_1\dots i_l}}{\partial x^p_\alpha}\, x^p_{\alpha i} + \frac{\partial a_{i_1\dots i_l}}{\partial X^p_\sigma}\, X^p_{\sigma i}\right) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_l}\,. \tag{9}$$

3 The functional bundle $S(E,Q)$

We shall use the following simplified version, [1], of the theory of smooth spaces by A. Frölicher, [3]. An F-smooth space is a set S along with a set C_S of maps $\gamma : \mathbb{R} \to S$, which are called F-smooth curves, satisfying

(i) each constant curve $\mathbb{R} \to S$ belongs to C_S,

(ii) if $\gamma \in C_S$ and $\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$, then $\gamma \circ \varepsilon \in C_S$.

Every subset $\bar{S} \subset S$ is also an F-smooth space, if we define $C_{\bar{S}} \subset C_S$ to be the subset of all curves with values in $\bar{S}$. If $(S', C_{S'})$ is another F-smooth space, a map $f : S \to S'$ is said to be F-smooth, if $f \circ \gamma$ is an F-smooth curve on S' for every F-smooth curve γ on S. So we obtain the category $\mathcal{S}$ of F-smooth spaces.

In particular, every smooth manifold M turns out to be an F-smooth space by assuming as F-smooth curves just the smooth curves. Moreover, a map between smooth manifolds is F-smooth, if and only if it is smooth. An F-smooth bundle is a triple of an F-smooth space S, a smooth manifold M and a surjective F-smooth map $p : S \to M$. If $p' : S' \to M'$ is another F-smooth bundle, then a morphism of S into S' is a pair of an F-smooth map $f : S \to S'$ and a smooth map $\underline{f} : M \to M'$ satisfying $\underline{f} \circ p = p' \circ f$. So we obtain the category $\mathcal{SB}$ of F-smooth bundles.

If $p_1 : Y_1 \to M$, $p_2 : Y_2 \to M$ are two fibered manifolds, we write

$$\mathcal{F}(Y_1, Y_2) = \bigcup_{x\in M} C^\infty(Y_{1x}, Y_{2x})$$

and denote by $p : \mathcal{F}(Y_1,Y_2) \to M$ the canonical projection. A curve $\widehat{c} : \mathbb{R} \to \mathcal{F}(Y_1,Y_2)$ is said to be F-smooth, if $\underline{c} := p \circ \widehat{c} : \mathbb{R} \to M$ is a smooth curve and the induced map

$$c : \underline{c}^* Y_1 \to Y_2, \quad c(t,y) = \widehat{c}(t)(y), \quad p_1(y) = \underline{c}(t)\,,$$

is also smooth, [2]. The F-smooth sections of $\mathcal{F}(Y_1,Y_2)$ are identified with the base preserving morphisms $s : Y_1 \to Y_2$. We write $\widehat{s} : M \to \mathcal{F}(Y_1,Y_2)$ for the F-smooth section induced by s.

The tangent bundle $T\mathcal{F}(Y_1,Y_2) \to TM$ is defined as follows, [2]. For every F-smooth curve $\widehat{f} : \mathbb{R} \to \mathcal{F}(Y_1,Y_2)$, we first construct the tangent vector $X = \frac{\partial}{\partial t}|_0 (p \circ \widehat{f}) \in TM$. Write

$$T_X Y_1 = (Tp_1)^{-1}(X) \subset TY_1, \quad T_X Y_2 = (Tp_2)^{-1}(X) \subset TY_2\,.$$

Then $\widehat{f}$ defines a map $T_0\widehat{f} : T_X Y_1 \to T_X Y_2$ by

$$T_0\widehat{f}(\frac{\partial}{\partial t}|_0 h(t)) = \frac{\partial}{\partial t}|_0 \widehat{f}(t)(h(t))\,, \tag{10}$$

where we may assume that $h : \mathbb{R} \to Y_1$ satisfies $p \circ \widehat{f} = p_1 \circ h$. We say that $\widehat{f}$ and another F-smooth curve $\widehat{g} : \mathbb{R} \to \mathcal{F}(Y_1, Y_2)$ satisfying $\frac{\partial}{\partial t}|_0 (p \circ \widehat{g}) = X$ determine the same tangent vector at $f(0) = g(0) \in \mathcal{F}(Y_1, Y_2)$, if $T_0 \widehat{f} = T_0 \widehat{g} : T_X Y_1 \to T_X Y_2$. The set $T\mathcal{F}(Y_1, Y_2)$ of all equivalence classes is called the tangent bundle of $\mathcal{F}(Y_1, Y_2)$. The map $T_0 \widehat{f}$ is said to be the associated map of the tangent vector $\frac{d}{dt}|_0 \widehat{f}$.

Since $T\mathcal{F}(Y_1, Y_2) \subset \mathcal{F}(TY_1 \to TM, TY_2 \to TM)$, this is also an F-smooth bundle. The vertical tangent bundle $V\mathcal{F}(Y_1, Y_2) \to M$ is the subbundle of $T\mathcal{F}(Y_1, Y_2)$ of all elements projected by Tp into a zero vector on M.

Given a 2-fibered manifold $Q \xrightarrow{q} E \xrightarrow{p} M$, we denote by $S(E, Q) \subset \mathcal{F}(E, Q \to M)$ the F-smooth bundle of all sections $s : E_x \to Q_x$ of q.

A 2-fibered manifold morphism is a triple (f, f_1, f_0) such that the following diagram commutes

$$\begin{array}{ccccc} Q & \xrightarrow{q} & E & \xrightarrow{p} & M \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\ \bar{Q} & \xrightarrow{\bar{q}} & \bar{E} & \xrightarrow{\bar{p}} & \bar{M} \end{array}$$

So we obtain the category $2\mathcal{FM}$. Write $2\mathcal{FM}^I \subset 2\mathcal{FM}$ for the category defined by the requirement that f_1 is a diffeomorphism on each fiber. If $f \in 2\mathcal{FM}^I$, we have the induced map

$$S(f) : S(E, Q) \to S(\bar{E}, \bar{Q})$$

transforming $s : E_x \to Q_x$ into

$$f_x \circ s \circ (f_{1x})^{-1} : \bar{E}_{f_0(x)} \to \bar{Q}_{f_0(x)} \, .$$

Clearly, S is a functor on $2\mathcal{FM}^I$ with values in $\mathcal{SB}$.

If we have another 2-fibered manifold $P \to E \to M$, then a $2\mathcal{FM}$-morphism over id_E will be called an E-morphism. In this case we shall also write $\widehat{f} = S(f) : S(E, Q) \to S(E, P)$.

Consider a vertical curve $\widehat{\gamma} : \mathbb{R} \to S(E, Q)$ over $x \in M$. Then $\gamma(t) : E_x \to Q_x$, $t \in \mathbb{R}$, and $\gamma(t)(y)$ is a vertical curve on $Q \to E$ for every $y \in E_x$. Hence $\frac{d}{dt}|_0 \gamma(t)(y) \in V_y(Q \to E)$. Using the standard globalization procedure, [11], we deduce

$$VS(E, Q) = S(E, V(Q \to E)) \, . \tag{11}$$

We have a canonical injection

$$i : J^r S(E, Q) \to S(E, J^r(Q \to E)) \tag{12}$$

defined as follows. Consider a section $\widehat{s} : M \to S(E,Q)$, so that $s : E \to Q$. Then $j^r\widehat{s}$ determines $j^r s : E \to J^r(Q \to E)$. We have $j^r_x\widehat{s} \in J^r_x S(E,Q) \subset J^r_x\mathcal{F}(E,Q)$ and we set

$$i(j^r_x\widehat{s}) = j^r s|E_x : E_x \to J^r_x(Q \to E)\,.$$

We shall consider some local fiber coordinates x^i, x^p on Y_1 and local fiber coordinates x^i, z^a on Y_2. In the case of $Q \to E \to M$, x^i, x^p, z^a will mean the corresponding fiber coordinates on Q. Hence the coordinate expression of $j^r_x\widehat{s}$ are the functions

$$z^a_\alpha(x^p), \quad 0 \le \|\alpha\| \le r\,,$$

where α is a multiindex of the range m, [2]. On the other hand, the coordinate expression of $i(j^r_x\widehat{s})$ are some functions $z^a_{\alpha\beta}(x^p)$, $0 \le \|\alpha\| + \|\beta\| \le r$, where β is a multiindex of the range $m+1, \dots, m+n$. Our definition implies

$$z^a_{\alpha\beta} = \partial_\beta z^a_\alpha(x^p)\,. \tag{13}$$

3 Remark. We remark that (13) describes also a general injection

$$S(E, J^r(Q \to B)) \hookrightarrow S(E, J^r(Q \to E))\,. \tag{14}$$

Consider another 2-fibered manifold $P \to E \to M$ and an E-morphism $f : Q \to P$. Then we have the induced maps

$$J^r f : J^r(Q \to E) \to J^r(P \to E), \quad J^r f(j^r_y s) = j^r_y(f \circ s)$$

and $S(f) : S(E,Q) \to S(E,P)$.

4 Lemma. *The following diagram commutes*

$$\begin{array}{ccc} J^r S(E,Q) & \xrightarrow{J^r S(f)} & J^r S(E,P) \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle i}\big\downarrow \\ S(E, J^r(Q \to E)) & \xrightarrow{S(J^r f)} & S(E, J^r(P \to E)) \end{array} \tag{15}$$

Proof. For $j^r_x\widehat{s} \in J^r S(E,Q)$, we obtain clockwise $i(j^r_x(\widehat{f} \circ \widehat{s})) = i(j^r_x(\widehat{f \circ s}))$ $= j^r(f \circ s)|E_x$ and counterclockwise $S(J^r f)(j^r s|E_x) = j^r(f \circ s)|E_x$. QED

The classical exchange map $\varkappa_r : VJ^rY \to J^rVY$ is defined by

$$\tfrac{\partial}{\partial t}|_0 j^r_x s(t,u) \mapsto j^r_x \tfrac{\partial}{\partial t}|_0 s(t,u), \quad t \in \mathbb{R}, u \in M\,,$$

[8]. In the functional case, we have an exchange map

$$K_r : VJ^r\mathcal{F}(Y_1,Y_2) \to J^rV\mathcal{F}(Y_1,Y_2)$$

defined by the analogous formula

$$K_r\big(\tfrac{\partial}{\partial t}\,|_0 j^r_x \widehat{s}(t,u)\big) = j^r_x \tfrac{\partial}{\partial t}\,|_0 \widehat{s}(t,u)\,. \tag{16}$$

If we consider $S(E,Q)$ instead of $\mathcal{F}(Y_1,Y_2)$, then the values of $\widehat{s}$ in (16) are the sections of q, so that we have a restricted and corestricted map

$$K_r : VJ^rS(E,Q) \to J^rVS(E,Q)\,.$$

The same character of the definitions of $\varkappa_r$ and K_r implies that the following diagram commutes

$$\begin{array}{ccc} VJ^rS(E,Q) & \xrightarrow{K_r} & J^rVS(E,Q) \\ \downarrow & & \downarrow \\ S(E,VJ^rQ) & \xrightarrow{S(\varkappa_r)} & S(E,J^rVQ) \end{array} \tag{17}$$

where the left and right arrows are the canonical injections induced by (12) in combination with (11).

4 The operator order on $J^r\mathcal{F}(Y_1,Y_2)$

In [2] there was discussed, in fact, the operator order on an F-smooth morphism

$$A : \mathcal{F}(Y_1,Y_2) \to \mathcal{F}(Y_1,Y)\,, \tag{18}$$

where $Y \to M$ is another fibered manifold. We say that A is of the operator order k, if, for every $\varphi,\psi \in C^\infty(Y_{1x},Y_{2x})$,

$$j^k_y\varphi = j^k_y\psi \quad \text{implies} \quad A(\varphi)(y) = A(\psi)(y)\,.$$

Then A determines the associated map

$$\mathcal{A} : \mathcal{F}J^k(Y_1,Y_2) \to Y\,, \quad \mathcal{A}(j^k_y\varphi) = A(\varphi)(y)\,, \tag{19}$$

where

$$\mathcal{F}J^k(Y_1,Y_2) = \bigcup_{x\in M} J^k(Y_{1x},Y_{2x})$$

is a classical manifold. By [2], $\mathcal{A}$ is a smooth map.

Let x^i, x^p and z^a be the local coordinates on Y_1 and Y_2 from Section 2. Then the induced coordinates on $\mathcal{F}J^k(Y_1,Y_2)$ are z^a_β, $0 \leq \|\beta\| \leq k$, where β is a multiindex of range $(m+1,\dots,m+n)$. If x^i, w^s, $s=1,\dots,\dim Y-\dim M$, are local fiber coordinates on Y, then the coordinate expression of $\mathcal{A}$ is

$$w^s = f^s(x^i, x^p, z^a_\beta)\,.$$

If we consider an $\mathcal{SB}$-morphism

$$A : J^r\mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_1, Y)\,, \tag{20}$$

we have take into account that $\varphi, \psi \in J^r_x\mathcal{F}(Y_1, Y_2)$ are characterized by the associated maps

$$\bar\varphi, \bar\psi : J^r_x Y_1 \to J^r_x Y_2\,.$$

So we have $j^k\bar\varphi, j^k\bar\psi : J^r_x Y_1 \to J^k(J^r_x Y_1, J^r_x Y_2)$.

5 Definition. We say that A is of the operator order k, if

$$j^k\bar\varphi|J^r_y Y_1 = j^k\bar\psi|J^r_y Y_1 \quad \text{implies} \quad A(\varphi)(y) = A(\psi)(y)\,.$$

To characterize the associated map of A in this situation, we introduce a new concept.

6 Definition. For a base preserving morphism $f : Y_1 \to Y_2$, its fiberwise r-jet prolongation $(\mathcal{F}j^r)f$ is defined by

$$(\mathcal{F}j^r)f : Y_1 \to \mathcal{F}J^r(Y_1, Y_2)\,, \quad (\mathcal{F}j^r)f(y) = j^r_y(f_x)\,,$$

where $f_x : Y_{1x} \to Y_{2x}$ is the restricted and corestricted map, $x = p_1(y)$. The k-jet $j^k_y(\mathcal{F}j^r)f$ is called the *fiberwise (k,r)-jet* of f at y.

Let α be a multiindex of the range m and $\gamma = (\alpha, \beta)$. Let $z^a = f^a(x^i, x^p)$ be the coordinate expression of f. Then the coordinate expression of $(\mathcal{F}j^r)f$ is

$$z^a_\beta = \partial_\beta f^a\,, \quad 0 \le \|\beta\| \le r\,.$$

We write $\mathcal{F}J^{k,r}(Y_1, Y_2) = J^k(\mathcal{F}J^r(Y_1, Y_2) \to Y_1)$ for the space of all fiberwise (k,r)-jets of Y_1 to Y_2. This is a classical manifold with the induced coordinates

$$z^a_{\beta\gamma}\,, \quad 0 \le \|\beta\| \le r\,, \; 0 \le \|\gamma\| \le k\,.$$

Clearly, we have

$$\mathcal{F}J^{k,0}(Y_1, Y_2) \simeq J^k(Y_1 \underset{M}{\times} Y_2 \to Y_1)\,. \tag{21}$$

Indeed, $(\mathcal{F}j^0)f = f$, which we identify with its graph $Y_1 \to Y_1 \underset{M}{\times} Y_2$, $y \mapsto (y, f(y))$.

7 Proposition. *If $A : J^r\mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_1, Y)$ is of operator order k, then $A(j^r_x\widehat{f})(y)$ depends on $j^k_y(\mathcal{F}j^r)f$ only.*

PROOF. For $r = 1$, the associated map $h : J^1_x Y_1 \to J^1_x Y_2$ of an element of $J^1_x\mathcal{F}(Y_1, Y_2)$ is

$$z^a = f^a(x^i_0, x^p), \;\; z^a_i = \partial_i f^a(x^i_0, x^p) + \partial_p f^a(x^i_0, x^p)\, x^p_i \,, \tag{22}$$

where x^p and x^p_i are the variables on $J^1_x Y_1$, $x = (x^i_0) \in M$. Hence $j^k_y h|J^1_y Y$, $y = (x^i_0, x^p_0)$, depends on

$$\partial_\beta f^a(x^i_0, x^p_0), \;\; \partial_\beta\partial_i f^a(x^i_0, x^p_0), \;\; \partial_\beta\partial_p f^a(x^i_0, x^p_0), \;\; 0 \leq\|\beta\|\leq k \,.$$

These are the coordinates of $j^k_y(\mathcal{F}j^1)f$. For $r > 1$ we proceed by iteration using the facts J^r is an r-th order functor and the coordinate formula for $J^r f$ is of a specific polynomial character in the induced jet coordinates. QED

Hence A determines the associated map

$$\mathcal{A} : \mathcal{F}J^{k,r}(Y_1, Y_2) \to Y \,, \quad \mathcal{A}(j^k_y(\mathcal{F}j^r)f) = A(j^r_x\widehat{f})(y) \,.$$

Analogously to (19), $\mathcal{A}$ is a smooth map.

We remark that the concept of fiberwise (k, r)-jet can be incorporated into the general framework of the concept of (r, s, q)-jet of fibered manifold morphisms, [8]. But this is somewhat sophisticated for our purposes, so that we prefer our direct approach here.

8 Remark. It is interesting that a similar approach can be applied to an arbitrary fiber product preserving bundle functor G on $\mathcal{FM}_m$. In [1] we clarified that G can be extended to $\mathcal{F}(Y_1, Y_2)$ as follows. If G is of the base order r, it can be identified with a triple (A, H, t), where A is a Weil algebra, $H : G^r_m \to \operatorname{Aut} A$ is a group homomorphism and $t : \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. In [1] we defined $G\mathcal{F}(Y_1, Y_2)$ as the subset of the F-smooth associated bundle $P^r M[T^A\mathcal{F}(Y_1, Y_2)]$ of all equivariance classes $\{u, Z\}$, $u \in P^r M, \; Z \in T^A\mathcal{F}(Y_1, Y_2)$ satisfying $t_M(u) = T^A p(Z)$.

Analogously to the tangent case, Z can be interpreted as a map

$$\bar{Z} : T^A_X Y_1 \to T^A_X Y_2 \,, \quad X \in T^A p(Z) \in T^A M \,.$$

We know that GY_i, $i = 1, 2$, is the subset of $P^r M[T^A Y_i]$ of all $\{u, Z_i\}$ satisfying $t_M(u) = Tp_i(Z_i)$. Then we construct a well defined inclusion

$$G\mathcal{F}(Y_1, Y_2) \subset \mathcal{F}(GY_1, GY_2)$$

by transforming $\{u, Z\} \in G\mathcal{F}(Y_1, Y_2)$ into the map

$$\overline{\{u, Z\}}(\{u, Z_1\}) = \{u, \bar{Z}(Z_1)\} \,, \quad \{u, Z_1\} \in GY_1 \,.$$

Thus, for every G we can treat the operator order of an $\mathcal{SB}$-morphism

$$G\mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_1, Y)$$

similarly to the case $G = J^r$.

5 The formal exterior differential over $\mathcal{F}(Y_1, Y_2)$

We recall that, given two other fibered manifolds $Y_3 \to M$, $Y_4 \to M$, an $\mathcal{SB}$-morphism $A : \mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_3, Y_4)$ is called J^k-differentiable, if the rule

$$(j^k_x \widehat{s}) \mapsto j^k_x(A \circ \widehat{s}), \quad \widehat{s} : M \to \mathcal{F}(Y_1, Y_2)$$

defines an F-smooth map

$$J^k A : J^k\mathcal{F}(Y_1, Y_2) \to J^k\mathcal{F}(Y_3, Y_4)\,.$$

In general, consider three F-smooth bundles S_1, S_2, S_3 over M and two surjective $\mathcal{SB}$-morphisms $\pi_1 : S_1 \to S_3$, $\pi_2 : S_2 \to S_3$. We write

$$S_1 \underset{S_3}{\times} S_2 = \{(u_1, u_2) \in S_1 \times S_2, \pi_1(u_1) = \pi_2(u_2)\}\ .$$

Clearly, this is also an F-smooth bundle over M.

Consider a J^k-differentiable morphism, $s \leq r$,

$$A : J^r\mathcal{F}(Y_1, Y_2) \underset{J^s\mathcal{F}(Y_1,Y_2)}{\times} VJ^s\mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_1, Y)\,. \tag{23}$$

Using the exchange map K_s, see (16), we can define

$$\mathcal{J}^k_{\mathrm{hol}}A : J^{k+r}\mathcal{F}(Y_1, Y_2) \underset{J^{k+s}\mathcal{F}(Y_1,Y_2)}{\times} VJ^{k+s}\mathcal{F}(Y_1, Y_2) \to J^k\mathcal{F}(Y_1, Y) \tag{24}$$

in the same way as in Section 1.

To introduce the formal exterior differential, we have to consider $S(Y_1, \bigwedge^l T^*Y_1)$ on the right hand side of (23). So, let

$$A : J^r\mathcal{F}(Y_1, Y_2) \underset{J^s\mathcal{F}(Y_1,Y_2)}{\times} VJ^s\mathcal{F}(Y_1, Y_2) \to S(Y_1, \bigwedge^l T^*Y_1) \tag{25}$$

be a J^1-differentiable morphism. Then we construct $\mathcal{J}^1_{\mathrm{hol}}A$, use the inclusion

$$i : J^1 S(Y_1, \bigwedge^l T^*Y_1) \to S(Y_1, J^1 \bigwedge^l T^*Y_1)$$

on the right hand side and add $S(\delta) : S(Y_1, J^1 \bigwedge^l T^*Y_1) \to S(Y_1, \bigwedge^{l+1} T^*Y_1)$, where δ is the formal version of the exterior differential.

9 Definition. The F-smooth morphism

$$\mathbb{D}A := S(\delta) \circ i \circ \mathcal{J}^1_{\text{hol}} A : \tag{26}$$
$$: J^{r+1}\mathcal{F}(Y_1, Y_2) \underset{J^{s+1}\mathcal{F}(Y_1,Y_2)}{\times} VJ^{s+1}\mathcal{F}(Y_1, Y_2) \to S(Y_1, \bigwedge^{l+1} T^*Y_1)$$

will be called the *formal exterior differential* of (25).

Clearly, the construction of $J^rY_1 \underset{J^sY_1}{\times} VJ^sY_1$ and of the induced maps is a fiber product preserving bundle functor on $\mathcal{FM}_m$. According to Remark 8, we can introduce the concept of operator order of the morphism (23). However, we shall not go into details in this paper.

6 The restriction of $\mathbb{D}$ to $S(E, Q)$

Now we consider $S(E, Q)$ in the role of $\mathcal{F}(Y_1, Y_2)$. Let $P \to E \to M$ be another 2-fibered manifold and

$$A : J^rS(E, Q) \underset{J^sS(E,Q)}{\times} VJ^sS(E, Q) \to S(E, P) \tag{27}$$

be a J^k-differentiable morphism. Then (24) restricts to a morphism

$$\mathcal{J}^k_{\text{hol}} A : J^{k+s}S(E, Q) \underset{J^{k+s}S(E,Q)}{\times} VJ^{k+s}S(E, Q) \to J^kS(E, P)\,. \tag{28}$$

In the case $k = 1$ and $P = \bigwedge^l T^*E$, (26) yields a morphism

$$\mathbb{D}A : J^{r+1}S(E, Q) \underset{J^{s+1}S(E,Q)}{\times} VJ^{s+1}S(E, Q) \to S(E, \bigwedge^{l+1} T^*E)\,. \tag{29}$$

An important fact is that (29) and the finite dimensional formal exterior differential over E are related as follows. From now on we always consider Q as a fibered manifold over E. Let

$$B : J^rQ \underset{J^sQ}{\times} VJ^sQ \to P \tag{30}$$

be a smooth E-morphism. On one hand, we construct

$$S(B) : S(E, J^rQ) \underset{S(E,J^sQ)}{\times} S(E, VJ^sQ) \to S(E, P)\,. \tag{31}$$

The injection $J^r S(E,Q) \to S(E,J^r Q)$ induces, including holonomization, an injection

$$I : J^k S(E, J^r Q) \underset{J^k S(E,J^s Q)}{\times} J^k S(E, V J^s Q) \to \\ \to S(E, J^{k+r} Q) \underset{S(E,J^{k+s}Q)}{\times} S(E, V J^{k+s} Q)\,. \tag{32}$$

On the other hand, we can construct

$$\mathcal{J}^k_{\mathrm{hol}} B : J^{k+r} Q \underset{J^{k+s}Q}{\times} V J^{k+s} Q \to J^k P\,. \tag{33}$$

So we have a diagram

$$\begin{array}{ccc} J^k S(E, J^r Q) \underset{J^k S(E,J^s Q)}{\times} J^k S(E, V J^s Q) & \xrightarrow{J^k S(B)} & J^k S(E,P) \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle i} \\ S(E, J^{k+r} Q) \underset{S(E,J^{k+s}Q)}{\times} S(E, V J^{k+s} Q) & \xrightarrow{S(\mathcal{J}^k_{\mathrm{hol}} B)} & S(E, J^k P) \end{array} \tag{34}$$

Then the proofs of (15) and (17) imply

10 Lemma. *(34) is a commutative diagram.* QED

In the case of $B : J^r Q \underset{J^s Q}{\times} V J^s Q \to \bigwedge^l T^* E$, $S(B)$ induces

$$\mathbb{D}S(B) : J^1 S(E, J^r Q) \underset{J^1 S(E,J^s Q)}{\times} J^1 S(E, V J^s Q) \to S(E, \bigwedge^{l+1} T^* E)\,. \tag{35}$$

On the other hand, we have $DB : J^{r+1} Q \times_{J^{s+1}Q} V J^{s+1} Q \to \bigwedge^{l+1} T^* E$. Then Lemma 10 implies

11 Proposition. *We have $\mathbb{D}(S(B)) = S(DB) \circ I$.* QED

7 The Euler-Lagrange morphism

We first recall a suitable construction of the Euler-Lagrange morphism of a first order Lagrangian on a fibered manifold $Y \to M$, [6,10]. We shall discuss a slightly more general case of a morphism

$$\lambda : J^1 Y \to \bigwedge^l T^* M\,. \tag{36}$$

If $l = m = \dim M$, we obtain a classical first order Lagrangian on Y.

The vertical differential of λ is a map

$$d_V\lambda : J^1Y \to V^*J^1Y \otimes \bigwedge^l T^*M\,. \tag{37}$$

The well-known exact sequence

$$0 \to VY \otimes T^*M \to VJ^1Y \to VY \to 0$$

induces the dual map $V^*J^1Y \to V^*Y \otimes TM$. If we add the classical tensor contraction $\text{åy} : TM \otimes \bigwedge^l T^*M \to \bigwedge^{l-1} T^*M$, we obtain the composed map

$$\rho_Y : V^*J^1Y \otimes \bigwedge^l T^*M \to V^*Y \otimes \bigwedge^{l-1} T^*M\,. \tag{38}$$

Hence $\rho_Y \circ d_V\lambda$ can be interpreted as a morphism

$$B(\lambda) = \rho_Y \circ d_V\lambda : J^1Y \underset{Y}{\times} VY \to \bigwedge^{l-1} T^*M\,. \tag{39}$$

Then

$$DB(\lambda) : J^2Y \underset{J^1Y}{\times} VJ^1Y \to \bigwedge^l T^*M\,.$$

In coordinates, if

$$\lambda \equiv L_{i_1\dots i_l}(x^i, x^p, x^p_i)\, dx^{i_1} \wedge \dots \wedge dx^{i_l}\,,$$

then

$$d_V\lambda \equiv \frac{\partial L_{i_1\dots i_l}}{\partial x^p}\, dx^p \otimes dx^{i_1} \wedge \dots \wedge dx^{i_l} + \frac{\partial L_{i_1\dots i_l}}{\partial x^p_i}\, dx^p_i \otimes dx^{i_1} \wedge \dots \wedge dx^{i_l}\,.$$

Hence

$$B(\lambda) \equiv \frac{\partial L_{i_1\dots i_l}}{\partial x^p_i}\, dx^p \otimes \frac{\partial}{\partial x^i}\, \text{åy}\, dx^{i_1} \wedge \dots \wedge dx^{i_l}\,.$$

Using (9), we obtain

$$DB(\lambda) \equiv D_i \frac{\partial L_{i_1\dots i_l}}{\partial x^p_i}\, dx^p \otimes dx^{i_1} \wedge \dots \wedge dx^{i_l} + \frac{\partial L_{i_1\dots i_l}}{\partial x^p_i}\, dx^p_i \otimes dx^{i_1} \wedge \dots \wedge dx^{i_l}\,,$$

where D_i denotes the standard formal partial derivative with respect to x^i, [6]. This implies that the difference

$$\mathcal{E}(\lambda) := (d_V\lambda) \circ \pi^2_1 - DB(\lambda)\,, \tag{40}$$

where $\pi^2_1 : J^2Y \to J^1Y$ is the jet map, is projectable to VY. Hence it can be interpreted as a morphism

$$\mathcal{E}(\lambda) : J^2Y \to V^*Y \otimes \bigwedge^l T^*M\,. \tag{41}$$

Its coordinate expression implies that for $m = l$ we obtain the Euler-Lagrange morphism of λ. A more geometric explanation of this fact can be found in [6].

Consider now a smooth E-morphism

$$L : J^1Q \to \bigwedge^l T^*E\,, \tag{42}$$

which is a first order Lagrangian on $Q \to E$ in the case $l = m + n$. We can interpret L as an $\mathcal{SB}$-morphism

$$\widehat{L} : S(E, J^1Q) \to S(E, \bigwedge^l T^*E) \tag{43}$$

or as a section, denoted by the same symbol,

$$\widehat{L} : M \to \mathcal{F}(J^1Q, \bigwedge^l T^*E)\,. \tag{44}$$

Under the functional approach, the vertical differential is a map

$$\widehat{d_V} : \mathcal{F}(J^1Q, \bigwedge^l T^*E) \to S(J^1Q, V^*J^1Q \otimes \bigwedge^l T^*E)\,. \tag{45}$$

Then we take into account

$$\rho_Q : V^*J^1Q \otimes \bigwedge^l T^*E \to V^*Q \otimes \bigwedge^{l-1} T^*E\,.$$

This induces, fiberwise, a map

$$\mathcal{F}(\mathrm{id}_{J^1Q}, \rho_Q) : \mathcal{F}(J^1Q, V^*J^1Q \otimes \bigwedge^l T^*E) \to \mathcal{F}(J^1Q, V^*Q \otimes \bigwedge^{l-1} T^*E)\,.$$

Then $\widehat{B}(\widehat{L}) := \mathcal{F}(\mathrm{id}_{J^1Q}, \rho_Q) \circ \widehat{d_V} \circ \widehat{L}$ can be viewed as a morphism

$$\widehat{B}(\widehat{L}) : S(E, J^1Q \underset{Q}{\times} VQ) \to S(E, \bigwedge^{l-1} T^*E)$$

and we can construct

$$\mathbb{D}(\widehat{B}(\widehat{L})) : J^1S(E, J^1Q) \underset{J^1S(E,Q)}{\times} J^1S(E, VJ^1Q) \to S(E, \bigwedge^l T^*E)\,.$$

Since $\widehat{B}(\widehat{L}) = S(B(L))$, Proposition 11 yields $\mathbb{D}(\widehat{B}(\widehat{L})) = S(D(B(L))) \circ I$. This implies that $\mathbb{D}(\widehat{B}(\widehat{L}))$ can be viewed as an F-smooth section

$$M \to \mathcal{F}(J^2Q, V^*J^1Q \otimes \bigwedge^l T^*E)$$

and $\widehat{d}_V \circ \widehat{L} \circ \pi^2_1 - \mathbb{D}(\widehat{B}(\widehat{L}))$ corestricts to an F-smooth section

$$\widehat{\mathcal{E}}(\widehat{L}) = \widehat{d}_V \circ \widehat{L} \circ \pi^2_1 - \mathbb{D}(\widehat{B}(\widehat{L})) : M \to \mathcal{F}(J^2Q, V^*Q \otimes \bigwedge^l T^*E)\,. \tag{46}$$

This can be viewed as an $\mathcal{SB}$-morphism, denoted by the same symbol,

$$\widehat{\mathcal{E}}(\widehat{L}) : S(E, J^2Q) \to S(E, V^*Q \otimes \bigwedge^l T^*E)\,. \tag{47}$$

By construction, $\widehat{\mathcal{E}}(\widehat{L}) = S(\mathcal{E}(L))$.

Thus, in the case $l = m+n$ our construction represents a functional approach to the Euler-Lagrange morphism of a first order Lagrangian on $Q \to E$.

12 Remark. Given two fibered manifolds $Y_1 \to M$ and $Y_2 \to M$, one can study variational calculus for the base preserving morphisms $Y_1 \to Y_2$. Since these morphisms are identified with the sections of the fibered manifold $Y_1 \times_M Y_2 \to Y_1$, from the abstract point of view the morphism problem reduces to the variational calculus for the sections of the latter bundle. However, the geometry of the morphism problem should be more rich. It seems to be reasonable to discuss this subject in more details elsewhere.

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