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Analytic semigroups generated in L^p by elliptic operators with high order degeneracy at the boundary

Simona Fornaro
Dipartimento di Matematica "F. Casorati",
Università di Pavia,
27100, Pavia, Italy.
simona.fornaro@unipv.it

Giorgio Metafune
Dipartimento di Matematica "E. De Giorgi",
Università del Salento,
73100, Lecce, Italy.
giorgio.metafune@unisalento.it

Diego Pallara
Dipartimento di Matematica "E. De Giorgi",
Università del Salento,
73100, Lecce, Italy.
diego.pallara@unisalento.it

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Abstract. Given a regular bounded domain, we show that an elliptic operator whose diffusion coefficients degenerate at the boundary at least as the square of the distance from the boundary, endowed with a suitable domain, generates an analytic semigroup in L^p, $1 < p < \infty$. The proof relies on the discussion of the model case of the half-space and the derivation of the main result via local charts.

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Dedicated to the memory of our colleague and friend
Vincenzo Bruno Moscatelli

Introduction

Let Ω be a bounded open subset of $\mathbf{R}^{N+1}$ with C^2 boundary and let ϱ be a function in $C^2(\mathbf{R}^{N+1})$ such that

$$\varrho > 0 \text{ in } \Omega, \quad \varrho = 0 \text{ on } \partial\Omega \quad \text{and} \quad \nabla\varrho(\xi) = \hat{n}(\xi), \text{ for every } \xi \in \partial\Omega.$$

Here, $\hat{n}(\xi)$ is the inward unitary normal vector to $\partial\Omega$ at ξ. We study the operator

$$A = -\varrho^\alpha \sum_{i,j=1}^{N+1} a_{ij} D_{ij} + \varrho^{\frac{\alpha}{2}} \sum_{i=1}^{N+1} b_i D_i \tag{1}$$

with $\alpha \geq 2$, under the following conditions on the coefficients:

(H1) a_{ij} are real continuous functions on $\overline{\Omega}$, $a_{ij} = a_{ji}$, and satisfy the ellipticity condition $\sum_{i,j=1}^{N+1} a_{ij}(\xi)\zeta_i\zeta_j \geq \nu|\zeta|^2$ for every $\xi \in \overline{\Omega}$, $\zeta \in \mathbf{R}^{N+1}$ and some $\nu > 0$.

(H2) b_i are real continuous functions on $\overline{\Omega}$.

Set $M = \max_{i,j}\{\|a_{ij}\|_\infty, \|b_i\|_\infty\}$. For $1 < p < \infty$, we endow A with the domain

$$D_p(A) = \big\{u \in W^{2,p}_{\rm loc}(\Omega) \cap L^p(\Omega) : \varrho^\alpha D^2 u, \varrho^{\frac{\alpha}{2}} \nabla u \in L^p(\Omega)\big\},$$

which is a Banach space with respect to the norm $\|u\|_{D_p(A)} = \|u\|_{L^p(\Omega)} + \|\varrho^{\frac{\alpha}{2}}\nabla u\|_{L^p(\Omega)} + \|\varrho^\alpha D^2 u\|_{L^p(\Omega)}$. In [10] it has been proved, among other things, that $(A, D_p(A))$ generates an analytic semigroup in $L^p(\Omega)$, and the aim of the present paper is to give a different proof of this result. In [10] the proof goes as follows: the Author gives first a direct proof of the result in $L^2(\Omega)$ based on variational estimates, then uses Stewart's estimates to get the result in $C(\overline{\Omega})$ and deduces the L^p case by an interpolation argument for $2 < p < \infty$. Finally, a duality argument allows him to prove the result for $1 < p < 2$. Our approach is different: we prove directly, via variational estimates, the L^p case, $1 < p < \infty$ in the special case of $a_{ij} = \delta_{ij}$ and $\Omega = \{(x, y) : \ x \in \mathbf{R}^N, 0 < y < 1\}$ with Dirichlet boundary conditions for $y = 1$ and subsequently we use local charts to treat the case of a bounded domain. This is basically the same approach used in [6] and [7], where first order global (resp. *tangencial*) degeneracy of the diffusion is considered, in both cases under Dirichlet boundary conditions. Among these, that of high order degeneracy is the simplest one, because no boundary conditions have to be imposed and the generation result holds with the domain $D_p(A)$. From a probabilistic point of view, the strong degeneracy prevents the stochastic process governed by the operator L defined in (2) to reach the boundary $\mathbf{R}^N \times \{y = 0\}$ in a finite time. In terms of Feller's theory (see e.g. [5, Section IV.4.c]), the boundary is made by *inaccessible*, and more precisely *natural*, points.

1 The model problem

Let us introduce the operator

$$L = -y^\alpha \Delta + a y^{\frac{\alpha}{2}} \cdot \nabla_x + b\, y^{\frac{\alpha}{2}} D_y, \tag{2}$$

where $a \in \mathbf{R}^N$, $b \in \mathbf{R}$ and $\alpha \geq 2$. We study a suitable realization of L in the space $L^p(S)$ with $1 < p < +\infty$ and $S = \mathbf{R}^N \times (0, 1)$, with Dirichlet boundary conditions on $\mathbf{R}^N \times \{y = 1\}$. As explained in the Introduction, we do not prescribe any boundary condition for $y = 0$.

We consider the set

$$D_p = \big\{u \in L^p(S) \cap W^{2,p}_{\rm loc}(S) : y^{\frac{\alpha}{2}} \nabla u,\, y^\alpha D^2 u \in L^p(S), u(\cdot, 1) = 0\big\}$$

which is a Banach space when endowed with its canonical norm:

$$\|u\|_{D_p} := \|u\|_{L^p(S)} + \|y^{\frac{\alpha}{2}} \nabla u\|_{L^p(S)} + \|y^\alpha D^2 u\|_{L^p(S)}.$$

If $\varepsilon > 0$, we define

$$S_\varepsilon = \{(x,y) : x \in \mathbf{R}^N, \varepsilon < y < 1\}$$

and

$$\begin{aligned} D_{p,\varepsilon} &= W^{2,p}(S_\varepsilon) \cap W_0^{1,p}(S_\varepsilon), \\ \mathcal{D}_\varepsilon &= \{u \in C_c^\infty(\mathbf{R}^{N+1}) : u(\cdot,\varepsilon) = u(\cdot,1) = 0\}. \end{aligned}$$

To unify the notation, we agree that $D_{p,0} = D_p$ and we simply write S instead of S_0 and $\mathcal{D}$ instead of $\mathcal{D}_0$. Clearly, $\mathcal{D}_\varepsilon$ is dense in $D_{p,\varepsilon}$ for any $\varepsilon > 0$. A similar result also holds for D_p, as shown by the following lemma.

Lemma 1. *$\mathcal{D}$ is dense in D_p.*

Proof. We prove the statement in two steps.

First step. Let $u \in D_p$. We introduce two cut–off functions $\eta \in C^\infty(\mathbf{R}), \phi \in C^\infty(\mathbf{R}^N)$, such that $0 \le \eta, \phi \le 1$ and

$$\eta = \begin{cases} 0 & \text{in } (-\infty, 1) \\ 1 & \text{in } (2, +\infty) \end{cases} \qquad \phi = \begin{cases} 1 & \text{in } B_1(0) \\ 0 & \text{outside } B_2(0). \end{cases} \tag{3}$$

Set $u_n(x,y) = \eta_n(y)\phi_n(x)u(x,y)$, for $(x,y) \in S$, where $\eta_n(y) = \eta(ny)$, $\phi_n(x) = \phi(\frac{x}{n})$. Then $u_n \in W^{2,p}(S) \cap W_0^{1,p}(S)$ and u_n has compact support in $\mathbf{R}^N \times (0,1]$. We claim that u_n converge to u in D_p. It is easily seen that u_n, $y^{\alpha/2}\nabla_x u_n$ and $y^\alpha D_x^2 u_n$ converge to u, $y^{\alpha/2}\nabla_x u$ and $y^\alpha D_x^2 u$, respectively, in $L^p(S)$ as $n \to +\infty$. Concerning the derivatives with respect to y, we have

$$y^{\frac{\alpha}{2}} D_y u_n = n y^{\frac{\alpha}{2}} \eta'(ny)\phi_n u + \eta_n \phi_n y^{\frac{\alpha}{2}} D_y u.$$

The second addend clearly converge to $y^{\alpha/2}D_y u$ in $L^p(S)$, as $n \to +\infty$. Regarding the first one, we note that $\eta'(ny)$ equals 0 outside the interval $\left[\frac{1}{n}, \frac{2}{n}\right]$ and it tends to 0, pointwise, for every $y \neq 0$. Therefore, by the estimate $|ny^{\alpha/2}\eta'(ny)\phi_n u| \le 2^{\alpha/2} n^{(2-\alpha)/2} |\eta'(ny)u|$ and the Dominated Convergence Theorem we find that $ny^{\alpha/2}\eta'(ny)\phi_n u$ converges to 0 in $L^p(S)$. The term containing the second order y–derivative can be handled similarly.

Second step. Let $u \in D_p$ and assume that it has compact support in $\mathbf{R}^N \times (0,1]$. Then $u \in W^{2,p}(S) \cap W_0^{1,p}(S)$ and it is well–known that there exists a sequence of functions u_n in $C_c^\infty(\mathbf{R}^{N+1})$, vanishing on the boundary of S and converging to u in $W^{2,p}(S)$. In particular, u_n converge to u in D_p. QED

Some preliminary L^p-estimates for L are easy consequences of Calderón-Zygmund inequalities.

Lemma 2. *There exists $C > 0$, depending on N, p, α, such that for every $u \in D_{p,\varepsilon}$ and $0 \le \varepsilon \le \frac{1}{2}$*

$$\|y^\alpha D^2 u\|_{L^p(S_\varepsilon)} \le C\big(\|y^\alpha \Delta u\|_{L^p(S_\varepsilon)} + \|y^{\frac{\alpha}{2}} \nabla u\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)}\big). \tag{4}$$

Proof. First we consider the case $\varepsilon = 0$. Let $u \in \mathcal{D}$ and let us apply the Calderón-Zygmund estimate to the function $v = y^\alpha u$ in S

$$\|D^2 v\|_{L^p(S)} \le C(\|\Delta v\|_{L^p(S)} + \|v\|_{L^p(S)}). \tag{5}$$

By computing explicitly the second order derivatives we get

$$\begin{aligned} D_x^2 v &= y^\alpha D_x^2 u, \\ D_y^2 v &= y^\alpha D_y^2 u + 2\alpha y^{\alpha-1} D_y u + \alpha(\alpha-1) y^{\alpha-2} u, \\ D_y \nabla_x v &= y^\alpha D_y \nabla_x u + \alpha y^{\alpha-1} \nabla_x u, \end{aligned}$$

which imply the statement together with the estimate $y^{\alpha-1} \le y^{\frac{\alpha}{2}}$, holding in $[0,1]$. By density, (4) follows for every $u \in D_p$.

If $\varepsilon > 0$, we proceed analogously by considering a function $u \in \mathcal{D}_\varepsilon$. A scaling argument shows that in this case the constant C in (5) with S_ε instead of S can be chosen independent of ε, as the strips S_ε have uniformly bounded widths. QED

To get rid of the first order term on the right hand side of (4) we need an interpolative inequality.

Lemma 3. *There exist two positive constants C, η_0 such that for every $u \in D_{p,\varepsilon}$, $0 \le \varepsilon \le \frac{1}{2}$ and $0 < \eta \le \eta_0$ the following inequality holds*

$$\|y^{\frac{\alpha}{2}} \nabla u\|_{L^p(S_\varepsilon)} \le \eta \|y^\alpha D^2 u\|_{L^p(S_\varepsilon)} + \frac{C}{\eta}\|u\|_{L^p(S_\varepsilon)}. \tag{6}$$

Proof. We deal only with the case $u \in C_c^\infty(\mathbf{R}^{N+1})$, as the general case can be handled by density. Let $h \in \mathbf{R}$. Then by Taylor formula we have

$$u(x, y+h) - u(x,y) = hD_y u(x,y) + h^2 \int_0^1 (1-s) D_y^2 u(x, y+sh)\, ds.$$

Set $S_\varepsilon^1 = \{(x,y) : x \in \mathbf{R}^N, \varepsilon < y < (\varepsilon+1)/2\}$, $S_\varepsilon^2 = S_\varepsilon \setminus S_\varepsilon^1$. Choosing $h = \eta y^{\alpha/2}$ in S_ε^1 and $h = -\eta y^{\alpha/2}$ in S_ε^2, we find that if $0 < \eta \le \eta_0$, for a suitable η_0 depending on α, the points $(x, y + \eta y^{\alpha/2})$ and $(x, y - \eta y^{\alpha/2})$ belong to S_ε, whenever (x,y) belongs to S_ε^1, S_ε^2, respectively. Therefore, we obtain

$$y^{\frac{\alpha}{2}} D_y u(x,y) = \pm\frac{1}{\eta}\big(u(x, y \pm \eta y^{\frac{\alpha}{2}}) - u(x,y)\big) \mp \eta \int_0^1 (1-s) y^\alpha D_y^2 u(x, (y \pm s\eta y^{\frac{\alpha}{2}})\, ds,$$

in $S_\varepsilon^1, S_\varepsilon^2$, respectively. Integrating over S_ε the previous formula and noting that the changes of variable $[\varepsilon, (\varepsilon+1)/2]y \to y + \eta y^{\frac{\alpha}{2}}$ and $[(\varepsilon+1)/2, 1]y \to y - \eta y^{\frac{\alpha}{2}}$ produce quantities that can be bounded by constants independent of ε, we have the statement for the y–derivative. As regards the x–derivatives, we can argue as before but choosing $h = \eta y^{\alpha/2}$ in the whole of S_ε and noting that the change of variables $(x,y) \mapsto (x + \eta y^{\alpha/2} e_i, y)$ is measure-preserving and leaves $\mathbf{R}^N$ invariant. Thus the proof is complete. QED

Combining Lemma 2 and Lemma 3 it follows that

Proposition 1. *There exists $C_1 > 0$ depending on N, p, α such that for every $u \in D_{p,\varepsilon}$ and $0 \le \varepsilon \le \frac{1}{2}$*

$$\|y^\alpha D^2 u\|_{L^p(S_\varepsilon)} \le C_1\big(\|y^\alpha \Delta u\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)}\big).$$

Moreover, there exists $C_2 > 0$ depending also on b, a such that

$$\|y^{\frac{\alpha}{2}} \nabla u\|_{L^p(S_\varepsilon)} + \|y^\alpha D^2 u\|_{L^p(S_\varepsilon)} \le C_2\big(\|Lu\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)}\big). \tag{7}$$

In particular, it follows that the operator (L, D_p) is closed in $L^p(S)$.

The next proposition deals with the quasi–dissipativity of $(-L, D_p)$.

Proposition 2. *There exists $\omega > 0$, depending on p, α, b such that for every λ with $Re\,\lambda > \omega$, for every $u \in D_{p,\varepsilon}$, $0 \le \varepsilon \le \frac{1}{2}$*

$$(\mathrm{Re}\,\lambda - \omega)\|u\|_{L^p(S_\varepsilon)} \le \|\lambda u + Lu\|_{L^p(S_\varepsilon)}. \tag{8}$$

Proof. By density, we may assume that $u \in \mathcal{D}_\varepsilon$. Multiplying the equation $\lambda u + Lu = f$ by $u^* = \bar{u}|u|^{p-2}$ and integrating by parts on S_ε, all boundary terms vanish and we have

$$\begin{aligned}\int_{S_\varepsilon} fu^* &= \lambda \|u\|^p_{L^p(S_\varepsilon)} + \int_{S_\varepsilon} y^\alpha |u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\nabla u)|^2 + |\mathrm{Im}\,(\bar{u}\nabla u)|^2\Big)\\ &\quad + i(p-2)\int_{S_\varepsilon} y^\alpha |u|^{p-4}\Big(\mathrm{Re}\,(\bar{u}\nabla u)\cdot(\mathrm{Im}\,(\bar{u}\nabla u)\Big)\\ &\quad + \int_{S_\varepsilon} y^{\frac{\alpha}{2}}(a\cdot\nabla_x u)u^* + \alpha\int_{S_\varepsilon} y^{\alpha-1}(D_y u)u^* + b\int_{S_\varepsilon} y^{\frac{\alpha}{2}}(D_y u)u^*.\end{aligned} \tag{9}$$

Taking the real parts we deduce

$$\begin{aligned}\mathrm{Re}\int_{S_\varepsilon} fu^* &= (\mathrm{Re}\,\lambda)\|u\|^p_{L^p(S_\varepsilon)} + \int_{S_\varepsilon} y^\alpha |u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\nabla u)|^2 + |\,\mathrm{Im}\,(\bar{u}\nabla u)|^2\Big)\\ &\quad + \frac{\alpha}{p}\int_{S_\varepsilon} y^{\alpha-1}D_y|u|^p + \frac{b}{p}\int_{S_\varepsilon} y^{\frac{\alpha}{2}}D_y|u|^p\\ &\geq (\mathrm{Re}\,\lambda)\|u\|^p_{L^p(S_\varepsilon)} - \frac{\alpha(\alpha-1)}{p}\int_{S_\varepsilon} y^{\alpha-2}|u|^p - \frac{\alpha\, b}{2p}\int_{S_\varepsilon} y^{\frac{\alpha}{2}-1}|u|^p\end{aligned} \tag{10}$$

and consequently

$$(\mathrm{Re}\,\lambda - \omega)\|u\|_{L^p(S_\varepsilon)} \leq \|f\|_{L^p(S_\varepsilon)},$$

where $\omega = \dfrac{\alpha(\alpha-1)}{p} + \dfrac{\alpha\, b^+}{2p}$. QED

We are ready to prove the generation result.

Theorem 1. *$(-L, D_p)$ generates a quasi–contractive positive semigroup $(T_p(t))$ in $L^p(S)$. If, moreover, $1 < q < +\infty$ then $T_p(t)f = T_q(t)f$ for every $f \in L^p(S) \cap L^q(S)$.*

Proof. Let us prove that there exists $\lambda \in \mathbf{R}$ such that $(\lambda + L)D_p = L^p(S)$. Let $f \in L^p(S)$ and fix $\varepsilon \in (0, \frac{1}{2}]$. By classical results, see e.g. [8, Theorem 3.1.2], the equation $(\lambda + L)u = f$ in S_ε admits a unique solution $u_\varepsilon \in D_{p,\varepsilon}$, for any λ large enough. By (8), we can choose λ larger than ω and independent of ε and we have for every ε

$$\|u_\varepsilon\|_{L^p(S_\varepsilon)} \leq (\lambda - \omega)^{-1}\|f\|_{L^p(S)}.$$

Moreover, taking Proposition 1 into account we have

$$\|y^{\frac{\alpha}{2}}\nabla u_\varepsilon\|_{L^p(S_\varepsilon)} + \|y^\alpha D^2 u_\varepsilon\|_{L^p(S_\varepsilon)} \leq C\|f\|_{L^p(S)},$$

for every ε and for some $C > 0$ independent of ε. By weak compactness, there exists a suitable sequence $\varepsilon_n \to 0$ such that u_{ε_n} converges to u weakly in $W^{2,p}(O)$, for every open set O having compact closure in $\mathbf{R}^N \times (0, 1]$. Then $u \in D_p$ with $\|u\|_{L^p(S)} \leq (\lambda - \omega)^{-1}\|f\|_{L^p(S)}$, $\|u\|_{D_p} \leq C\|f\|_{L^p(S)}$ and $\lambda u + Lu = f$. In view of Propositions 1 and 2, we can apply the Lumer–Phillips Theorem which shows that $(-L, D_p)$ generates a quasi–contractive semigroup $(T_p(t))_{t\geq 0}$ in $L^p(S)$.

Concerning the positivity preserving property, it suffices to observe that if f is positive then u_ε is positive, by the maximum principle, and then, passing to the limit, u is positive. Moreover, u_ε, hence u, do not depend on p. Therefore the resolvent of $-L$ is positive and p-independent and the proof is complete. QED

Since $T_p(t) = T_q(t)$ in $L^p(S) \cap L^q(S)$, in the sequel we write simply $T(t)$.

Theorem 2. *The semigroup $(T(t))_{t\geq 0}$ is analytic in $L^p(S)$.*

Proof. By density, we limit ourselves to considering $u \in \mathcal{D}$. From (10) it follows that for every λ with $\mathrm{Re}\,\lambda > 2\omega$

$$(\mathrm{Re}\,\lambda-\omega)\int_S |u|^p+\int_S y^\alpha|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar u\nabla u)|^2+|\mathrm{Im}\,(\bar u\nabla u)|^2\Big) \le \|u\|^{p-1}_{L^p(S)}\|f\|^p_{L^p(S)}, \quad (11)$$

where $f = \lambda u + Lu$. Taking the imaginary parts in the identity (9) with $\varepsilon = 0$, we find

$$\begin{aligned}|\mathrm{Im}\,\lambda|\,\|u\|^p_{L^p(S)} &\le \|f\|_{L^p(S)}\|u\|^{p-1}_{L^p(S)}\\ &+|p-2|\Big(\int_S y^\alpha|u|^{p-4}|\mathrm{Re}\,(\bar u\nabla u)|^2\Big)^{\frac12}\Big(\int_S y^\alpha|u|^{p-4}|\mathrm{Im}\,(\bar u\nabla u)|^2\Big)^{\frac12} \qquad (12)\\ &+(|a|+|b|+\alpha)\|y^{\frac{\alpha}{2}}\nabla u\|_{L^p(S)}\|u\|^{p-1}_{L^p(S)}.\end{aligned}$$

We estimate the second and third addends on the right hand side as follows. Using (11) we get

$$\begin{aligned}&\Big(\int_S y^\alpha|u|^{p-4}|\mathrm{Re}\,(\bar u\nabla u)|^2\Big)^{1/2}\Big(\int_S y^\alpha|u|^{p-4}|\mathrm{Im}\,(\bar u\nabla u)|^2\Big)^{1/2}\\ &\le \frac{1}{2\sqrt{p-1}}\Big((p-1)\int_S y^\alpha|u|^{p-4}|\mathrm{Re}\,(\bar u\nabla u)|^2+\int_S y^\alpha|u|^{p-4}|\mathrm{Im}\,(\bar u\nabla u)|^2\Big)\\ &\le \frac{1}{2\sqrt{p-1}}\|f\|_{L^p(S)}\|u\|^{p-1}_{L^p(S)}.\end{aligned}$$

On the other hand, by Lemma 3, (7) and (8), for every $\eta \le \eta_0$ we have

$$\begin{aligned}\|y^{\frac{\alpha}{2}}\nabla u\|_{L^p(S)} &\le C\Big(\eta\|Lu\|_{L^p(S)}+\frac{1}{\eta}\|u\|_{L^p(S)}\Big)\\ &\le C\Big[\eta\|f\|_{L^p(S)}+\eta(\mathrm{Re}\,\lambda-\omega)\|u\|_{L^p(S)}+\eta|\mathrm{Im}\,\lambda|\|u\|_{L^p(S)}\\ &\qquad+\Big(\omega\eta+\frac{1}{\eta}\Big)\|u\|_{L^p(S)}\Big]\\ &\le C\Big[\eta\|f\|_{L^p(S)}+\eta|\mathrm{Im}\,\lambda|\|u\|_{L^p(S)}+c(\eta,\omega)\|f\|_{L^p(S)}\Big].\end{aligned}$$

Thus, (12) yields

$$\begin{aligned}|\mathrm{Im}\,\lambda|\,\|u\|_{L^p(S)} &\le \Big(1+\frac{|p-2|}{2\sqrt{p-1}}\Big)\|f\|_{L^p(S)}\\ &\qquad+C\Big[\eta\|f\|_{L^p(S)}+\eta|\mathrm{Im}\,\lambda|\|u\|_{L^p(S)}+c(\eta,\omega)\|f\|_{L^p(S)}\Big],\end{aligned}$$

for some constant $C > 0$ depending on a, b, α, p, N. Choosing η small enough it follows that

$$|\mathrm{Im}\,\lambda|\,\|u\|_{L^p(S)} \le C\|f\|_{L^p(S)},$$

for a possibly different value of C. Thus, we have proved the analyticity estimate $\|(\lambda+L)^{-1}\| \le \frac{C}{|\mathrm{Im}\,\lambda|}$, for every $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda > 2\omega$. *QED*

Our next result establishes optimal parabolic regularity for the solution of the Cauchy problem associated with the operator L via purely functional analytic tools. We recall that an analytic semigroup $(T(t))_{t\ge0}$ on a Banach space X with generator $-B$ has *maximal regularity of type* L^q $(1 < q < \infty)$ if for each $f \in L^q([0,T],X)$ the function $t \mapsto u(t) = \int_0^t T(t-s)f(s)\,ds$ belongs to $W^{1,q}([0,T],X)\cap L^q([0,T],D(B))$. This means that the mild solution of the evolution equation

$$u'(t)+Bu(t)=f(t), \quad t>0, \qquad u(0)=0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < +\infty$ and $T > 0$. If X is an L^p–space, with $1 < p < +\infty$, which is our case, then the operator $-B$ has maximal regularity of type L^q if its imaginary powers satisfy $\|B^{is}\| \le Me^{a|s|}$ for some $a \in [0, \pi/2)$ and all $s \in \mathbf{R}$ thanks to the Dore–Venni theorem, see e.g. [1, Theorem II.4.10.7].

Proposition 3 (Maximal regularity)**.** *The operator $(-L, D_p)$ has maximal regularity of type L^q on $L^p(S)$.*

PROOF. By Proposition 2 and Theorem 1 $-L - 2\omega$ generates a positive contraction semigroup on $L^p(S)$. Then $\|(L + 2\omega)^{is}\| \le M_\varepsilon \exp((\varepsilon + \pi/2)|s|)$ for each $\varepsilon > 0$ and $s \in \mathbf{R}$ because of the transference principle [4, Section 4], see [3, Theorem 5.8]. In addition, by Theorem 2 $-L - 2\omega$ is sectorial in $L^2(S)$ and then $\|(L + 2\omega)^{is}\| \le Me^{a|s|}$ for $a = \pi/2 - \phi$ and some $\phi \in (0, \pi/2]$, by a result due to McIntosh, [9]. If we combine these facts with the Riesz-Thorin interpolation theorem, we obtain the thesis. QED

Remark 1. We point out that all the results proved so far can be easily adapted to a strip of arbitrary width k by performing the standard change of variables $(x, y) \in \mathbf{R}^N \times (0, 1) \to (kx, ky) \in \mathbf{R}^N \times (0, k)$ (instead of $(x, y) \in \mathbf{R}^N \times (0, 1) \to (x, ky) \in \mathbf{R}^N \times (0, k)$). Of course the constants involved in all the estimates will also depend on k.

In order to deal with degenerate operators with variable coefficients, we need to consider, as an intermediate step, operators with constant coefficients of the form

$$\hat{L} = -y^\alpha \sum_{i,j=1}^{N+1} a_{ij} D_{ij} + y^{\frac{\alpha}{2}} a \cdot \nabla_x + y^{\frac{\alpha}{2}} b\, D_y, \qquad D_p(\hat{L}) = D_p,$$

where $a_{ij} = a_{ji} \in \mathbf{R}$, $\sum_{i,j=1}^{N+1} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2$ for all $\xi \in \mathbf{R}^{N+1}$, for some $\nu > 0$. Set $M = \max |a_{ij}|$.

Lemma 4. *There exist two positive constants $C, \hat{\omega}$ that can be determined in terms of $N, p, \alpha, a, b, M, \nu$ such that for every $\operatorname{Re} \lambda > \hat{\omega}$, the estimate $\|(\lambda + \hat{L})^{-1}\| \le C|\lambda|^{-1}$ holds.*

Moreover, for every $u \in D_p$, $\|u\|_{D_p} \le C(\|\hat{L}u\|_p + \|u\|_p)$.

PROOF. The proof is very similar to that of [6, Lemma 2.13]. For the reader's convenience, we sketch it. Let Q be a non-singular matrix such that $\sum_{i,j=1}^{N+1} a_{ij} D_{ij} u(z) = \Delta v(Qz)$ whenever $u(z) = v(Qz)$, $z = (x, y)$, and $\mathbf{R}_+^{N+1}$ is invariant under Q. It follows that $Q^* e_{N+1} = ke_{N+1}$ with $k^2 a_{N+1N+1} = 1$. Then the equation $\lambda u(z) + \hat{L}u(z) = f(z)$ with $z \in S$, is equivalent to

$$\lambda k^\alpha v(\zeta) - \eta^\alpha \Delta v(\zeta) + \eta^{\frac{\alpha}{2}} a_1 \cdot \nabla_\xi v(\zeta) + \eta^{\frac{\alpha}{2}} b_1 D_\eta v(\zeta) = k^\alpha f(Q^{-1}\zeta),$$

where $\zeta = (\xi, \eta)$, $\xi \in \mathbf{R}^N$, $\eta \in (0, k)$ and a_1, b_1 are suitable constant coefficients in $\mathbf{R}^N$ and $\mathbf{R}$, respectively. By Theorem 2, Proposition 1 and Remark 1 we end the proof. QED

As in [6, Corollary 2.14] we easily have the following estimate.

Corollary 1. *There exists a constant $C = C(N, p, \alpha, M, \nu, \eta_0)$, η_0 being given in Lemma 3, such that for all $u \in D_p$ and all $\lambda \in \boldsymbol{C}$ with $\operatorname{Re} \lambda > \hat{\omega}$ and $|\lambda| \ge 1/\eta_0^2$*

$$\|y^{\frac{\alpha}{2}} \nabla u\|_p \le C|\lambda|^{-1/2} \|(\lambda + \hat{L})u\|_p.$$

2 General bounded domains

The present section is devoted to the study of degenerate operators with variable coefficients in bounded domains, as presented in the Introduction, to which we refer for the notation and the hypotheses. We also briefly recall the setting introduced in [6] and refer to [6, Section 3] for all the details concerning the geometric construction.

The main result of the section is stated in the next theorem.

Theorem 3. *Under assumptions (H1) and (H2) the operator* $(-A, D_p(A))$ *generates an analytic semigroup in* $L^p(\Omega)$. *In particular, there exists* $\omega_p > 0$, *such that*

$$\sup_{\mathrm{Re}\lambda \geq \omega_p} \|\lambda(\lambda + A)^{-1}\| < +\infty.$$

Our approach to prove Theorem 3 is based on the classical argument of straightening the boundary via local charts. Before starting the proof we need to settle precisely our geometric framework.

Let $\xi_0 \in \partial\Omega$ be fixed and without loss of generality, assume that at the point ξ_0 the ξ_{N+1} coordinate axis lies in the direction of $\hat{n}(\xi_0)$. By definition of a C^2 boundary, there exist an open neighborhood U of ξ_0 and a C^2-diffeormorphism J from U onto $\widetilde{U} = J(U)$ satisfying

$$J(U \cap \Omega) = \widetilde{U} \cap \mathbf{R}^{N+1}_+,$$
$$J(U \cap \partial\Omega) = \widetilde{U} \cap \partial\mathbf{R}^{N+1}_+.$$

By compactness of $\partial\Omega$, all the derivatives of J and $H := J^{-1}$ up to the second order may be assumed to be bounded by a certain constant L independent of ξ_0. Moreover, the coordinate transformation J is *admissible* at ξ_0, which means that the tangent space $T_{\partial\Omega,\xi_0}$ and the normal direction $\hat{n}(\xi_0)$ at ξ_0 are mapped into the tangent space $T_{\partial\mathbf{R}^{N+1}_+,z_0}$ and the normal direction at $z_0 = J(\xi_0) = (x_0, 0)$, respectively.

Define $\phi(z) = \varrho(Hz)$, for $z \in \widetilde{U} \cap \mathbf{R}^{N+1}_+$. By using Taylor formula one can find

$$\phi(z) = yh(z), \tag{13}$$

where h is a continuous function which is bounded from above and below by positive constants, still independent of ξ_0, and $h(z_0) = 1$.

Finally, given a function $u : U \cap \Omega \to \mathbf{R}$, we introduce the change of variables $u \to \mathcal{T}u = u \circ H$ on $\widetilde{U} \cap \mathbf{R}^{N+1}_+$. $\mathcal{T}$ is an isomorphism from $L^p(U \cap \Omega)$ onto $L^p(\widetilde{U} \cap \mathbf{R}^{N+1}_+)$. Moreover, $\varrho^{\frac{\alpha}{2}} \nabla u$ (resp. $\varrho^\alpha D^2 u$) belongs to $L^p(U \cap \Omega)$ if and only if $y^{\frac{\alpha}{2}} \nabla \mathcal{T}u$ (resp. $y^\alpha D^2 \mathcal{T}u$) belongs to $L^p(\widetilde{U} \cap \mathbf{R}^{N+1}_+)$, with equivalence of the norms through constants independent of ξ_0.

The differential operator A is locally transformed into the operator $\mathcal{A}$ given by

$$\mathcal{A} = -\phi^\alpha(z) \sum_{h,k=1}^{N+1} \alpha_{hk}(z) D_{z_h z_k} - \phi^\alpha(z) \sum_{k=1}^{N+1} \beta_k(z) D_{z_k} + \phi^{\frac{\alpha}{2}}(z) \sum_{k=1}^{N+1} \gamma_k(z) D_{z_k} \tag{14}$$

with

$$\begin{aligned} \alpha_{hk}(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz) D_{\xi_j} J_h(Hz) D_{\xi_i} J_k(Hz), \\ \beta_k(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz) D_{\xi_j \xi_i} J_k(Hz), \\ \gamma_k(z) &= \sum_{i=1}^{N+1} b_i(Hz) D_{\xi_i} J_k(Hz). \end{aligned} \tag{15}$$

In order to deal with the class of operators studied in the previous section, we freeze the coefficients of $\mathcal{A}$ at the point z_0, recalling (13), as follows

$$\mathcal{A}^0 = -y^\alpha \sum_{h,k=1}^{N+1} \alpha_{hk}(z_0) D_{z_h z_k} + y^{\frac{\alpha}{2}} \sum_{k=1}^{N+1} \gamma_k(z_0) D_{z_k}. \tag{16}$$

Note that the coefficients $\alpha_{hk}(z_0)$ preserve the ellipticity condition with a constant independent of ξ_0.

For the sequel we need the following interpolative estimate, which is given without proof, as it is very similar to that of [6, Lemma 3.3].

Lemma 5. *There exist $\varepsilon_0, C > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and every $u \in D_p(A)$ one has*

$$\|\varrho^{\frac{\alpha}{2}} \nabla u\|_{L^p(\Omega)} \le \varepsilon \|u\|_{D_p(A)} + \frac{C}{\varepsilon} \|u\|_{L^p(\Omega)}. \tag{17}$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. For every $\xi_0 \in \partial\Omega$, let U_{ξ_0} be the open neighborhood of ξ_0 and J_{ξ_0} the corresponding coordinate transformation described at the beginning of the section. Given $\varepsilon > 0$, choose a ball $B_{r(\xi_0)}(\xi_0) \subset U_{\xi_0}$ such that if $z \in J_{\xi_0}(B_{r(\xi_0)}(\xi_0)) \cap \mathbf{R}^{N+1}_+$, then for every $h, k = 1, \dots, N+1$

$$\begin{aligned} |h^\alpha(z)\alpha_{hk}(z) - \alpha_{hk}(z_0)| &< \varepsilon, \\ |\phi^\alpha(z)\beta_k(z)| &< \varepsilon, \\ |h^{\alpha/2}\gamma_k(z) - \gamma_k(z_0)| &< \varepsilon, \end{aligned} \tag{18}$$

where $z_0 = J_{\xi_0}(\xi_0)$, $\alpha_{hk}, \beta_k, \gamma_k$ are given in (15) and h, ϕ in (13). Set $\mathcal{F}_\varepsilon = \{B_{r(\xi)}(\xi) : \xi \in \partial\Omega\}$. By means of a suitable covering argument (see e.g. [2, Theorem 2.18]), recalling that $\partial\Omega$ is compact, we can extract a finite subcovering $\mathcal{F}'_\varepsilon = \{B_{r(\xi_i)}(\xi_i) : i = 1, \dots, m\}$ such that at most c_N among the balls of $\mathcal{F}'_\varepsilon$ overlap. Here c_N is a natural number which depends only on the dimension. Set $U_i = B_{r(\xi_i)}(\xi_i)$, $J_i = J_{\xi_i | B_{r(\xi_i)}(\xi_i)}$ and $\widetilde{U}_i = J_i(U_i)$, $z_i = J_i(\xi_i)$. Let κ be a fixed positive number such that $J_i(U_i \cap \Omega) \subset \mathbf{R}^N \times (0, \kappa)$ for every $i \in \{1, \dots, m\}$ and for every ε sufficiently small. To simplify the notation we assume that $\kappa = 1$.

Finally, let $U_0 \subset\subset \Omega$ be an open set with boundary of class C^2 such that $\{U_0, U_1, \dots, U_m\}$ is a covering of $\overline{\Omega}$.

To prove the statement it suffices to show that $(-A, D_p(A))$ is a sectorial operator in $L^p(\Omega)$. We split the proof in two steps.

Step 1. We first deal with the surjectivity of the operator $\lambda + A : D_p(A) \to L^p(\Omega)$. To be definite, we show that there exist $\omega'_p, C > 0$ such that for every $\operatorname{Re}\lambda \ge \omega'_p$ and $f \in L^p(\Omega)$ there is $u \in D_p(A)$ satisfying $\lambda u + Au = f$ and $|\lambda|\, \|u\|_{L^p(\Omega)} \le C\, \|f\|_{L^p(\Omega)}$.

Consider the open covering $\{U_0, U_1, \dots, U_m\}$ of $\overline{\Omega}$, as above, with ε to be determined. Let $H_i = J_i^{-1}$ and $\mathcal{T}_i : L^p(U_i) \to L^p(\widetilde{U}_i)$, $\mathcal{T}_i\varphi = \varphi \circ H_i$. Set $\Omega_i = U_i \cap \Omega$. Let $\{\eta_i^2\}_{i=0}^m$ be a partition of unity subordinate to such a covering, with $0 \le \eta_i \le 1$. To simplify the notation, in the constants appearing in the estimates below we make only the dependence on U_i explicit, whereas we omit the dependence on other quantities.

Let $f \in L^p(\Omega)$ be fixed. Since the operator A is nondegenerate in U_0, it is well-known that if $\operatorname{Re}\lambda \ge \lambda_0$, for a suitable $\lambda_0 \in \mathbf{R}$, then there exists a unique solution $u_0 \in W^{2,p}(U_0) \cap W^{1,p}_0(U_0)$ of the equation $\lambda u_0 + Au_0 = \eta_0 f$. Set $R_0(\lambda)f = \eta_0 u_0$. Then $R_0(\lambda)f \in D_p(A)$ and

$$(\lambda + A)R_0(\lambda)f = \eta_0^2 f + [A, \eta_0]u_0 = \eta_0^2 f + E_0 f,$$

where the symbol $[\cdot\,,\cdot]$ denotes the commutator of two operators. It is easily seen that

$$\|E_0 f\|_{L^p(\Omega)} \le \frac{C_0}{|\lambda|^{1/2}} \|f\|_{L^p(U_0)}, \tag{19}$$

where the constant C_0 depends on U_0.

Now, fix $i \geq 1$. Denote by $\mathcal{A}_i$, $\mathcal{A}_i^0$ the operators obtained from $\mathcal{A}$, $\mathcal{A}^0$, defined in (14), (16), replacing J, H, z_0 with J_i, H_i, z_i, respectively. From Section 1 for every $\mathrm{Re}\,\lambda > \hat{\omega}$, for some $\hat{\omega}$, there exists a unique solution $v_i \in D_p$ of $\lambda v_i + \mathcal{A}_i^0 v_i = \mathcal{T}_i(\eta_i f)$ in S. Let $R_i(\lambda)f$ be the trivial extension to Ω of the function $\mathcal{T}_i^{-1}\big(\mathcal{T}_i(\eta_i)v_i\big)$. Then $R_i(\lambda)f \in D_p(A)$ and it has compact support contained in U_i. Since $A = \mathcal{T}_i^{-1}\mathcal{A}_i\mathcal{T}_i$ in $L^p(\Omega_i)$, we easily get

$$\begin{aligned}(\lambda + A)R_i(\lambda)f &= \mathcal{T}_i^{-1}(\lambda + \mathcal{A}_i)\big(\mathcal{T}_i(\eta_i)v_i\big)\\ &= \eta_i^2 f + B_i f + E_i f,\end{aligned}$$

where we have set

$$\begin{aligned}B_i f &= \eta_i \mathcal{T}_i^{-1}\big((\mathcal{A}_i - \mathcal{A}_i^0)v_i\big)\\ E_i f &= \mathcal{T}_i^{-1}\Big(\big[\mathcal{A}_i, \mathcal{T}_i\eta_i\big]v_i\Big).\end{aligned}$$

Now, we are going to estimate the L^p-norms of $B_i f$ and $E_i f$. Concerning $B_i f$, we observe that

$$\|B_i f\|_{L^p(\Omega)} \leq C\,\|(\mathcal{A}_i - \mathcal{A}_i^0)v_i\|_{L^p(\widetilde{U}_i \cap \mathbf{R}_+^{N+1})} \leq C\varepsilon\|v_i\|_{D_p},$$

where, in the last step, we have used the fact that U_i has been constructed in such a way that (18) is satisfied. Applying Lemma 4 to the operator $\mathcal{A}_i^0$, it turns out that

$$\begin{aligned}\|v_i\|_{D_p} &\leq C\Big(\|\mathcal{A}_i^0 v_i\|_{L^p(S)} + \|v_i\|_{L^p(S)}\Big)\\ &\leq C\Big(\|\mathcal{T}_i(\eta_i f)\|_{L^p(S)} + (|\lambda|+1)|\lambda|^{-1}\|\mathcal{T}_i(\eta_i f)\|_{L^p(S)}\Big) \leq C\|f\|_{L^p(\Omega_i)}.\end{aligned}$$

Thus, we have established that

$$\|B_i f\|_{L^p(\Omega)} \leq C\varepsilon\|f\|_{L^p(\Omega_i)}. \tag{20}$$

Concerning the norm of $E_i f$, we have

$$\|E_i f\|_{L^p(\Omega)} \leq C\,\big\|\big[\mathcal{A}_i, \mathcal{T}_i\eta_i\big]v_i\big\|_{L^p(\widetilde{U}_i \cap \mathbf{R}_+^{N+1})} \leq C_i\Big(\|y^{\frac{\alpha}{2}}\nabla v_i\|_{L^p(S)} + \|v_i\|_{L^p(S)}\Big).$$

If $|\lambda| \geq 1/\eta_0^2$, then from Corollary 1 and Lemma 4 applied to $\mathcal{A}_i^0$, it follows that

$$\|E_i f\|_{L^p(\Omega)} \leq \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}. \tag{21}$$

Setting $R(\lambda)f = \sum_{i=0}^m R_i(\lambda)f$ and $S(\lambda)f = \sum_{i=1}^m \big(B_i f + E_i f\big) + E_0 f$ we find that

$$(\lambda + A)R(\lambda)f = f + S(\lambda)f. \tag{22}$$

Estimates (19), (20) and (21) imply that

$$\|S(\lambda)f\|_{L^p(\Omega)} \leq \sum_{i=1}^m C\varepsilon\|f\|_{L^p(\Omega_i)} + \sum_{i=0}^m \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}.$$

Since at most c_N among the U_i's overlap, we get

$$\|S(\lambda)f\|_{L^p(\Omega)} \leq c_N C\varepsilon\|f\|_{L^p(\Omega)} + \sum_{i=0}^m \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}.$$

Now, it is clear that we can choose $\varepsilon > 0$ sufficiently small and λ large enough to get $\|S(\lambda)\| \leq 1/2$. This shows that there exists $\omega_p' \geq \max\{\lambda_0, \hat{\omega}, 1/\eta_0^2\} > 0$ such that for every $\mathrm{Re}\,\lambda \geq \omega_p'$,

$I + S(\lambda) : L^p(\Omega) \to L^p(\Omega)$ is invertible and, denoted by $V(\lambda)$ its inverse, $\|V(\lambda)\| \le 2$. By (22), with $V(\lambda)f$ instead of f, we infer that $u = R(\lambda)V(\lambda)f$ is a function in $D_p(A)$ and solves the equation $(\lambda + A)u = f$. Moreover,

$$\|u\|_{L^p(\Omega)} \le \sum_{i=0}^{m} \|R_i(\lambda)V(\lambda)f\|_{L^p(\Omega)} \le \frac{C}{|\lambda|}\|V(\lambda)f\|_{L^p(\Omega)} \le \frac{2C}{|\lambda|}\|f\|_{L^p(\Omega)}.$$

Hence, the first step is done.

Step 2. Now, we study the injectivity of $\lambda + A$. According to the notation introduced in the first step, if $u \in D_p(A)$ and $\operatorname{Re}\lambda > \max\{\hat{\omega}, \lambda_0\}$, we can write

$$\begin{aligned} R_i(\lambda)(\lambda + A)u &= \eta_i^2 u + F_i u + G_i u, \quad i \ge 1, \\ R_0(\lambda)(\lambda + A)u &= \eta_0^2 u + Hu \end{aligned}$$

where

$$\begin{aligned} F_i u &= \eta_i \mathcal{T}_i^{-1}\Big((\lambda + \mathcal{A}_i^0)^{-1}(\mathcal{A}_i - \mathcal{A}_i^0)\mathcal{T}_i(\eta_i u)\Big) \\ G_i u &= \eta_i \mathcal{T}_i^{-1}\Big((\lambda + \mathcal{A}_i^0)^{-1}\mathcal{T}_i\big([\eta_i, A]u\big)\Big), \end{aligned}$$

and, if A_0 denotes the realization of A in $L^p(U_0)$ with Dirichlet boundary conditions

$$Hu = \eta_0(\lambda + A_0)^{-1}([A, \eta_0]u).$$

Summing over i, it turns out that

$$\sum_{i=0}^{m} R_i(\lambda)(\lambda + A)u = u + \sum_{i=1}^{m}(F_i u + G_i u) + Hu,$$

for every $u \in D_p(A)$. Let $u \in D_p(A)$ be such that $(\lambda + A)u = 0$. Then, the expression above implies that

$$u = -\sum_{i=1}^{m}\big(F_i u + G_i u\big) - Hu \tag{23}$$

We claim that $u = 0$. To prove this, we need to estimate the norms of u in $D_p(A)$ and in $L^p(\Omega)$. It is useful to set

$$\begin{aligned} \|\cdot\|_{p,i} &= \|\cdot\|_{L^p(\Omega_i)}, \\ \|\cdot\|_{D_{p,i}} &= \|\cdot\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla(\cdot)\|_{p,i} + \|\varrho^{\alpha}D^2(\cdot)\|_{p,i}. \end{aligned}$$

The easiest term to be estimated is Hu, since it involves a nondegenerate operator. To this aim, we observe that, as Hu is supported in U_0, its norm in $D_p(A)$ is equivalent to the $W^{2,p}$-norm, therefore the classical L^p estimates yield

$$\|Hu\|_{D_p(A)} \le C_0\|[A, \eta_0]u\|_{p,0}.$$

Since $[A, \eta_0]$ is a first-order operator, for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|Hu\|_{D_p(A)} \le C_0\delta\|u\|_{D_{p,0}} + C_\delta\|u\|_{p,0}. \tag{24}$$

On the other hand

$$\|Hu\|_{L^p(\Omega)} \le \frac{C_0}{|\lambda|}\|u\|_{D_{p,0}}. \tag{25}$$

Here, C_0 denotes a suitable constant depending on η_0. Now, we estimate F_iu and G_iu, for every $i \geq 1$. To keep the notation simpler, we set

$$f_i = (\mathcal{A}_i - \mathcal{A}_i^0)\mathcal{T}_i(\eta_i u), \quad g_i = \mathcal{T}_i[\eta_i, A]u$$

and we define

$$\varphi_i = \mathcal{T}_i^{-1}(\lambda + \mathcal{A}_i^0)^{-1} f_i, \quad \psi_i = \mathcal{T}_i^{-1}(\lambda + \mathcal{A}_i^0)^{-1} g_i.$$

As a consequence, we can write $F_iu = \eta_i\varphi_i$ and $G_iu = \eta_i\psi_i$. It is easily seen that

$$\|F_iu\|_{D_p(A)} \leq \|\varphi_i\|_{D_{p,i}} + C_i(\|\varphi_i\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla\varphi_i\|_{p,i}). \tag{26}$$

We estimate separately each term of the right hand side as follows.

$$\|\varphi_i\|_{D_{p,i}} \leq C\|(\lambda + \mathcal{A}_i^0)^{-1} f_i\|_{D_p} \leq C\|f_i\|_p \leq C\varepsilon\|\eta_i u\|_{D_{p,i}}$$

and

$$\|\varphi_i\|_{p,i} \leq \frac{C}{|\lambda|}\|f_i\|_p \leq \frac{C}{|\lambda|}\varepsilon\|\eta_i u\|_{D_{p,i}}.$$

Moreover, thanks to Corollary 1, if $|\lambda| \geq 1/\eta_0^2$ then

$$\|\varrho^{\frac{\alpha}{2}}\nabla\varphi_i\|_{p,i} \leq C\|y^{\frac{\alpha}{2}}\nabla(\lambda + \mathcal{A}_i^0)^{-1} f_i\|_p \leq \frac{C}{|\lambda|^{1/2}}\|f_i\|_p \leq \frac{C}{|\lambda|^{1/2}}\varepsilon\|\eta_i u\|_{D_{p,i}}. \tag{27}$$

As

$$\|\eta_i u\|_{D_{p,i}} \leq \|u\|_{D_{p,i}} + C_i(\|u\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla u\|_{p,i}),$$

we finally obtain

$$\begin{aligned}\|F_iu\|_{D_p(A)} &\leq \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|\eta_i u\|_{D_{p,i}} \\ &\leq \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|u\|_{D_{p,i}} + C_i(\|u\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla u\|_{p,i}). \end{aligned} \tag{28}$$

For our purposes, we need to estimate the L^p norm of F_iu independently. This is much easier; indeed, we immediately have

$$\|F_iu\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|}\|f_i\|_p \leq \frac{C_i}{|\lambda|}\|u\|_{D_{p,i}}. \tag{29}$$

Next, we consider the term G_iu. Replacing φ_i, f_i with ψ_i, g_i, respectively, in (26)–(27) and observing that

$$\|g_i\|_p \leq C_i(\|u\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla u\|_{p,i}),$$

we infer

$$\|G_iu\|_{D_p(A)} \leq C_i(\|u\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla u\|_{p,i}), \tag{30}$$

and

$$\|G_iu\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|}\|g_i\|_p \leq \frac{C_i}{|\lambda|}\|u\|_{D_{p,i}}. \tag{31}$$

Now, by (23), (24), (28) and (30) we derive

$$\begin{aligned}\|u\|_{D_p(A)} \quad \leq \quad & \sum_{i=1}^m \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|u\|_{D_{p,i}} + \sum_{i=1}^m C_i(\|u\|_{p,i} + \|\varrho^{\frac{\alpha}{2}}\nabla u\|_{p,i}) \\ & + C_0\delta\|u\|_{D_{p,0}} + C_\delta\|u\|_{p,0}.\end{aligned}$$

At this point, arguing as in the end of the first step, choose ε, δ sufficiently small and λ sufficiently large to obtain

$$\|u\|_{D_p(A)} \le C(\|u\|_{L^p(\Omega)} + \|\varrho^{\frac{\alpha}{2}} \nabla u\|_{L^p(\Omega)}).$$

Using the interpolative estimate (17) we get

$$\|u\|_{D_p(A)} \le C\|u\|_{L^p(\Omega)}.$$

Moreover, from (23), (25), (29) and (31) it follows that

$$\|u\|_{L^p(\Omega)} \le \frac{C}{|\lambda|}\|u\|_{D_p(A)}.$$

Combining the last two estimates we obtain

$$\|u\|_{D_p(A)} \le \frac{C}{|\lambda|}\|u\|_{D_p(A)},$$

which leads to a contradiction, for λ large, unless $u = 0$. Therefore, there exists $\omega_p'' > 0$ such that $\lambda + A : D_p(A) \to L^p(\Omega)$ is injective for every $\operatorname{Re}\lambda \ge \omega_p''$. Hence, the second step is complete.

Now, we are immediately led to the conclusion. Indeed, from Steps 1,2 it follows that $\lambda + A$ is bijective from $D_p(A)$ onto $L^p(\Omega)$, for every $\operatorname{Re}\lambda \ge \omega_p = \max\{\omega_p', \omega_p''\}$ and, in addition, $\sup_{\operatorname{Re}\lambda \ge \omega_p} \|\lambda(\lambda + A)^{-1}\| < +\infty$. QED

Remark 2. The inclusion $D_q(A) \subset D_p(A)$ holding when $1 < p < q < +\infty$ implies that the resolvents of $(-A, D_p(A))$ and $(-A, D_q(A))$ are consistent. It follows that also the semigroups $(T_p(t))_{t\ge 0}$ and $(T_p(t))_{t\ge 0}$ are consistent.

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