Note di Matematica **24,** n. 2, 2005, 59–73.

The translation planes of order 81 admitting $SL(2,5)$

Vikram Jha
Mathematics Dept., Caledonian University,
Cowcaddens Road, Glasgow, Scotland
v.jha@gcal.ac.uk

Norman L. Johnson [i]
Mathematics Dept., University of Iowa,
Iowa City, Iowa 52242, USA
njohnson@math.uiowa.edu

Received: 9/12/2004; accepted: 9/12/2004.

Abstract. Johnson and Prince have classified all translation planes of order 81 that admit $SL(2,5)$, where the 3-elements are elations. In this article, it is shown that whenever $SL(2,5)$ acts as a collineation group on a translation plane of order 81, the problem can always be reduced to the elation case. Hence, there is a complete classification of all translation planes of order 81 admitting $SL(2,5)$; there are exactly 14 translation planes of order 81 admitting $SL(2,5)$.

Keywords: spread, SL(2,5) elation group, Baer group

MSC 2000 classification: primary 51E23, secondary 51A40

1 Introduction

Recently, Johnson and Prince [4] determined the translation planes of order 81 admitting $SL(2,5)$, where the 3-elements are elations. In this situation, the set of 10 elation axes form a derivable net, so upon derivation, there are translation planes of order 81 admitting $SL(2,5)$, where the 3-elements are Baer. However, in a general situation, it would not necessarily be known whether the 3-elements were Baer, or if they were Baer that the Baer axes would be disjoint or even if they were that these axes would belong to a derivable net. In fact, we show that all 3-elements are either elation or Baer and in the Baer case, both of these latter possibilities occur.

There are exactly 14 translation planes determined by Johnson and Prince, of which exactly 10 planes are new. In this case, the subspaces fixed pointwise by 3-elements are mutually disjoint.

[i]This article was written when the second author was visiting Caledonian University in May of 2004. The authors are grateful to the University and to the London Mathematical Society for support on this research and the visit to Scotland.

In this article, we consider the more general case: Translation planes of order 81 admitting $SL(2,5)$ as a collineation group in the translation complement, but without further assumptions.

1 Theorem. *Let π be a translation plane of order 81 admitting $SL(2,5)$ as a collineation group of the translation complement.*

(1) Then the 3-elements are either elations or Baer.

(2) If the 3-elements are Baer, the Baer axes are disjoint, and form a derivable subnet within the translation plane π.

(3) There are exactly 14 translation planes of order 81 admitting $SL(2,5)$ as a collineation group, as listed in Johnson and Prince [4].

2 The main result

In this section, we assume the conditions in the statement of the theorem listed in the introduction. We assume that the translation plane admits a group isomorphic to $SL(2,5)$, in the translation complement.

2.1 The Baer case

2 Lemma. *Suppose two 3-Baer groups generate $SL(2,3)$ and fix a subspace X pointwise on a component L. Then X cannot be 2-dimensional over $GF(3)$.*

Proof. L is a 4-dimensional $GF(3)$-space. Each Baer group σ, τ is a generalized elation on the entire space so is a generalized elation group on any fixed component L. Choose a basis for the pointwise fixed subspace X so that σ or τ may be represented in the following form:

$$\begin{bmatrix} C & A \\ 0 & B \end{bmatrix}.$$

Note that either element fixes X and L/X pointwise. Letting X be denoted by $\{(0,y)\}$, this implies that $B = I$. But then (a,b) is mapped to $(aC, aA + bB)$, which is equal to (a,b) modulo X, implying that $a(C - I) = 0$ for all a, so that $C = I$. So, both elements σ and τ have the form:

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix},$$

acting on L. This means that the group generated by σ and τ has order 3 on L, since $\langle\sigma,\tau\rangle$ is isomorphic to $SL(2,3)$. So, there is a normal subgroup Q_8 of order 8 fixing L pointwise.

Since the central involution is in $SL(2,5)$, this means that $SL(2,5)$ must fix L. However, if there is a quaternion group of order 8 fixing L pointwise, then $SL(2,5)$ is forced to fix L pointwise, a contradiction, since the 3-elements are Baer. Hence, X cannot be 2-dimensional. $\boxed{QED}$

3 Lemma. *Assume that the* 3*-elements of* $SL(2,5)$ *are Baer. Then the center of* $SL(2,5)$ *is either*

(1) the kernel involution, or

(2) an affine homology, and in the latter case the subgroups isomorphic to $SL(2,3)$ *fix* 1*-spaces over* $GF(3)$ *pointwise on* L.

Proof. Let ρ be the unique involution in the center of $SL(2,5)$. If ρ is not the kernel involution, then ρ is either Baer or an affine homology. If ρ is Baer then the group induced on $\operatorname{Fix}\rho$ is A_5. Let K_4 denote a Klein 4-group. Since K_4 is in A_4 and all involutions are conjugate, it follows that the involutions of K_4 are conjugate in A_5. If all involutions are central collineations, one of these will have axis the line at infinity. Since the involutions are conjugate, this would force the line at infinity to be moved to the other affine axes, a contradiction. Hence, all involutions are Baer and conjugate. By Ostrom's Baer-Trick of Theorem 20.3.1, p. 301 of Biliotti, Jha and Johnson [1], the dimension of the translation plane is at least $2 \cdot 4$, where $\operatorname{Fix}\rho$ is either Hall or Desarguesian of order 9, a contradiction.

Thus, we may assume that ρ is an affine homology with axis L and coaxis M. If ρ is affine and fixes L pointwise and fixes M, the coaxis, then $SL(2,5)$ fixes two components and induces $SL(2,5)$ on M and A_5 on L. Since the 3-elements are Baer, each 3-element fixes 9 points on L and on M. There are 10 3-groups. Suppose that is no fixed point on L. Then we obtain a spread where the 3-elements are elations and these must generate $SL(2,5)$, a contradiction since A_5 is induced.

Hence, on L, two 3-groups have overlapping fixed-point subspaces.

Assume that the overlaps are 2-dimensional over $GF(3)$. If these overlap in a 2-dimensional subspace over $GF(3)$, we either have $SL(2,3)$ or $SL(2,5)$ fixing 9 points of L. If $SL(2,5)$ then since 5 is a 3-primitive divisor of 3^4-1, it follows that the 5-elements are affine homologies, a contradiction. Note that in any case, if $SL(2,5)$ fixes a 1-space over $GF(3)$ pointwise, then the 5-elements are affine homologies.

Hence, the overlaps are 1-dimensional and in this case, we may apply the previous lemma. $\boxed{QED}$

Since we are also interested in the overlap problem on a translation plane of order 3^{2r}, we assume the more general situation in the following subsection.

2.2 The overlap theorem

4 Theorem. *Let π be a translation plane of order 3^{2r} that admits a collineation group in the translation complement isomorphic to $SL(2,5)$, generated by Baer 3-elements. Then the Baer axes are mutually disjoint.*

Proof. There are ten Baer 3-elements in $SL(2,5)$ and any two Baer 3-elements from distinct 3-groups will generate $SL(2,3)$ or $SL(2,5)$. Choose any two non-trivial Baer 3-elements from distinct 3-groups, say σ and τ. Let $F(\sigma)$ and $F(\tau)$ denote the fixed-point Baer subplanes of σ and τ, respectively. Let $F(\sigma)\cap F(\tau) = X$. Assume that X is a non-zero subspace over $GF(3)$ of dimension t. We furthermore assume that σ and τ have been chosen within $\langle\sigma,\tau\rangle$ so that $F(\sigma)$ and $F(\tau)$ intersect in a subspace X of maximum possible dimension.

We recall that a generalized elation g on a vector space W is an element that induces the identity on $W/Fixg$ (see, e. g., Section 23 of [1]). By Lemma 24.2.1 of [1], $\langle\sigma,\tau\rangle$ leaves $F(\sigma)+F(\tau)$ invariant and furthermore, each of σ and τ are generalized elations on $F(\sigma)+F(\tau)$. Thus, it follows that $F(\sigma)+F(\tau)$ has dimension $4r-t$ over $GF(3)$ By Lemma 24.2.6 of [1], we have $F(\sigma) = (\sigma-1)F(\tau)\oplus X$ and $F(\tau) = (\tau-1)F(\sigma)\oplus X$. Thus, $(\tau-1)F(\sigma)$ and $(\sigma-1)F(\tau)$ both have dimension $2r-t$ over $GF(3)$. Also by Lemma 24.2.6 of [1],

$$V_2 = (\sigma-1)F(\tau)\oplus(\tau-1)F(\sigma)$$

has dimension $2(2r-t)$ over $GF(3)$ and is invariant under $\langle\sigma,\tau\rangle$. Moreover, there is a $\langle\sigma,\tau\rangle$-invariant partial spread of degree at least four consisting of the $\langle\sigma,\tau\rangle$-images of $(\tau-1)F(\sigma)$ by Lemma 24.28. Since the central involution of $\langle\sigma,\tau\rangle$ fixes each of these images (components of the partial spread), the central involution θ acts like -1 on V_2. If X is non-trivial and since $\langle\sigma,\tau\rangle$ fixes X pointwise, then θ cannot be the kernel involution of the superspace V. Hence, $-\theta$ fixes V_2 pointwise. Since the kernel involution $-1I_8$ and θ are collineations of π, so is $-\theta$. Since the number of fixed points of a collineation is bounded by 3^{2r} (i. e., if V_2 does not lie on a component of π then $-\theta$ is planar and must be contained in a Baer subplane (order 3^r)), we thus obtain:

$$2(2r-t)\le 2r,$$

so that $t\ge r$.

But X is a subspace of dimension t contained in a Baer subplane and is pointwise fixed by θ, which also leaves $F(\sigma)$ invariant. If $t>r$ then θ is forced to fix $F(\sigma)$ pointwise. However, also θ must fix $F(\tau)$ pointwise. Hence, θ fixes a subspace of dimension $4r-t$ pointwise, $4r-t\le 2r$. Hence, $t=2r$, a contradiction since $F(\sigma)$ and $F(\tau)$ are chosen to be distinct. Hence, $t=r$. Therefore, if there are maximal overlaps X must have dimension r. Since θ fixes X pointwise and

X has cardinality 3^r, either X is contained in a component (i. e., is a Baer subline), or X is a subplane of order $3^{r/2}$. Hence, θ is a Baer involution or affine homology, respectively as X is a subplane or Baer subline. However, it might be possible that θ is a Baer involution and still X is a Baer subline.

If θ is a Baer involution then $SL(2,5)$ induces A_5 on $\mathrm{Fix}\,\theta$ (we note that $SL(2,5)$ can't fix $\mathrm{Fix}\,\theta$ pointwise in this case since the group fixing a Baer subplane pointwise has order dividing $3^r(3^r-1)$ and there is a normal subgroup of order 3^r semidirect a cyclic group of order 3^r-1). If X is a Baer subplane of $\mathrm{Fix}\,\theta$ then there is an elementary Abelian 2-group of order 4 fixing X pointwise and acting faithfully on $\mathrm{Fix}\,\theta$. However, an Abelian group of order 4 fixing the Baer subplane X of $\mathrm{Fix}\,\theta$ pointwise then can't act faithfully on $\mathrm{Fix}\,\theta$, since the structure of the subgroup pointwise fixing X can only have a cyclic subgroup of order dividing $3^{r/2}-1$. Hence, X is a Baer subline contained in the component L. It is still possible that θ is a Baer involution. We know that $\langle\sigma,\tau\rangle$ fixes L and induces a generalized elation group of order 3 on the component L. This implies that there is at least a quaternion affine homology group in $\langle\sigma,\tau\rangle$ with axis L. Either $\langle\sigma,\tau\rangle$ is isomorphic to $SL(2,3)$ or $SL(2,5)$. In the latter case, $SL(2,5)$ is forced to fix L pointwise, a contradiction (for several reasons, θ is a Baer involution, the 3-elements are Baer). Hence, $\langle\sigma,\tau\rangle$ is isomorphic to $SL(2,3)$. Since there is a subgroup isomorphic to $SL(2,3)$ fixing L and L is not $SL(2,5)$ invariant then there are five images of L under $SL(2,5)$. However, there is an affine homology group of order 8 with axis L, implying that such a group fixes $\mathrm{Fix}\,\theta$, has coaxis coL in $\mathrm{Fix}\,\theta$ and has orbits of length 8 on the remaining components of $\mathrm{Fix}\,\theta$. If $SL(2,5)$ moves L then since $SL(2,5)$ leaves $\mathrm{Fix}\,\theta$, any image of L must lie in $\mathrm{Fix}\,\theta$. Note that an element ρ_5 of order 5 cannot interchange L with its coaxis so must map L into an orbit of length 8 under an affine homology group in $\langle\sigma,\tau\rangle$ with axis L, a contradiction, since the orbit length can be either 1 or 5, as $SL(2,3)$ leaves L invariant. Hence, $SL(2,5)$ does leave L invariant, a contradiction as seen previously.

Now assume that there are overlaps and θ is an affine homology. Then, we know that the overlaps have dimension r and must lie on components. So X lies in the component L, which is fixed pointwise by θ, implying that $SL(2,5)$ fixes L pointwise, just as above. However, again this is a contradiction.

Thus, Baer subplanes from $SL(2,3)$-subgroups are mutually disjoint. If initially, $\langle\sigma,\tau\rangle$ is $SL(2,5)$, and there are overlapping subplanes, overlapping in X, then $SL(2,3)$ fixes X pointwise and we may obtain a contradiction as above. Hence, the ten Baer subplanes are mutually disjoint. $\boxed{QED}$

2.3 The non-Baer case

Now assume that the 3-elements are neither Baer nor elation.

5 Lemma. *Each* 3*-element fixes* 27 *points on a component.*

Proof. Since we are in a 3-dimensional vector space, it follows that the minimal polynomial of a 3-element σ is $(x-1)^3$. We have an 8-dimensional $GF(3)$-vector space, implying that 8 is a sum of cyclic σ-modules of dimensions $1,2,3$ and there is at least one dimension-3 submodule. Therefore, $8 = 3a+2b+c$, a at least 1 and $a+b+c$ is the dimension of the fixed-point subspace. If $a = 2$ then b is 1 or c is 2. So, the dimension of the fixed-point subspace is either 3 or 4. In the 4 case, we have an elation or a Baer collineation, since the action is on an affine plane. If the 3-element is planar then the fixed-point space has dimension 2 over $GF(3)$, since the Baer case has been considered. If $a = 1$ then we have $2b + c = 5$, implying that c is non-zero. If b is non-zero, we have dimension at least 3 of the fixed-point space and hence exactly three. So, $b = 0$, implying the fixed-point space has dimension 6, a contradiction. Hence, the only possibility is that the 3-elements fix exactly 3^3 points on some component. $\boxed{QED}$

6 Lemma. *The overlaps are* 2*-dimensional,* $SL(2,3)$ *fixes a* 2*-space over* $GF(3)$ *pointwise.*

Proof. Suppose two 3-subgroups share their unique fixed component. The fixed-point sets cannot be disjoint since there are four 3-groups in $SL(2,3)$. Suppose $SL(2,3)$ fixes 27 points on L. Then, 24 must divide $3^4 - 3^3$, just as before, a contradiction again. So, $SL(2,3)$ fixes pointwise either a 1-space or a 2-space over $GF(3)$. If a 1-space then there must be at least $(27-3)4+3$ points on L, a contradiction. If a 2-space then there must be $(27-9)4+9 = 81$, so this is possible. So, $SL(2,3)$ fixes a 2-space pointwise on L. $\boxed{QED}$

7 Lemma. $SL(2,5)$ *fixes a unique component* L*.*

Proof. So, there are either 1 or 5 components in an orbit under $SL(2,5)$ since there are 5 groups isomorphic to $SL(2,3)$. If there are 5, then this set of components contains all the fixed points for all of the 3-elements. Therefore, 3 must divide $81-5$, a contradiction. Hence, $SL(2,5)$ must fix a component L. $\boxed{QED}$

8 Lemma. *The* 3*-elements do not share fixed points.*

Proof. If so, note that the minimal degree of A_5 is 5, and A_5 is the action on the line at infinity. Therefore, since the 3-elements cannot fix another component, the orbit lengths must be strictly larger than 3 and hence are either $6, 12$ or 15, 30 or 60. Every 5-element must fix an extra component, so that there are even-order orbits. Hence, $81 = 6a + 15b$, implying $27 = 2a + 5b$. Suppose that an element of even order in $SL(2,3)$ fixes a component. Then since this element fixes 9 points on L, it follows that there is an involution which is either Baer or an affine homology. In the affine homology case, the 3-elements are forced to be planar. In the Baer case, we have A_5 induced on a subplane of order 9. Since we have considered this situation previously, we then have a contradiction.

Thus, the orbit lengths under the $SL(2,3)$'s are all 12's. Thus, since 5-element fixes at least one extra component, then the orbit lengths are either 12 or 60. Hence, $81 = 12a + 60b$, a contradiction. Thus, the 3-elements do not share fixed points. $\boxed{QED}$

9 Lemma. *There is an orbit of components of length* 10 *under* $SL(2,5)$.

Proof. There are exactly 10 Sylow 3-subgroups, each of which fixes a unique component. Suppose that $SL(2,5)$ fixes a component L. Since each 3-group fixes exactly 27 points, it follows that some pair of 3-elements share fixed points. Hence, no two Sylow 3-subgroups fix the same component. Therefore, there is an orbit of length 10 of components. $\boxed{QED}$

10 Lemma. *The* 3*-elements are Baer or elation.*

Proof. If not, we then have a set of exactly 10 components permuted transitively by $SL(2,5)$. This means that the stabilizer of one of these components L has order 12, which is the normalizer of the 3-element in $SL(2,5)$. But this 3-element fixes 26 non-zero points, so since 4 cannot divide 26, there must be an involution which is not kernel. If ρ is an affine involution with axis L, then since the 3-elements cannot be planar, they will acts semi-regularly on the coaxis M, which does not occur since the characteristic is 3. Hence, the center is Baer. So, A_5 induces on the either the Hall plane or the Desarguesian plane of order 9. If Desarguesian, then the 3-elements are elations, generating $SL(2,5)$, so the plane is Hall and the 3-elements are Baer acting on the Hall plane. However, this does not occur as if there is an elementary Abelian 2-group K_4, the elements are conjugate in A_5, so if there are affine homologies in the group, one of the involutions would have the line at infinity as its axis, but since the elements are conjugate this would force moving the line at infinity. Hence, all involutions are Baer and we may apply Ostrom's theorem forcing the dimension to be at least 4, a contradiction. $\boxed{QED}$

11 Conclusion. The 3-elements are Baer or elations.

2.4 The main theorem

Hence, we may now assume that the 3-elements are Baer and the Baer axes are disjoint.

12 Theorem. *If the Baer axes are disjoint then the translation planes of order* 81 *admitting* $SL(2,5)$ *are completely determined by the list of Johnson and Prince* [4].

The proof will be given by a series of lemmas. Assume that the 3-elements are Baer and the axes disjoint. $SL(2,5)$ acts transitively on the 10 Baer axes and $SL(2,3)$ has orbits of length 4 and 6.

13 Lemma. *Assume that e and h in $SL(2,5)$ generate $SL(2,3)$. Then we may choose the Baer axes so that e and h are represented by*

$$\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

respectively, where the elements of the matrices are the 4×4 zero and identity matrices.

Proof. If the axes are chosen to be $x = 0$, $y = 0$, we may choose the e-image of $y = 0$ to be $y = x$. Hence, e has the required form. Since $\langle e, h\rangle$ has a $(y = 0)$-orbit or $(x = 0)$-orbit of length 4, we may assume that the h image of $(x = 0)$ is $y = x$. Hence, h has the required form. $\boxed{QED}$

14 Lemma. *The union of the orbit of length 6 together with the Baer axes of $SL(2,3)$ defines a derivable net.*

Proof. Since all of the elements of order 4 are conjugate, we may select the element of order 4 to be eh. If $y = xM$ is a fixed Baer axis, it follows easily that $M^2 + M - I = 0$. Since $x^2 + x - 1$ is irreducible over $GF(3)$, it follows that $M\alpha + \beta I$ for all α, β in $GF(3)$ is a field of order 9. Hence, this set of matrices union $x = 0$ defines a derivable net. $\boxed{QED}$

15 Proposition. *Suppose a translation plane admits a collineation group isomorphic to $SL(2,5)$, where the 3-elements are Baer. Then*

(1) the set of 10 Baer axes defines a derivable net.

(2) Then we may represent the group as generated by h (above) and

$$g = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix},$$

where

$$B = \mathrm{Diag}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(3) Note that the group representation of $SL(2,5)$ contains the representation of $SL(2,3)$ given in Lemma 13 and that $y = x(C + I) = xM$.

PROOF. Since the 3-elements are elations generating $SL(2,5)$, there are 10 Baer axes. Since $SL(2,3)$ is a subgroup of $SL(2,5)$, generated by two Baer 3-elements with distinct axes, $SL(2,3)$ has an orbit of length 4 and permutes the remaining 6 Baer axes. Since $SL(2,5)$ is transitive on the set of 10 Baer axes, it follows that the stabilizer of a given Baer axis has order 12. The remaining 6 Baer axes are in two orbits of length 3 under an elation group of order 3 in $SL(2,3)$. We note that $SL(2,3)$ is transitive on the remaining set of 6. It remains to show that we may choose the representation as claimed. Since we have a derivable net, we may choose bases appropriately so that the matrices of the partial spread are diagonal matrices consisting of equal 2×2 submatrices (since the vector space is of dimension 8 over $GF(3)$, the matrices of a partial spread are 4×4 over $GF(3)$). Since

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$$

maps $y = 0$ to $y = xB$, then B, as given within the representation of g, must be a diagonal matrix

$$\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},$$

for C a 2×2 matrix over $GF(3)$, which hence is in $GL(2,3)$. Since the two elations g and h of order 3 generate $SL(2,5)$, then hh^g has order 4; the square is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix};$$

(see, e. g., Lüneburg [5] p. 161 part D). If this product is worked out, it then follows that $B^2 = -I$. Thus, $C^2 = -I$ and C is in $GL(2,3)$. There is a matrix in $GL(2,3)$, W, so that

$$W^{-1}CW = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

since all elements in $GL(2,3)$ of order 4 are conjugate. Then

$$\begin{bmatrix} W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & W \end{bmatrix}^{-1} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & W \end{bmatrix} = \begin{bmatrix} I & \operatorname{Diag} W^{-1}CW \\ 0 & I \end{bmatrix}.$$

Furthermore, the elation form:

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix},$$

is invariant under the basis change. Hence, B may be chosen as maintained. $\boxed{QED}$

Now, the group representation can be made the same as in the elation case, which was the one considered in Johnson and Prince [4]. In the computer program, all spreads invariant under the group represented as in the preceding lemma were determined by computer. This was done as follows: First it was noted in the elation case that there must be an orbit of components of length 12, Γ_{12}. In this situation, it turned out that all partial spreads compatible to the given Γ_{12} and admitting the group $SL(2,5)$ as represented have orbit length 1, 12 or 60.

We record this result in a more formal manner:

16 Remark. Let $SL(2,5)$ act on an 8-dimensional $GF(3)$-vector space as a subgroup of $GL(8,3)$, in the manner explicated above. Assume that there is a partial spread of 12 4-dimensional $GF(3)$-subspaces Γ_{12} which is an orbit under $SL(2,5)$. Then all partial spreads $\mathcal{P}$ that are invariant under $SL(2,5)$ such that $\mathcal{P} \cup \Gamma_{12}$ is a partial spread have the property that the cardinality of $\mathcal{P}$ is either 1, 12 or 60.

We now show that the 10 Baer subplanes must line up in a derivable net.

By derivation, we are then returned to the elation case and may apply the classification given there. However, there are some details that remain to be resolved. In particular, in order to apply the computer program, we need that the partial spreads that are to be considered unioned to the net of Baer axes are disjoint from the net of Baer axes. If the Baer axes do not lie in the same net of degree 10 of the plane considered, we have a potential problem, especially if there are not Γ_{12} $SL(2,5)$-orbits.

2.5 The Baer axes line up

Recall that since we have a derivable net, lying across that derivable net is a set of 10 line-sized subspaces that are fixed by $SL(2,5)$. Let τ be an element of order 5 in $SL(2,5)$. Since 5 is a 3-primitive divisor of 3^4-1, it follows by Johnson LIST, that there is an associated Desarguesian spread Σ^{τ} of τ-invariant line-sized subspaces. Furthermore, taking a set of $q-1$ reguli of Σ^{τ}, relative to some field F isomorphic to $GF(9)$, one of which is the opposite regulus of the derivable net of Baer subspaces fixed pointwise by the Sylow 3-subgroups of $SL(2,5)$, we may multiply derived to create a Desarguesian plane Σ containing the Baer axes as components and admitting $SL(2,5)$ as a collineation group, generated by elations. It follows easily that there are orbits of length $10, 12, 60$ on Σ, say $\Gamma_{10}, \Gamma_{12}, \Gamma_{60}$. In particular, each 1-dimensional F-subspace not in Γ_{10} is in an orbit of length 60 under $SL(2,5)$.

Now consider the plane π admitting $SL(2,5)$, where the 3-elements are Baer but the Baer subplanes pointwise fixed by the 3-elements may not lie in the same net of degree 10 in π; that is, they might not line up. We know that somewhere in the vector space defined by π are exactly 10 $SL(2,5)$-invariant 4-dimensional $GF(3)$, subspaces, that form the opposite regulus, abstractly, to the set of 10 Baer subplanes, considered as F-subspaces. Note that we are not assuming that π is an F-vector space, merely that we have overlaid Σ^{τ} on π as an 8-dimensional $GF(3)$-subspace.

Let ρ_o be a $SL(2,5)$-invariant 4-dimensional $GF(3)$ subspace that intersects each Baer subplane π_o in 9 points of a 2-dimensional $GF(3)$-subspace. Suppose that there is a component L of π that non-trivially intersects ρ_o. Note that since each Baer subplane π_o is, in fact, a Baer subplane of π, it follows that each Baer subplane lies on exactly 10 components. Take L to be such a component of a Baer subplane π_o. Furthermore, assume that ρ_o intersects π_o in a 2-dimensional $GF(3)$-subspace X_1. Let $L \cap \pi_o = X_2$ and without loss of generality, assume that X_1 and X_2 non-trivially intersect. Assume that X_1 and X_2 are not equal. Then there are exactly four components $L_1 = L, L_2, L_3, L_4$ that non-trivially intersect X_1. This means that these four components are four of the 10 components that nontrivially intersect π_o. Now take another subspace ρ_1 which is $SL(2,5)$ invariant that also non-trivially intersects π_o in a 2-dimensional $GF(3)$-subspace. Repeating the previous argument shows that the 10 components non-trivially intersecting π_o must be partitioned into disjoint sets of 4 components, a contradiction. Hence, there is a subspace ρ_o such that $\rho_o \cap \pi_o = L \cap \pi_o$. Now ρ_o is a 4-dimensional $GF(3)$-subspace that intersects L in at least a 2-dimensional $GF(3)$-subspace. This subspace is the intersection with ρ_o and π_o, both of which are F-subspaces. Hence, the intersection is a 1-dimensional F-subspace. Note that τ-invariant subspaces of line size are mutually disjoint.

Hence, L cannot be τ-invariant unless $L = \rho_o$. If L is not ρ_o then ρ_o intersects at least 5 components in 1-dimensional F-subspaces in five different Baer subplanes. Moreover, ρ_o is $SL(2,5)$-invariant, $SL(2,5)$ is transitive on the Baer subplanes fixed pointwise by 3-elements and acts as a collineation group of π. Thus, it follows that ρ_o lies across exactly 10 components, each of which intersects a Baer subplane in a 1-dimensional F-subspace. Hence, ρ_o becomes a Baer subplane of π. If a component M_1 of ρ_o of π intersects at least two Baer subplanes then $SL(2,3)$ fixes M_1, but then there could not be an orbit of length 10. Hence, each component M_1 has exactly 72 points that do not lie in any Baer subplane. This means that there are $SL(2,5)$-point orbits of length 60 on the $72 \cdot 10$ points. This is combinatorially possible, but it also means that there are twelve orbits of length 6 on each component M_1. For the moment, assume that the spread is in $PG(3,9)$. Then, there are 9 remaining 1-dimensional $GF(9)$-subspaces in each component L so that the 90 1-dimensional $GF(9)$-subspaces must be partitioned into sets of 60. A contradiction. Hence, we note the following:

17 Remark. Without computer analysis, if the spread is in $PG(3,9)$ and the 3-elements are Baer, then the Baer axes line up in a derivable net.

Proof. We note that all $SL(2,5)$-invariant subspaces ρ_o not are forced to be components. Hence, the opposite regulus to the regulus of Baer subplanes is a set of 10 components of π. $\boxed{QED}$

We now continue with our general analysis, without the assumption that the spread is in $PG(3,9)$.

Consider another subspace ρ_1 that is $SL(2,5)$-invariant. This subspace is disjoint from ρ_o and non-trivially intersects each Baer subplane fixed by 3-elements. So, ρ_1 cannot intersect any of the components M_i, $i = 1, 2, \ldots, 10$ lying across ρ_o.

The above argument can be repeated for any component N that non-trivially intersects a Baer subplane pointwise fixed by a 3-element. Hence, it follows that the $SL(2,5)$-invariant subspaces are either components of π or are Baer subplanes. Suppose that k of these 10 subspaces are components and so $10 - k$ are Baer subplanes that lie on components that mutually intersect exactly one Baer subplane. This accounts for

$$k + (10 - k)10$$

components. In particular, we note that when the spread is in $PG(3,9)$, there are exactly 9 remaining 1-spaces on the 10 components that must lie in orbits of length 60, a contradiction. So, the spread cannot be in $PG(3,9)$, in this scenario. Hence, we have $40 - 4 = 36 \cdot 10$ $GF(3)$-subspaces in orbits of length 60, so we have 6 orbits of 1-dimensional $GF(3)$-subspaces. Note that τ-invariant subspaces intersect multiples of 5 components, if the components lie in the ρ_o subplanes.

Suppose that there is a component L of π that does not intersect any of these $k + (10 - k)10$ subspaces. It is conceivable that L lies in an $SL(2,5)$-orbit of length 15. Thus, the stabilizer of any component in Γ_{15} would be a Sylow 2-subgroup. Suppose that two distinct Sylow 2-subgroups fix the same component of Γ_{15}. Then, $SL(2,5)$ would fix that component. But each of the 15 components of Γ_{15} are fixed by Sylow 2-subgroups. In this situation, since there are exactly 5 Sylow 2-subgroups, it follows that every Sylow 2-subgroup will fix exactly three components of the orbit Γ_{15} of length 15. The normalizer of a Sylow 2-subgroup is a Sylow 3-subgroup so this 3-group will permute the three fixed components of S_2. But, since we are outside of the fixed-point-Baer subplanes, it follows that these three components are in an orbit under a group of order 3, normalizing the Sylow 2-subgroup.

Hence, we have a group isomorphic to $SL(2,3)$ that does have a component orbit of length 3. Since the component L is external to the fixed-point spaces and since these form a derivable partial spread, we may consider the union of these two partial spreads with the group generated by elations on this union. Hence, we may choose $x = 0$, $y = 0$, $y = x$, $y = -x$ to be the four Baer subplanes corresponding as elation axes to the group $SL(2,3)$. Furthermore, we take $x = 0$, $y = 0$ as the fixed-point subspaces of elations

$$\sigma = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

and

$$\rho = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}.$$

Since the components fixed by the Sylow 2-subgroup lie outside the Baer (elation) net, we may assume that any such component has the form $y = xM$, where M is a non-singular matrix not $\pm I$, as we may also assume that $y = x, y = -x$ represent Baer subplanes external to L. It follows that the product of these two collineations fixes the images $y = xM$ under either of the groups. Hence, we may assume that (x, xM) maps to $(x, x(M + I))$ under σ and then maps to $(x(M + 2I), x(M + I))$ under ρ so that

$$(M + 2I)M = M + I.$$

This implies that

$$M^2 + M = I.$$

But also $y = x(M + I)$ is similarly fixed. Hence, we have that $(x, x(M + I))$ is mapped to $(x, x(M + 2I))$ by σ and then to $(x(M + 3I), x(M + 2I))$ by ρ. Hence,

$$M(M + I) = M + 2I,$$

implying that

$$M^2 = 2I$$

or rather that $M = I$, a contradiction.

Hence, there cannot be a 15-orbit.

Since we cannot have a 15-orbit and there are no orbits of length 1 outside the set of $k + (10 - k)10$ components, we see that every orbit of components outside must have length divisible by 3, so is at least 6. Suppose there is an orbit of length 6. Then the stabilizer of a component L has order $5 \cdot 4$, the normalizer of a Sylow 5-subgroup. Thus, we may assume that τ fixes L, so that L is a component of Σ^τ, the Desarguesian spread of τ-invariant line-sized subspaces. However, then there is a group of order $5 \cdot 4$ that acts on a Desarguesian plane, fixes a component and induces a dihedral group on the line at infinity. The involutions of such groups must invert the two components fixed by the stem of the group. Hence, the orbit lengths of components are $12, 30$ or 60. So, the 82 components are decomposed in the $k + (10 - k)10$ components lying over the Baer subplanes fixed pointwise by Baer 3-elements and components orbits of lengths $12, 30$ or 60.

$$82 = k + (10 - k)10 + 12a + 30b + 60c.$$

Hence, k is even. If $k = 10$ then the Baer subplanes line up. When $k = 2$, then $a = b = c = 0$. The only other solution is when $k = 6$ and $a = 3, b = 0, c = 0$.

If $k = 6$, in this setting, we have three orbits of length 12. But also, we have 4 orbits of length 10. This cannot occur as noted above by the compatibility condition.

Thus, the only possibility is when $k = 2$ and we have two components and 8 Baer subplanes lying across the set of 82 components. We have orbits of length 10. This means that within the structure itself, there can be no 12-orbit that does not intersect each 10-orbit. Let π_o be a τ-invariant subspace. Note that this is a 4-dimensional $GF(3)$ subspace that must non-trivially intersect each 10-orbit, for otherwise, we would have a 12-orbit $SL(2, 5)\pi_o$ which is disjoint from a 10-orbit, a contradiction to the compatibility condition.

π_o has 80 non-zero points and lies on say a components that intersect in a 1-dimensional $GF(3)$-subspace and say b components that intersect in a 2-dimensional $GF(3)$-subspace. However, since we only have 10-orbits the 1- and 2-dimensional intersections occur in multiples of 5. Hence, we have $80 = 2a + 8b$, where a and b are both divisible by 5. The only possibilities are $(a, b) = (0, 10)$ or $(20, 5)$. Notice that the total number of 5's is either 2 or 5. Hence, in order that a τ-invariant subspace intersect each of the 8 10-orbits, we need at least 8 5's, in the τ-invariant subspace, a contradiction. Hence, we are back to the compatibility condition which does not allow orbits of length 10.

Hence, $k = 10$ and the Baer subplanes all line up on a derivable net.

This completes the proof of the theorem and also completes the proof of the main result stated in the introduction.

18 Remark. If it could be shown that if there is a 12-orbit of components, the only orbits of components have lengths 1, 12 or 60, then we would have a computer-free proof that the Baer subplanes line up in a derivable net, when $SL(2,5)$ is generated by Baer 3-elements. Furthermore, when the spread is in $PG(3,9)$, we have a computer-free proof that the Baer subplanes line up in a derivable net.

References

[1] M. Biliotti, V. Jha, N.L. Johnson: Foundations of Translation Planes, Monographs and Textbooks in Pure and Applied Mathematics, vol. 243, Marcel Dekker, New York 2001.

[2] D.A. Foulser: *Some translation planes of order* 81, pp. 114–118 in Finite Geometries and Designs (Chelwood Gate, 1980), London Math. Soc. Lecture Note Ser., vol. 49, Cambridge University Press, Cambridge-New York 1981.

[3] D.A. Foulser: *Baer p-elements in translation planes*, J. Algebra, **31** (1974), 354–366.

[4] N.L. Johnson, A. Prince: *The translation planes of order* 81 *that admit* $SL(2,5)$*, I: The elation case* (submitted).

[5] H. Lüneburg: Translation Planes, Springer-Verlag, Berlin-Heidelberg-New York 1980.