Note di Matematica **22,** n. 2, 2003, 179–208.

Selection principles in mathematics: A milestone of open problems

Boaz Tsaban[i]
Einstein Institute of Mathematics,
Hebrew University of Jerusalem,
Givat Ram, Jerusalem 91904, Israel
tsaban@math.huji.ac.il, http://www.cs.biu.ac.il/~tsaban

Received: 13/01/2003; accepted: 16/12/2003.

Abstract. We survey some of the major open problems involving selection principles, diagonalizations, and covering properties in topology and infinite combinatorics. Background details, definitions and motivations are also provided.

Keywords: Gerlits-Nagy γ property, Galvin-Miller strong γ property, Menger property, Hurewicz property, Rothberger property, Gerlits-Nagy $(*)$ property, Arkhangel'skiĭ property, Sakai property, Selection principles, Infinite game theory, Ramsey Theory.

MSC 2000 classification: primary: 37F20; secondary: 26A03, 03E75.

A mathematical discipline is alive and well if it has many exciting open problems of different levels of difficulty.

Vitali Bergelson [7]

1 Introduction

The general field in which the problems surveyed below arise is traditionally called *Selection Principles in Mathematics (SPM).*[1] It is at least as old as Cantor's works on the diagonalization argument. However, we concentrate on the study of diagonalizations of covers of topological spaces (and their relations to infinite combinatorics, Ramsey theory, infinite game theory, and function spaces) since these are the parts of this quickly-growing field with which we are more familiar. Example for an important area which is not covered here is that of topological groups. For problems on these and many more problems in the

[i]Partially supported by the Golda Meir Fund and the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

[1]Some other popular names are: Topological diagonalizations, infinite-combinatorial topology.

areas we do consider, the reader is referred to the papers cited in Scheepers' survey paper [48][2].

Many mathematicians have worked in the past on specific instances of these "topological diagonalizations", but it was only in 1996 that Marion Scheepers' paper [37] established a unified framework to study all of these sorts of diagonalizations. This pioneering work was soon followed by a stream of papers, which seems to get stronger with time. A new field was born.

As always in mathematics, this systematic approach made it possible to generalize and understand to a much deeper extent existing results, which were classically proved using ad-hoc methods and ingenious arguments which were re-invented for each specific question.

However, the flourishing of this field did not solve all problems. In fact, some of the most fundamental questions remained open. Moreover, since Scheepers' pioneering work, several new notions of covers where introduced into the framework, and some new connections with other fields of mathematics were discovered, which helped in solving some of the problems but introduced many new ones.

In the sequel we try to introduce a substantial portion of those problems which lie at the core of the field. All problems presented here are interesting enough to justify publication when solved. However, we do not promise that all solutions will be difficult – it could well be that we have overlooked a simple solution (there are too many problems for us to be able to give each of them the time it deserves). Please inform us of any solution[3] you find or any problem which you find important and which was not included here. It is our hope that we will be able to publish a complementary survey of these in the future.

Much of the material presented here is borrowed (without further notice) from the $\mathcal{SPM}$ **Bulletin**, a semi-monthly electronic bulletin dedicated to the field [52, 53, 54, 55, 56, 57]. Announcements of solutions and other problems sent to the author will be published in this bulletin: We urge the reader to subscribe to the $\mathcal{SPM}$ **Bulletin**[4] in order to remain up-to-date in this quickly evolving field of mathematics.

We thank the organizing committee of the *Lecce Workshop on Coverings, Selections and Games in Topology* (June 2002) for inviting this survey paper.

[2]Other topics which are very new, such as star selection principles and uniform selection principles, are not covered here and the reader is referred to the other papers in *this* issue of **Note di Matematica** for some information on these.

[3]See the new section NOTES ADDED IN PROOF at the end of the paper for new results obtained after the writing of the paper.

[4]E-mail us to get subscribed, free of charge.

1.1 A note to the reader

The definitions always appear before the *first problem* requiring them, and are not repeated later.

2 Basic notation and conventions

2.1 Selection principles

The following notation is due to Scheepers [37]. This notation can be used to denote many properties which were considered in the classical literature under various names (see Figure 1 below), and using it makes the analysis of the relationships between these properties very convenient.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be collections of covers of a space X. Following are selection hypotheses which X might satisfy or not satisfy.[5]

$\mathsf{S}_1(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for each n $U_n\in\mathcal{U}_n$, and $\{U_n\}_{n\in\mathbb{N}}\in\mathfrak{V}$.

$\mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that each $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\mathfrak{V}$.

$\mathsf{U}_{fin}(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$ *which do not contain a finite subcover*, there exists a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that for each n $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\{\cup\mathcal{F}_n\}_{n\in\mathbb{N}}\in\mathfrak{V}$.

Clearly, $\mathsf{S}_1(\mathfrak{U},\mathfrak{V})$ implies $\mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V})$, and for the types of covers that we consider here, $\mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V})$ implies $\mathsf{U}_{fin}(\mathfrak{U},\mathfrak{V})$.

2.2 Stronger subcovers

The following prototype of many classical properties is called "$\mathfrak{U}$ *choose* $\mathfrak{V}$" in [59].

$\binom{\mathfrak{U}}{\mathfrak{V}}$: For each $\mathcal{U}\in\mathfrak{U}$ there exists $\mathcal{V}\subseteq\mathcal{U}$ such that $\mathcal{V}\in\mathfrak{V}$.

Then $\mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V})$ implies $\binom{\mathfrak{U}}{\mathfrak{V}}$.

[5] Often these hypotheses are identified with the class of all spaces satisfying them.

2.3 The spaces considered

Many of the quoted results apply in the case that the spaces X in question are Tychonoff, perfectly normal, or Lindelöf in all powers. However, unless otherwise indicated, we consider spaces X which are (homeomorphic to) sets of reals. This is the case, e.g., for any separable zero-dimensional metrizable space.

This significantly narrows our scope, but since we are interested in finding *good problems* rather than proving general results, this may be viewed as a tool to filter out problems arising from topologically-pathological examples. However, most of the problems make sense in the general case and solutions in the general setting are usually also of interest.

Part 1: Classical types of covers

3 Thick covers

In this paper, by *cover* we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \cup\mathcal{U}$ and $X \notin \mathcal{U}$. $\mathcal{U}$ is:

(1) A *large cover* of X if each $x \in X$ is contained in infinitely many members of $\mathcal{U}$,

(2) An *ω-cover* of X if each finite subset of X is contained in some member of $\mathcal{U}$; and

(3) A *γ-cover* of X if $\mathcal{U}$ is infinite, and each $x \in X$ belongs to all but finitely many members of $\mathcal{U}$.

The large covers and the ω-covers are quite old. The term "γ-covers" was coined in a relatively new paper [22] but this type of covers appears at least as early as in [23].

Let $\mathcal{O}$, Λ, Ω, and Γ denote the collections of open covers, open large covers, ω-covers, and γ-covers of X, respectively. If we assume that X is a set of reals (or a separable, zero-dimensional metric space), then we may assume that all covers in these collections are countable [22, 58]. Similarly, let $\mathcal{B}, \mathcal{B}_\Lambda, \mathcal{B}_\Omega, \mathcal{B}_\Gamma$ be the corresponding *countable Borel* covers of X. Often the properties obtained by applying S_1, S_{fin}, or U_{fin} to a pair of the above families of covers are called *classical selection principles*.

4 Classification

The following discussion is based on [24, 47]. Recall that for the types of covers which we consider,

$$\mathsf{S}_1(\mathfrak{U},\mathfrak{V}) \to \mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V}) \to \mathsf{U}_{fin}(\mathfrak{U},\mathfrak{V}) \text{ and } \binom{\mathfrak{U}}{\mathfrak{V}}$$

and $\binom{\Lambda}{\Omega}$ does not hold for a nontrivial space X [24, 62]. This rules out several of the introduced properties as trivial. Each of our properties is monotone decreasing in the first coordinate and increasing in the second. In the case of U_{fin} note that for any class of covers $\mathfrak{V}$, $\mathsf{U}_{fin}(\mathcal{O},\mathfrak{V})$ is equivalent to $\mathsf{U}_{fin}(\Gamma,\mathfrak{V})$ because given an open cover $\{U_n\}_{n\in\mathbb{N}}$ we may replace it by $\{\bigcup_{i<n} U_i\}_{n\in\mathbb{N}}$, which is a γ-cover (unless it contains a finite subcover).

In the three-dimensional diagram of Figure 1, the double lines indicate that the two properties are equivalent. The proof of these equivalences can be found in [37, 24].

The analogous equivalences for the Borel case also hold, but in the Borel case more equivalences hold [47]: For each $\mathfrak{V} \in \{\mathcal{B}, \mathcal{B}_\Omega, \mathcal{B}_\Gamma\}$,

$$\mathsf{S}_1(\mathcal{B}_\Gamma,\mathfrak{V}) \Leftrightarrow \mathsf{S}_{fin}(\mathcal{B}_\Gamma,\mathfrak{V}) \Leftrightarrow \mathsf{U}_{fin}(\mathcal{B}_\Gamma,\mathfrak{V}).$$

After removing duplications we obtain Figure 2.

All implications which do not appear in Figure 2 where refuted by counter-examples (which are in fact sets of real numbers) in [37, 24, 47]. The only unsettled implications in this diagram are marked with dotted arrows.

1 Problem ([24, Problems 1 and 2]). (1) Is $\mathsf{U}_{fin}(\Gamma,\Omega)$ equivalent to $\mathsf{S}_{fin}(\Gamma,\Omega)$?

(2) And if not, does $\mathsf{U}_{fin}(\Gamma,\Gamma)$ imply $\mathsf{S}_{fin}(\Gamma,\Omega)$?

Bartoszyński (personal communication) suspects that an implication should be easy to prove if it is true, and otherwise it may be quite difficult to find a counter-example (existing methods do not tell these properties apart). However, the Hurewicz property $\mathsf{U}_{fin}(\Gamma,\Gamma)$ has some surprising disguises which a priori do not look equivalent to it [27, 63], so no definite conjecture can be made about this problem.

5 Classification in ZFC

Most of the examples used to prove non-implications in Figure 2 are ones using (fragments of) the Continuum Hypothesis. However, some non-implications

can be proved without any extra hypotheses. For example, every σ-compact space satisfies $\mathsf{U}_{fin}(\Gamma,\Gamma)$ and $\mathsf{S}_{fin}(\Omega,\Omega)$ (and all properties implied by these), but the Cantor set does not satisfy $\mathsf{S}_1(\Gamma,\mathcal{O})$ (and all properties implying it) [24].

It is not known if additional non-implications are provable without the help of additional axioms. We mention one problem which drew more attention then the others.

2 Problem ([24, Problem 3], [14, Problem 1], [12, Problem 1]). Does there exist (in ZFC) a set of reals X which has the Menger property $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ but not the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$?

The papers [24, 41, 6] (see also Section 6 below) deal with constructions in

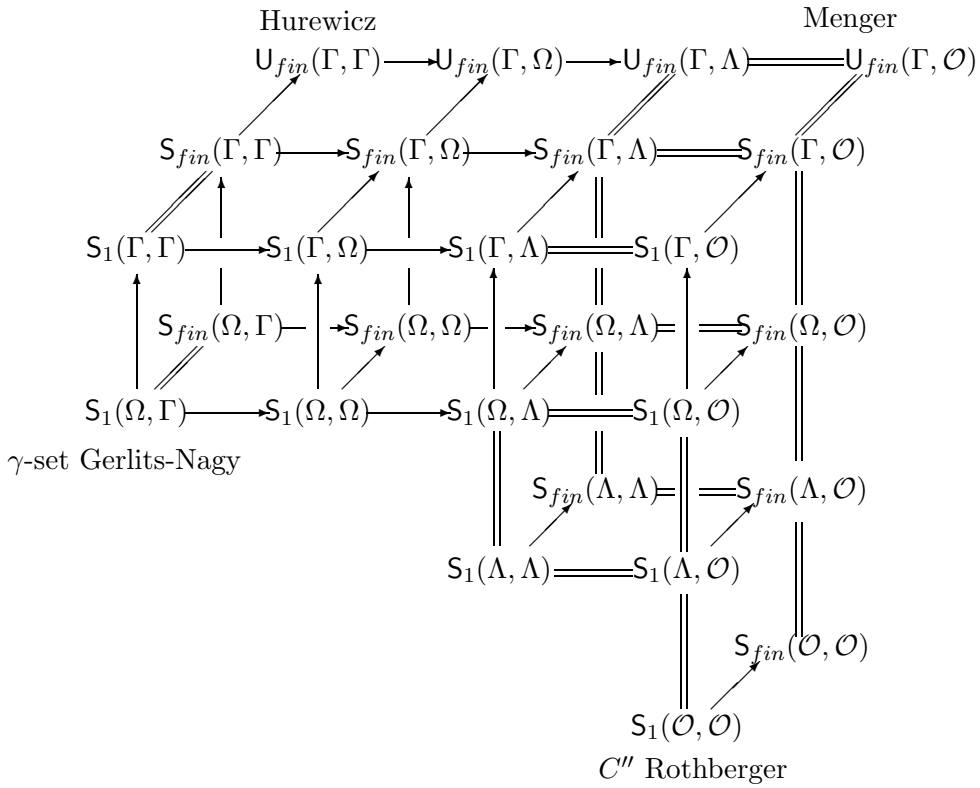


Figure 1.

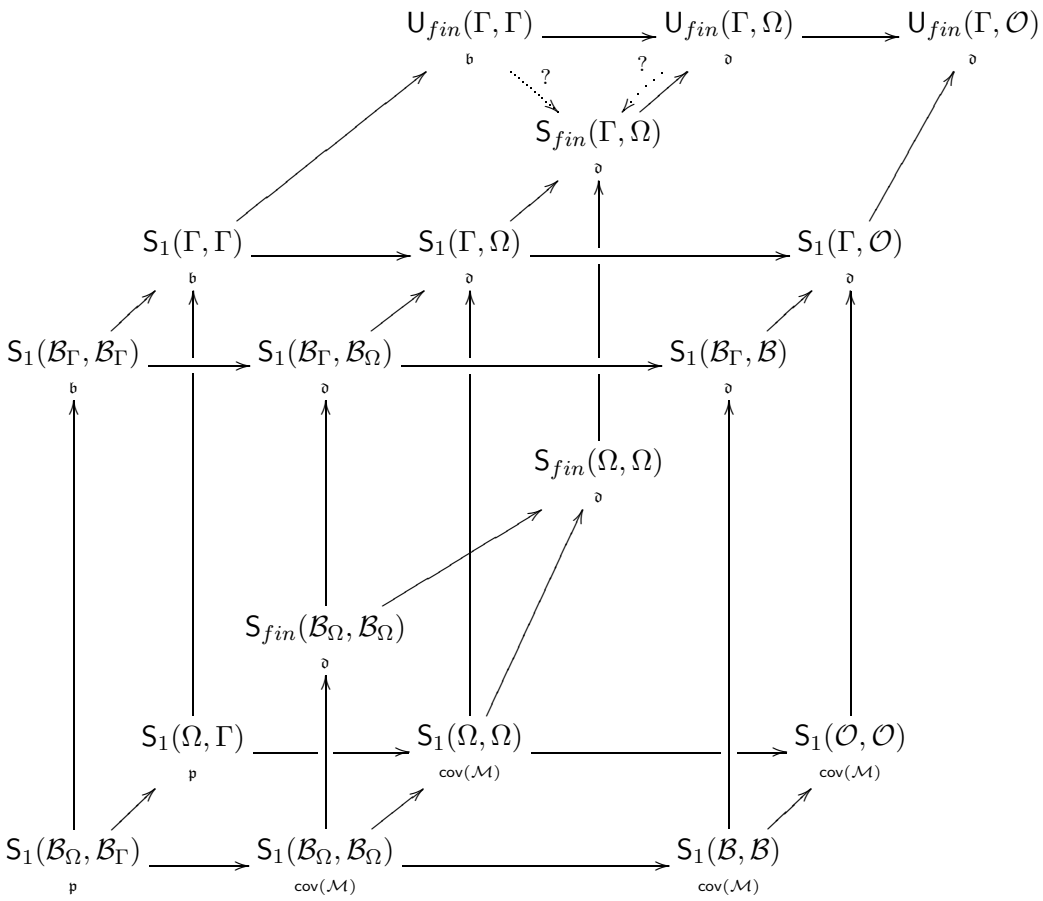

$\mathsf{U}_{fin}(\Gamma,\Gamma)$
$\mathfrak{b}$
$\mathsf{U}_{fin}(\Gamma,\Omega)$
$\mathfrak{d}$
$\mathsf{U}_{fin}(\Gamma,\mathcal{O})$
$\mathfrak{d}$
?
?
$\mathsf{S}_{fin}(\Gamma,\Omega)$
$\mathfrak{d}$
$\mathsf{S}_1(\Gamma,\Gamma)$
$\mathfrak{b}$
$\mathsf{S}_1(\Gamma,\Omega)$
$\mathfrak{d}$
$\mathsf{S}_1(\Gamma,\mathcal{O})$
$\mathfrak{d}$
$\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$
$\mathfrak{b}$
$\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Omega)$
$\mathfrak{d}$
$\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B})$
$\mathfrak{d}$
$\mathsf{S}_{fin}(\Omega,\Omega)$
$\mathfrak{d}$
$\mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$
$\mathfrak{d}$
$\mathsf{S}_1(\Omega,\Gamma)$
$\mathfrak{p}$
$\mathsf{S}_1(\Omega,\Omega)$
$\mathsf{cov}(\mathcal{M})$
$\mathsf{S}_1(\mathcal{O},\mathcal{O})$
$\mathsf{cov}(\mathcal{M})$
$\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma)$
$\mathfrak{p}$
$\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$
$\mathsf{cov}(\mathcal{M})$
$\mathsf{S}_1(\mathcal{B},\mathcal{B})$
$\mathsf{cov}(\mathcal{M})$

Figure 2.

ZFC of sets of reals with the Hurewicz property, and may be relevant to the problem.

Not much is known about the situation when arbitrary topological spaces are considered rather than sets of reals.

1 Project ([24, Problem 3]). *Find, without extra hypotheses, (general) topological spaces that demonstrate non-implications among the classical properties. Do the same for* Lindelöf *topological spaces.*

6 Uncountable elements in ZFC

We already mentioned the fact that the Cantor set satisfies all properties of type S_{fin} or U_{fin} in the case of open covers, but none of the remaining ones. It turns out that some S_1-type properties can be shown to be satisfied by uncountable elements without any special hypotheses.

This is intimately related to the following notions. The *Baire space* ${}^{\mathbb{N}}\mathbb{N}$ is equipped with the product topology and (quasi)ordered by eventual dominance: $f \leq^* g$ if $f(n) \leq^* g(n)$ for all but finitely many n. A subset of ${}^{\mathbb{N}}\mathbb{N}$ is *dominating* if it is cofinal in ${}^{\mathbb{N}}\mathbb{N}$ with respect to $\leq^*$. If a subset of ${}^{\mathbb{N}}\mathbb{N}$ is unbounded with respect to $\leq^*$ then we simply say that it is *unbounded*. Let $\mathfrak{b}$ (respectively, $\mathfrak{d}$) denote the minimal cardinality of an unbounded (respectively, dominating) subset of ${}^{\mathbb{N}}\mathbb{N}$.

The *critical cardinality* of a nontrivial family $\mathcal{J}$ of sets of reals is $\mathsf{non}(\mathcal{J}) = \min\{|X| : X \subseteq \mathbb{R} \text{ and } X \notin \mathcal{J}\}$. Then $\mathfrak{b}$ is the critical cardinality of $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, $\mathsf{S}_1(\Gamma, \Gamma)$, and $\mathsf{U}_{fin}(\Gamma, \Gamma)$, and $\mathfrak{d}$ is the critical cardinality of the classes in Figure 2 which contain $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ [24, 47].

6.1 The open case

In [24, 41] it was shown (in ZFC) that there exists a set of reals of size $\aleph_1$ which satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ as well as $\mathsf{S}_{fin}(\Omega, \Omega)$. In [45] this is improved to show that there always exists a set of size $\mathfrak{t}$ which satisfies these properties. In both cases the proof uses a dichotomy argument (two different examples are given in two possible extensions of ZFC).

In [6] the following absolute ZFC examples are given. Let $\mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$. (A subset $A \subseteq \mathbb{N} \cup \{\infty\}$ is open if: $A \subseteq \mathbb{N}$, or $\infty \in A$ and A is cofinite.) Let $\mathcal{Z} \subseteq {}^{\mathbb{N}}(\mathbb{N} \cup \{\infty\})$ consist of the functions f such that

(1) For all n, $f(n) \leq f(n+1)$; and

(2) For all n, if $f(n) < \infty$, then $f(n) < f(n+1)$.

($\mathcal{Z}$ is homeomorphic to the Cantor set of reals.) For each increasing finite sequence s of natural numbers, let $q_s \in \mathcal{Z}$ be defined as

$$q_s(k) = \begin{cases} s(k) & \text{if } k < |s| \\ \infty & \text{otherwise} \end{cases}$$

for each $k \in \mathbb{N}$. Note that the set

$$Q = \{q_s : s \text{ an increasing finite sequence in } \mathbb{N}\}$$

is dense in $\mathcal{Z}$.

Let $B = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ be a $\leq^*$-unbounded set of strictly increasing elements of ${}^{\mathbb{N}}\mathbb{N}$ which forms a $\mathfrak{b}$-*scale* (that is, for each $\alpha < \beta$, $f_\alpha \leq^* f_\beta$), and set $H = B \cup Q$. In [6] it is proved that all finite powers of H satisfy $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$.

3 Problem ([6, Problem 17]). Does H satisfy $\mathsf{S}_1(\Gamma, \Gamma)$?

6.2 The Borel case

Borel's Conjecture, which was proved to be consistent by Laver, implies that each set of reals satisfying $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (and the classes below it) is countable. From our point of view this means that there do not exist ZFC examples of sets satisfying $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. A set of reals X is a *σ-set* if each G_δ set in X is also an F_σ set in X. In [47] it is proved that every element of $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is a σ-set. According to a result of Miller [29], it is consistent that every σ-set of real numbers is countable. Thus, there do not exist uncountable ZFC examples satisfying $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. The situation for the other classes, though addressed by top experts, remains open. In particular, we have the following.

4 Problem ([33], [47, Problem 45], [6]). Does there exist (in ZFC) an uncountable set of reals satisfying $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$?

By [47], this is the same as asking whether it is consistent that each uncountable set of reals can be mapped onto a dominating subset of ${}^{\mathbb{N}}\mathbb{N}$ by a Borel function. This is one of the major open problems in the field.

7 Special elements under weak hypotheses

Most of the counter examples used to distinguish between the properties in the Borel case are constructed with the aid of the Continuum Hypothesis. The question whether such examples exist under weaker hypotheses (like Martin's Axiom) is often raised (e.g., [10, 31]). We mention some known results by indicating (by full bullets) all places in the diagram of the Borel case (the front

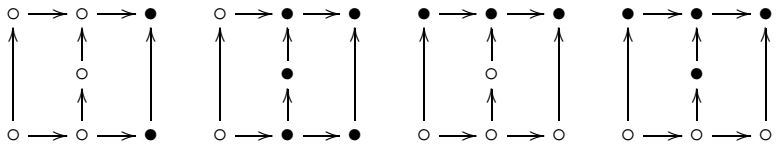
plane in Figure 2) which the example satisfies. All hypotheses we mention are weaker than Martin's Axiom.

Let us recall the basic terminology. $\mathcal{M}$ and $\mathcal{N}$ denote the collections of meager (=first category) and null (=Lebesgue measure zero) sets of reals, respectively. For a family $\mathcal{I}$ of sets of reals define:

$$\begin{aligned}
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \cup\mathcal{F} \notin \mathcal{I}\}\\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \cup\mathcal{F} = \mathbb{R}\}\\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } (\forall I \in \mathcal{I})(\exists J \in \mathcal{F})\ I \subseteq J\}
\end{aligned}$$

A set of reals X is a *κ-Luzin* set if $|X| \geq \kappa$ and for each $M \in \mathcal{M}$, $|X \cap M| < \kappa$. Dually, X is a *κ-Sierpiński* set if $|X| \geq \kappa$ and for each $N \in \mathcal{N}$, $|X \cap N| < \kappa$.

If $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$ then there exists a $\mathsf{cov}(\mathcal{M})$-Luzin set satisfying the properties indicated in Figure 3(a) [47] (in [34] this is proved under Martin's Axiom). Under the slightly stronger assumption $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, there exists a $\mathsf{cov}(\mathcal{M})$-Luzin set as in Figure 3(b) [5].[6] Dually, assuming $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$ there exists a $\mathfrak{b}$-Sierpiński set as in Figure 3(c) [47], and another one as in Figure 3(d) [64].[7]



(a) (b) (c) (d)

Figure 3. Some known configurations under Martin's Axiom

2 Project. *Find constructions, under Martin's Axiom or weaker hypotheses, for any of the consistent configurations not covered in Figure 3.*

8 Preservation of properties

8.1 Hereditariness

A property is (provably) *hereditary* if for each space X satisfying the property, all subsets of X satisfy that property. Most of the properties considered

[6] In fact, we can require that this Luzin set does not satisfy $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$ [59] – see Section 9 for the definition of $\mathcal{B}_\mathrm{T}$.

[7] The last Sierpiński set actually satisfies $\mathsf{S}_{fin}(\mathcal{B}_\Omega^{gp}, \mathcal{B}_\Omega)$ – see Section 10 for the definition of $\mathcal{B}_\Omega^{gp}$.

in this paper may be considered intuitively as notions of smallness, thus it is somewhat surprising that none of the properties involving open covers is hereditary [6]. However, the property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$ as well as all properties of the form $\Pi(\mathcal{B}_\Gamma,\mathcal{B})$ are hereditary [6] (but $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma)$ is not [31]).

5 Problem ([6, Problem 4], [31, Question 6]). Is any of the properties $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$ or $\mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$ hereditary?

This problem is related to Problem 7 below.

8.2 Finite powers

$\mathsf{S}_1(\Omega,\Gamma)$, $\mathsf{S}_1(\Omega,\Omega)$, and $\mathsf{S}_{fin}(\Omega,\Omega)$ are the only properties in the open case which are preserved under taking finite powers [24]. The only candidates in the Borel case to be preserved under taking finite powers are the following.

6 Problem ([47, Problem 50]). Is any of the classes

$$\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma),\quad \mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega),\quad \mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$$

closed under taking finite powers?

In [47] it is shown that if all finite powers of X satisfy $\mathsf{S}_1(\mathcal{B},\mathcal{B})$ (respectively, $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B})$), then X satisfies $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$ (respectively, $\mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$). Consequently, the last two cases of Problem 6 translate to the following.

7 Problem ([47, Problems 19 and 21]). (1) Is it true that if X satisfies $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$, then all finite powers of X satisfy $\mathsf{S}_1(\mathcal{B},\mathcal{B})$?

(2) Is it true that if X satisfies $\mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$, then all finite powers of X satisfy $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B})$?

The analogous assertion in the open case is true [35, 24]. Observe that a positive answer to this problem implies a positive answer to Problem 5 above.

It is worthwhile to mention that by a sequence of results in [42, 47, 5, 58], none of the properties in Figure 2 is preserved under taking finite products.

8.3 Unions

The question of which of the properties in Figure 2 is provably preserved under taking finite or countable unions (that is, finitely or countably additive) is completely settled in [24, 41, 42, 5]. Also, among the classes which are not provably additive, it is known that some are *consistently* additive [5]. Only the following problems remain open in this category.

8 Problem ([5, (full version) Problem 4.12]). Is any of the classes

$$\mathsf{S}_{fin}(\Omega,\Omega),\quad \mathsf{S}_1(\Gamma,\Omega),\quad \mathsf{S}_{fin}(\Gamma,\Omega)$$

consistently closed under taking finite unions?

9 Problem ([5, (full version) Problem 4.13]). Is $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ consistently closed under taking finite unions?

Another sort of problems which remain open is that of determining the *exact* additivity numbers of the (provably) additive properties. The general problem is to determine the additivity numbers of the properties in question in terms of well known cardinal characteristics of the continuum (like $\mathfrak{b}$, $\mathfrak{d}$, etc.). See [48] for a list of properties for which the problem is still open. Three of the more interesting ones among these are the following.

10 Problem ([5, (full version) Problem 2.4]). Is $\mathsf{add}(\mathsf{U}_{fin}(\Gamma, \mathcal{O})) = \mathfrak{b}$?

It is only known that $\mathfrak{b} \leq \mathsf{add}(\mathsf{U}_{fin}(\Gamma, \mathcal{O})) \leq \mathsf{cf}(\mathfrak{d})$, and that the additivity of the corresponding combinatorial notion of smallness is equal to $\mathfrak{b}$ [5, full version].

11 Problem. Is $\mathsf{add}(\mathsf{S}_1(\Gamma, \Gamma)) = \mathfrak{b}$?

It is known that $\mathfrak{h} \leq \mathsf{add}(\mathsf{S}_1(\Gamma, \Gamma)) \leq \mathfrak{b}$ [41]. This problem is related to Problem 45 below.

In [3] it is proved that $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathsf{S}_1(\mathcal{O}, \mathcal{O}))$.

12 Problem ([3, Problem 4]). Is it consistent that

$$\mathsf{add}(\mathcal{N}) < \mathsf{add}(\mathsf{S}_1(\mathcal{O}, \mathcal{O}))?$$

Part 2: Modern types of covers

In this part we divide the problems according to the involved type of covers rather than according to the type of problem.

9 τ-covers

$\mathcal{U}$ is a *τ-cover* of X [51] if it is a large cover of X,[8] and for each $x, y \in X$, either $\{U \in \mathcal{U} : x \in U, y \notin U\}$ is finite, or else $\{U \in \mathcal{U} : y \in U, x \notin U\}$ is finite. If all powers of X are Lindelöf (e.g, if X is a set of reals) then each τ-cover of X contains a countable τ-cover of X [58]. Let T denote the collection of open τ-covers of X. Then

$$\Gamma \subseteq \mathrm{T} \subseteq \Omega \subseteq \mathcal{O}.$$

The following problem arises in almost every study of τ-covers [51, 59, 58, 62, 61, 31]. By [22], $\mathsf{S}_1(\Omega, \Gamma) \Leftrightarrow \binom{\Omega}{\Gamma}$. As $\Gamma \subseteq \mathrm{T}$, this property implies $\binom{\Omega}{\mathrm{T}}$. Thus far, all examples of sets not satisfying $\binom{\Omega}{\Gamma}$ turned out not to satisfy $\binom{\Omega}{\mathrm{T}}$.

[8] Recall that by "cover of X" we mean one not containing X as an element.

13 Problem ([52, §4]). Is $\binom{\Omega}{\Gamma}$ equivalent to $\binom{\Omega}{\mathrm{T}}$?

A positive answer would imply that the properties $\mathsf{S}_1(\Omega,\Gamma)$, $\mathsf{S}_1(\Omega,\mathrm{T})$, and $\mathsf{S}_{fin}(\Omega,\mathrm{T})$ are all equivalent, and therefore simplify the study of τ-covers considerably. It would also imply a positive solution to Problems 15, 18(1), 25, and other problems.[9] The best known result in this direction is that $\binom{\Omega}{\mathrm{T}}$ implies $\mathsf{S}_{fin}(\Gamma,\mathrm{T})$ [59]. A modest form of Problem 13 is the following. If $\binom{\Omega}{\mathrm{T}}$ implies $\mathsf{S}_{fin}(\mathrm{T},\Omega)$, then $\binom{\Omega}{\mathrm{T}} \Leftrightarrow \mathsf{S}_{fin}(\Omega,\mathrm{T})$.

14 Problem ([59, Problem 2.9]). Is $\binom{\Omega}{\mathrm{T}}$ equivalent to $\mathsf{S}_{fin}(\Omega,\mathrm{T})$?

The notion of τ-covers introduces seven new pairs—namely, $(\mathrm{T},\mathcal{O})$, (T,Ω), (T,T), (T,Γ), $(\mathcal{O},\mathrm{T})$, (Ω,T), and (Γ,T)—to which any of the selection operators S_1, S_{fin}, and U_{fin} can be applied. This makes a total of 21 new selection hypotheses. Fortunately, some of them are easily eliminated. The surviving properties appear in Figure 4.

Below each property in Figure 4 appears a "serial number" (to be used in Table 1), and its critical cardinality. The cardinal numbers $\mathfrak{p}$ and $\mathfrak{t}$ are the well-known pseudo-intersection number and tower number (see, e.g., [15] or [9] for definitions and details). $\mathfrak{r}$ is the excluded-middle number, and is equal to $\max\{\mathfrak{s},\mathfrak{b}\}$, where $\mathfrak{s}$ is the splitting number [49].

As indicated in the diagram, some of the critical cardinalities are not yet known.

3 Project ([59, Problem 6.6]). *What are the unknown critical cardinalities in Figure 4?*

Recall that there are only two unsettled implications in the corresponding diagram for the *classical* types of open covers (Section 4). As there are many more properties when τ-covers are incorporated into the framework, and since this investigation is new, there remain *many* unsettled implications in Figure 4. To be precise, there are exactly 76 unsettled implications in this diagram. These appear as question marks in the *Implications Table* 1. Entry (i,j) in the table (ith row, jth column) is to be interpreted as follows: It is 1 if property i implies property j, 0 if property i does not imply property j (that is, consistently there exists a counter-example), and ? if the implication is unsettled.

4 Project ([59, Problem 6.5]). *Settle any of the unsettled implications in Table 1.*

Marion Scheepers asked us which single solution would imply as many other solutions as possible. The answer found by a computer program is the following: If entry $(12,5)$ is 1 (that is, $\mathsf{S}_{fin}(\Gamma,\mathrm{T})$ implies $\mathsf{S}_1(\mathrm{T},\mathrm{T})$), then there remain only

[9] This looks too good to be true, but a negative answer should also imply (through a bit finer analysis) a solution to several open problems.

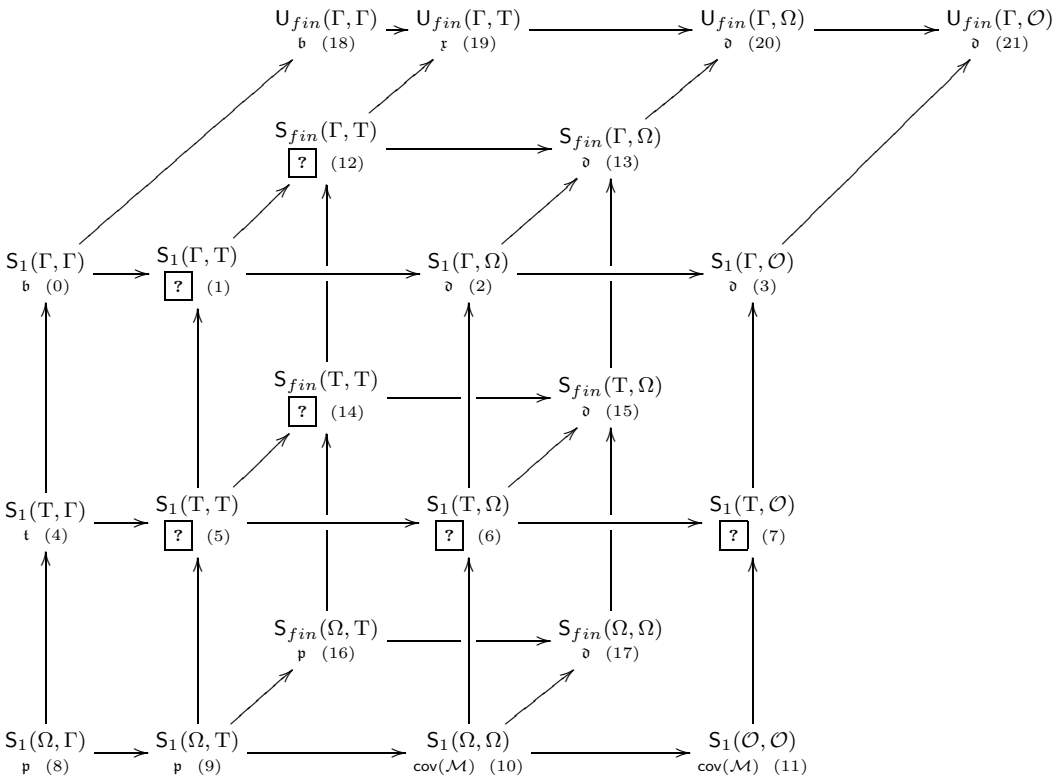


Figure 4. The diagram involving τ-covers

33 (!) open problems. The best possible negative entry is $(16, 3)$: If $\mathsf{S}_{fin}(\Omega, \mathrm{T})$ does not imply $\mathsf{S}_1(\Gamma, \mathcal{O})$, then only 47 implications remain unsettled.

Finally, observe that any solution in Problem 3 would imply several new nonimplications.

Scheepers chose the following problem out of all the problems discussed above as the most interesting.

15 Problem. Does $\mathsf{S}_1(\Omega, \mathrm{T})$ imply the Hurewicz property $\mathsf{U}_{fin}(\Gamma, \Gamma)$?

The reason for this choice is that if the answer is positive, then $\mathsf{S}_1(\Omega, \mathrm{T})$ implies the Gerlitz-Nagy $(*)$ property [22], which is equivalent to another modern selection property (see Section 10 below).

Not much is known about the preservation of the new properties under set theoretic operations. Miller [31] proved that assuming the Continuum Hypothesis, there exists a set X satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ and a subset Y of X such that Y does not satisfy $\binom{\Omega}{\mathrm{T}}$. Together with the remarks preceding Problem 5, we

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	1	1	1	1	0	?	?	?	0	0	0	0	1	1	?	?	0	0	1	1	1	1
1	?	1	1	1	0	?	?	?	0	0	0	0	1	1	?	?	0	0	?	1	1	1
2	0	0	1	1	0	0	?	?	0	0	0	0	0	1	0	?	0	0	0	0	1	1
3	0	0	0	1	0	0	0	?	0	0	0	0	0	0	0	0	0	0	0	0	0	1
4	1	1	1	1	1	1	1	1	0	0	?	?	1	1	1	1	0	?	1	1	1	1
5	?	1	1	1	?	1	1	1	0	0	?	?	1	1	1	1	0	?	?	1	1	1
6	0	0	1	1	0	0	1	1	0	0	?	?	0	1	0	1	0	?	0	0	1	1
7	0	0	0	1	0	0	0	1	0	0	0	?	0	0	0	0	0	0	0	0	0	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	?	1	1	1	?	1	1	1	?	1	1	1	1	1	1	1	1	1	?	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	1	0	1	0	1	0	0	1	1
11	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	?	?	?	?	0	?	?	?	0	0	0	0	1	1	?	?	0	0	?	1	1	1
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	?	0	0	0	0	1	1
14	?	?	?	?	?	?	?	?	0	0	?	?	1	1	1	1	0	?	?	1	1	1
15	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	?	0	0	1	1
16	?	?	?	?	?	?	?	?	?	?	?	?	1	1	1	1	1	1	?	1	1	1
17	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	1	1
18	0	0	0	0	0	0	0	0	0	0	0	0	0	?	0	?	0	0	1	1	1	1
19	0	0	0	0	0	0	0	0	0	0	0	0	0	?	0	?	0	0	0	1	1	1
20	0	0	0	0	0	0	0	0	0	0	0	0	0	?	0	?	0	0	0	0	1	1
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 1. Implications and nonimplications

have that the only classes (in addition to those in Problem 5) for which the hereditariness problem is not settled are the following ones.

16 Problem ([6, Problem 4]). Is any of the properties

$$\mathsf{S}_1(\mathcal{B}_\mathrm{T},\mathcal{B}_\Gamma),\quad \mathsf{S}_1(\mathcal{B}_\mathrm{T},\mathcal{B}_\mathrm{T}),\quad \mathsf{S}_1(\mathcal{B}_\mathrm{T},\mathcal{B}_\Omega),$$
$$\mathsf{S}_1(\mathcal{B}_\mathrm{T},\mathcal{B}),\quad \mathsf{S}_{fin}(\mathcal{B}_\mathrm{T},\mathcal{B}_\mathrm{T}),\quad \mathsf{S}_{fin}(\mathcal{B}_\mathrm{T},\mathcal{B}_\Omega),$$

hereditary?

Here are the open problems regarding unions.

17 Problem ([5, (full version) Problems 5.2 and 5.3]). Is any of the properties $\mathsf{S}_1(\mathrm{T},\mathrm{T})$, $\mathsf{S}_{fin}(\mathrm{T},\mathrm{T})$, $\mathsf{S}_1(\Gamma,\mathrm{T})$, $\mathsf{S}_{fin}(\Gamma,\mathrm{T})$, and $\mathsf{U}_{fin}(\Gamma,\mathrm{T})$ (or any of their Borel versions) preserved under taking finite unions?

And here are the open problems regarding powers.

18 Problem. Is any of the properties

(1) $\mathsf{S}_1(\Omega,\mathrm{T})$, or $\mathsf{S}_{fin}(\Omega,\mathrm{T})$,

(2) $\mathsf{S}_1(\mathrm{T},\Gamma)$, $\mathsf{S}_1(\mathrm{T},\mathrm{T})$, $\mathsf{S}_1(\mathrm{T},\Omega)$, $\mathsf{S}_{fin}(\mathrm{T},\mathrm{T})$, or $\mathsf{S}_{fin}(\mathrm{T},\Omega)$,

preserved under taking finite powers?

The answer to (1) is positive if it is for Problem 13.

10 Groupable covers

Groupability notions for covers appear naturally in the studies of selection principles [26, 27, 2, 63] and have various notations. Scheepers has standardized

the notations in [48]. We use Scheepers' notation, but take a very minor variant of his definitions which allows more simple definitions and does not make a difference in any of the theorems proved in the literature (since we always consider covers which do not contain finite subcovers).

Let ξ be γ, τ, or ω. A cover $\mathcal{U}$ of X is *ξ-groupable* if there is a partition of $\mathcal{U}$ into finite sets, $\mathcal{U}=\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$, such that $\{\cup\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a ξ-cover of X. Denote the collection of ξ-groupable open covers of X by $\mathcal{O}^{\xi\text{-}gp}$. Then $\mathcal{O}^{\gamma\text{-}gp}\subseteq\mathcal{O}^{\tau\text{-}gp}\subseteq\mathcal{O}^{\omega\text{-}gp}$. Observe that we must require in the definitions that the elements $\mathcal{F}_n$ are disjoint, as otherwise any cover of X would be γ-groupable.

Recall that $\mathsf{U}_{fin}(\Gamma,\mathcal{O})\Leftrightarrow\mathsf{S}_{fin}(\Omega,\mathcal{O})$. In [27] it is proved that $\mathsf{U}_{fin}(\Gamma,\Gamma)\Leftrightarrow\mathsf{S}_{fin}(\Omega,\mathcal{O}^{\gamma\text{-}gp})$, and in [2] it is proved that $\mathsf{U}_{fin}(\Gamma,\Omega)\Leftrightarrow\mathsf{S}_{fin}(\Omega,\mathcal{O}^{\omega\text{-}gp})$.

19 Problem. Is $\mathsf{U}_{fin}(\Gamma,\mathrm{T})$ equivalent to $\mathsf{S}_{fin}(\Omega,\mathcal{O}^{\tau\text{-}gp})$?

In [62] it was pointed out that $\mathsf{S}_1(\Omega,\mathcal{O}^{\omega\text{-}gp})$ is strictly stronger than $\mathsf{S}_1(\Omega,\mathcal{O})$ (which is the same as $\mathsf{S}_1(\mathcal{O},\mathcal{O})$). The following problem remains open.

20 Problem ([2, Problem 4]). Is $\mathsf{S}_1(\Omega,\Omega)$ equivalent to $\mathsf{S}_1(\Omega,\mathcal{O}^{\omega\text{-}gp})$?

If all powers of sets in $\mathsf{S}_1(\Omega,\mathcal{O}^{\omega\text{-}gp})$ satisfy $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, then we get a positive answer to Problem 20. In [62] it is shown that $\mathsf{S}_1(\Omega,\mathcal{O}^{\omega\text{-}gp})\Leftrightarrow\mathsf{U}_{fin}(\Gamma,\Omega)\cap\mathsf{S}_1(\mathcal{O},\mathcal{O})$, so the question can be stated in classical terms.

21 Problem. Is $\mathsf{S}_1(\Omega,\Omega)$ equivalent to $\mathsf{U}_{fin}(\Gamma,\Omega)\cap\mathsf{S}_1(\mathcal{O},\mathcal{O})$?

Surprisingly, it turns out that $\mathsf{U}_{fin}(\Gamma,\Gamma)\Leftrightarrow\binom{\Lambda}{\mathcal{O}^{\gamma\text{-}gp}}$ [63].

22 Problem. (1) Is $\mathsf{U}_{fin}(\Gamma,\Omega)$ equivalent to $\binom{\Lambda}{\mathcal{O}^{\omega\text{-}gp}}$?

(2) Is $\mathsf{U}_{fin}(\Gamma,\mathrm{T})$ equivalent to $\binom{\Lambda}{\mathcal{O}^{\tau\text{-}gp}}$?

It is often the case that properties of the form $\Pi(\Omega,\mathfrak{V})$ where $\Omega\subseteq\mathfrak{V}$ are equivalent to $\Pi(\Lambda,\mathfrak{V})$ [37, 24, 27, 2, 62]. But we do not know the answer to the following simple question.

23 Problem. Is $\binom{\Omega}{\mathcal{O}^{\gamma\text{-}gp}}$ equivalent to $\binom{\Lambda}{\mathcal{O}^{\gamma\text{-}gp}}$?

Let $\mathfrak{U}$ be a family of covers of X. Following [27], we say that a cover $\mathcal{U}$ of X is *$\mathfrak{U}$-groupable* if there is a partition of $\mathcal{U}$ into finite sets, $\mathcal{U}=\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$, such that for each infinite subset A of $\mathbb{N}$, $\{\cup\mathcal{F}_n\}_{n\in A}\in\mathfrak{U}$. Let $\mathfrak{U}^{gp}$ be the family of $\mathfrak{U}$-groupable elements of $\mathfrak{U}$. Observe that $\Lambda^{gp}=\mathcal{O}^{gp}=\mathcal{O}^{\gamma\text{-}gp}$.

In [27] it is proved that X satisfies $\mathsf{S}_{fin}(\Omega,\Omega^{gp})$ if, and only if, all finite powers of X satisfy $\mathsf{U}_{fin}(\Gamma,\Gamma)$, which we now know is the same as $\binom{\Lambda}{\Lambda^{gp}}$.

24 Problem ([60, Problem 8]). Is $\mathsf{S}_{fin}(\Omega,\Omega^{gp})$ equivalent to $\binom{\Omega}{\Omega^{gp}}$?

In [32, 27] it is proved that $\mathsf{S}_1(\Omega,\Lambda^{gp})\Leftrightarrow\mathsf{U}_{fin}(\Gamma,\Gamma)\cap\mathsf{S}_1(\mathcal{O},\mathcal{O})$, and is the same as the Gerlits-Nagy $(*)$ property. Clearly, $\mathsf{S}_1(\Omega,\mathrm{T})$ implies $\mathsf{S}_1(\Omega,\mathcal{O})$, which is the same as $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. Thus, a positive solution to Problem 15 would imply

that the property $\mathsf{S}_1(\Omega, \mathrm{T})$ lies between the $\mathsf{S}_1(\Omega, \Gamma)$ (γ-property) and $\mathsf{S}_1(\Omega, \Lambda^{gp})$ ($(*)$ property).

These notions of groupable covers are new and were not completely classified yet. Some partial results appear in [26, 27, 2, 63].

5 Project. *Classify the selection properties involving groupable covers.*

The studies of preservation of these properties under set theoretic operations are also far from being complete. Some of the known results are quoted in [48].

11 Splittability

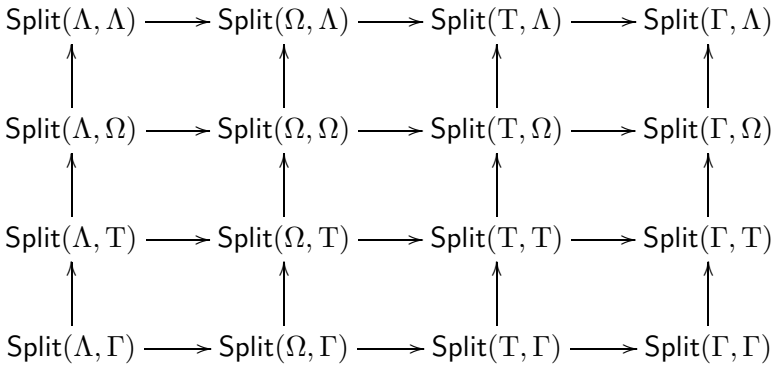
The following discussion is based on [58]. Assume that $\mathfrak{U}$ and $\mathfrak{V}$ are collections of covers of a space X. The following property was introduced in [37].

$\mathsf{Split}(\mathfrak{U}, \mathfrak{V})$: Every cover $\mathcal{U} \in \mathfrak{U}$ can be split into two disjoint subcovers $\mathcal{V}$ and $\mathcal{W}$ which contain elements of $\mathfrak{V}$.

This property is useful in the Ramsey theory of thick covers. Several results about these properties (where $\mathfrak{U}, \mathfrak{V}$ are collections of thick covers) are scattered in the literature. Some results relate these properties to classical properties. For example, it is known that the Hurewicz property and Rothberger's property each implies $\mathsf{Split}(\Lambda, \Lambda)$, and that the Sakai property (asserting that each finite power of X has Rothberger's property) implies $\mathsf{Split}(\Omega, \Omega)$ [37]. It is also known that if all finite powers of X have the Hurewicz property, then X satisfies $\mathsf{Split}(\Omega, \Omega)$ [27]. Let C_Ω denote the collection of all *clopen* ω-covers of X. By a recent characterization of the Reznichenko (or: weak Fréchet-Urysohn) property of $C_p(X)$ in terms of covering properties of X [36], the Reznichenko property for $C_p(X)$ implies that X satisfies $\mathsf{Split}(C_\Omega, C_\Omega)$.

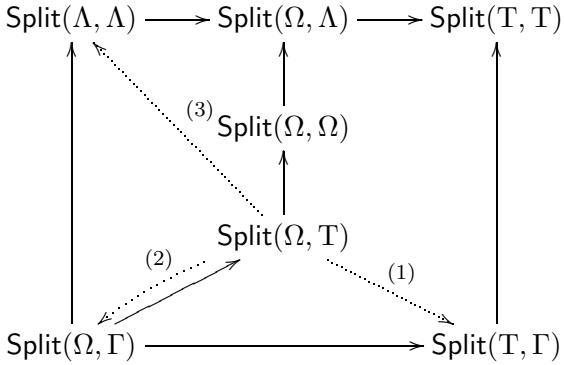
11.1 Classification

If we consider this prototype with $\mathfrak{U}, \mathfrak{V} \in \{\Lambda, \Omega, \mathrm{T}, \Gamma\}$ we obtain the following 16 properties.



But all properties in the last column are trivial in the sense that all sets of reals satisfy them. On the other hand, all properties but the top one in the first column imply $\binom{\Lambda}{\Omega}$ and are therefore trivial in the sense that no infinite set of reals satisfies any of them. Moreover, the properties $\mathsf{Split}(\mathrm{T}, \mathrm{T})$, $\mathsf{Split}(\mathrm{T}, \Omega)$, and $\mathsf{Split}(\mathrm{T}, \Lambda)$ are equivalent. It is also easy to see that $\mathsf{Split}(\Omega, \Gamma) \Leftrightarrow \binom{\Omega}{\Gamma}$, therefore $\mathsf{Split}(\Omega, \Gamma)$ implies $\mathsf{Split}(\Lambda, \Lambda)$. In [58] it is proved that no implication can be added to the diagram in the following problem, except perhaps the dotted ones.

25 Problem ([58, Problem 5.9]). Is the dotted implication (1) (and therefore (2) and (3)) in the following diagram true? If not, then is the dotted implication (3) true?



A positive answer to Problem 13 would imply a positive answer to this problem.

11.2 Preservation of properties

We list briefly the only remaining problems concerning preservation of the splittability properties mentioned in the last section under set theoretic operations. All problems below are settled for the properties which do not appear in them.

26 Problem ([58, Problem 6.8]). Is $\mathsf{Split}(\Lambda,\Lambda)$ additive?

27 Problem ([58, Problem 7.5]). Is any of the properties $\mathsf{Split}(\mathcal{B}_\Omega,\mathcal{B}_\Lambda)$, $\mathsf{Split}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$, $\mathsf{Split}(\mathcal{B}_\mathrm{T},\mathcal{B}_\mathrm{T})$, and $\mathsf{Split}(\mathcal{B}_\mathrm{T},\mathcal{B}_\Gamma)$ hereditary?

28 Conjecture ([58, Conjecture 8.7]). *None of the classes* $\mathsf{Split}(\mathrm{T},\mathrm{T})$ *and* $\binom{\mathrm{T}}{\Gamma}$ *is provably closed under taking finite products.*

29 Problem ([58, Problem 8.8]). Is any of the properties $\mathsf{Split}(\Omega,\Omega)$, $\mathsf{Split}(\Omega,\mathrm{T})$, or $\mathsf{Split}(\mathrm{T},\mathrm{T})$ preserved under taking finite powers?

12 Ultrafilter-covers

12.1 The δ-property

The following problem is classical by now, but it is related to the problems which follow. For a sequence $\{X_n\}_{n\in\mathbb{N}}$ of subsets of X, define $\liminf X_n = \bigcup_m \bigcap_{n\geq m} X_n$. For a family $\mathcal{F}$ of subsets of X, $L(\mathcal{F})$ denotes its closure under the operation $\liminf$. The following definition appears in the celebrated paper [22] just after that of the γ-property: X is a *δ-set* (or: has the *δ-property*) if for each ω-cover $\mathcal{U}$ of X, $X\in L(\mathcal{U})$. Observe that if $\{U_n\}_{n\in\mathbb{N}}$ is a γ-cover of X, then $X=\liminf U_n$. Thus, the γ-property implies the δ-property. Surprisingly, the converse is still open.

30 Problem ([22, p. 160]). Is the δ-property equivalent to the γ-property?

The δ-property implies Gerlits-Nagy ($*$) property [22], which is the same as $\mathsf{U}_{fin}(\Gamma,\Gamma)\cap\mathsf{S}_1(\mathcal{O},\mathcal{O})$ (or $\mathsf{S}_1(\Omega,\Lambda^{gp})$) and implies $\mathsf{S}_1(\Omega,\Omega)$ [32]. Miller (personal communication) suggested that if we could construct an increasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of γ-sets whose union is not a γ-set, then the union of these sets would be a δ-set which is not a γ-set.

For a sequence $\{X_n\}_{n\in\mathbb{N}}$ of subsets of X, define $p\text{-}\lim X_n = \bigcup_{A\in p}\bigcap_{n\in A} X_n$. For a family $\mathcal{F}$ of subsets of X, $L_p(\mathcal{F})$ denotes its closure under the operation p-lim. A space X satisfies the δ_M property if for each open ω-cover $\mathcal{U}$ of X, there exists $p\in M$ such that $X\in S_p(\mathcal{U})$. When $M=\{p\}$, we write δ_p instead of $\delta_{\{p\}}$.

The following problem is analogous to Problem 30.

31 Problem ([21, Problem 3.14]). Assume that X satisfies δ_p for each ultrafilter p. Must X satisfy γ_p for each ultrafilter p?

12.2 Sequential spaces

A space Y is *sequential* if for each non-closed $A \subseteq Y$ there exists $y \in Y \setminus A$ and a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A such that $\lim a_n = y$. This notion has a natural generalization.

An *ultrafilter on* $\mathbb{N}$ is a family p of subsets of $\mathbb{N}$ that is closed under taking supersets, is closed under finite intersections, does not contain the empty set as an element, and for each $A \subseteq \mathbb{N}$, either $A \in p$ or $\mathbb{N} \setminus A \in p$. An ultrafilter p on $\mathbb{N}$ is *nonprincipal* if it is not of the form $\{A \subseteq \mathbb{N} : n \in A\}$ for any n. In the sequel, by *ultrafilter* we always mean a *nonprincipal ultrafilter on* $\mathbb{N}$.

For an ultrafilter p, $\mathcal{O}_p$ denotes the collection of open γ_p-covers of X, that is, open covers $\mathcal{U}$ that can be enumerated as $\{U_n\}_{n\in\mathbb{N}}$ where $\{n : x \in U_n\} \in p$ for all $x \in X$. The property $\binom{\Omega}{\mathcal{O}_p}$ is called *the* γ_p*-property* in [21].

32 Problem ([19, Question 2.4]). Is the property $\binom{\Omega}{\mathcal{O}_p}$ additive for each ultrafilter p?

33 Problem ([21, Problem 3.14(2)]). Assume that X satisfies $\binom{\Omega}{\mathcal{O}_p}$ for each ultrafilter p. Must X satisfy $\binom{\Omega}{\Gamma}$?

In [21, Theorem 3.13] it is shown that the answer to this problem is positive under an additional set theoretic hypothesis.

For an ultrafilter p, we write $x = p\text{-}\lim x_n$ when for each neighborhood U of x, $\{n : x_n \in U\} \in p$. A space Y is *p-sequential* if we replace lim by p-lim in the definition of sequential.

34 Problem ([20], [21, Problem 0.10]). Assume that $C_p(X)$ is p-sequential. Must X satisfy $\binom{\Omega}{\mathcal{O}_p}$?

Kombarov [25] introduced the following two generalizations of p-sequentiality: Let M be a collection of ultrafilters. Y is *weakly* M*-sequential* if for each non-closed $A \subseteq Y$ there exists $y \in Y \setminus A$ and a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A such that $p\text{-}\lim a_n = y$ for *some* $p \in M$. Y is *strongly* M*-sequential* if *some* is replaced by *for all* in the last definition.

35 Problem ([21, Problem 0.6 (reformulated)]). Assume that X satisfies the δ_M property. Must $C_p(X)$ be weakly M-sequential?

Part 3: Applications

13 Infinite game Theory

Each selection principle has a naturally associated game. In the game $\mathsf{G}_1(\mathfrak{U},\mathfrak{V})$ ONE chooses in the nth inning an element $\mathcal{U}_n$ of $\mathfrak{U}$ and then TWO responds by choosing $U_n \in \mathcal{U}_n$. They play an inning per natural number. A play

$$(\mathcal{U}_0, U_0, \mathcal{U}_1, U_1 \dots)$$

is won by TWO if $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$; otherwise ONE wins. The game $\mathsf{G}_{fin}(\mathfrak{U},\mathfrak{V})$ is played similarly, where TWO responds with finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$ and wins if $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$.

Observe that if ONE does not have a winning strategy in $\mathsf{G}_1(\mathfrak{U},\mathfrak{V})$ (respectively, $\mathsf{G}_{fin}(\mathfrak{U},\mathfrak{V})$), then $\mathsf{S}_1(\mathfrak{U},\mathfrak{V})$ (respectively, $\mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V})$) holds. The converse is not always true; when it is true, the game is a powerful tool for studying the combinatorial properties of $\mathfrak{U}$ and $\mathfrak{V}$ – see, e.g., [27], [2], and references therein.

Let $\mathcal{D}$ denote the collection of all families $\mathcal{U}$ of open sets in X such that $\cup\mathcal{U}$ is dense in X. In [8], Berner and Juhász introduce the open-point game, which by [44] is equivalent to $\mathsf{G}_1(\mathcal{D},\mathcal{D})$ in the sense that a player has a winning strategy in the open-point game on X if, and only if, the *other* player has a winning strategy in $\mathsf{G}_1(\mathcal{D},\mathcal{D})$.

36 Problem ([8, Question 4.2], [42, footnote 1]). Does there exist in ZFC a space in which $\mathsf{G}_1(\mathcal{D},\mathcal{D})$ is undetermined?

$\mathcal{D}_\Omega$ is the collection of all $\mathcal{U} \in \mathcal{D}$ such that for each $U \in \mathcal{U}$, $X \not\subseteq \overline{U}$, and for each finite collection $\mathcal{F}$ of open sets, there exists $U \in \mathcal{U}$ which intersects all members of $\mathcal{F}$.

Problem 37 is not a game theoretical one, but it is related to Problem 38 which is a game theoretic problem. If all finite powers of X satisfy $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ (respectively, $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$), then X satisfies $\mathsf{S}_1(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ (respectively, $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$) [43]. If the other direction also holds, then the answer to the following is positive.

37 Problem ([43]). Are the properties $\mathsf{S}_1(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ or $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ preserved under taking finite powers?

The answer is "Yes" for a nontrivial family of spaces – see [43]. A positive answer for this problem implies a positive answer to the following one. If each finite power of X satisfies $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ (respectively, $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$), then ONE has no winning strategy in $\mathsf{G}_1(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ (respectively, $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$) [43].

38 Problem ([43, Problem 3]). Is any of the properties $\mathsf{S}_1(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ or $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ equivalent to ONE not having a winning strategy in the corresponding game?

Let $\mathcal{K}$ denote the families $\mathcal{U} \in \mathcal{D}$ such that $\{\overline{U} : U \in \mathcal{U}\}$ is a cover of X. In [50] Tkachuk shows that the Continuum Hypothesis implies that ONE has

a winning strategy in $\mathsf{G}_1(\mathcal{K},\mathcal{D})$ on any space of uncountable cellularity. In [39], Scheepers defines j as the minimal cardinal κ such that ONE has a winning strategy in $\mathsf{G}_1(\mathcal{K},\mathcal{D})$ on each Tychonoff space with cellularity at least κ, and shows that $\mathsf{cov}(\mathcal{M}) \leq \mathsf{j} \leq \mathsf{non}(SMZ)$.

39 Problem ([39, Problem 1]). Is j equal to any standard cardinal characteristic of the continuum?

Scheepers conjectures that j is not provably equal to $\mathsf{cov}(\mathcal{M})$, and not to $\mathsf{non}(SMZ)$ either.

13.1 Strong selection principles and games

The following prototype of selection hypotheses is described in [62]. Assume that $\{\mathfrak{U}_n\}_{n\in\mathbb{N}}$ is a sequence of collections of covers of a space X, and that $\mathfrak{V}$ is a collection of covers of X. Define the following selection hypothesis.

$\mathsf{S}_1(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ where $\mathcal{U}_n \in \mathfrak{U}_n$ for each n, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for each n, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

Similarly, define $\mathsf{S}_{fin}(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$. A cover $\mathcal{U}$ of a space X is an *n-cover* if each n-element subset of X is contained in some member of $\mathcal{U}$. For each n denote by $\mathcal{O}_n$ the collection of all open n-covers of a space X. Then X is a *strong γ-set* according to the definition of Galvin-Miller [18] if, and only if, X satisfies $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}},\Gamma)$ [62]. It is well known that the strong γ-property is strictly stronger than the γ-property, and is therefore not equivalent to any of the classical properties. However, for almost any other pair $(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$, $\mathsf{S}_1(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$ and $\mathsf{S}_{fin}(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$ turns out equivalent to some classical property [62]. The only remaining problem is the following.

40 Conjecture ([62, Conjecture 1]). $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}},\mathrm{T})$ *is strictly stronger than* $\mathsf{S}_1(\Omega,\mathrm{T})$.

If this conjecture is false, then we get a negative answer to Problem 13 of [61].

As in the classical selection principles, there exist game-theoretical counterparts of the new prototypes of selection principles [62]. Define the following games between two players, ONE and TWO, which have an inning per natural number. $\mathsf{G}_1(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$: In the nth inning, ONE chooses an element $\mathcal{U}_n \in \mathfrak{U}_n$, and TWO responds with an element $U_n \in \mathcal{U}_n$. TWO wins if $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$; otherwise ONE wins. $\mathsf{G}_{fin}(\{\mathfrak{U}_n\}_{n\in\mathbb{N}},\mathfrak{V})$: In the nth inning, ONE chooses an element $\mathcal{U}_n \in \mathfrak{U}_n$, and TWO responds with a finite subset $\mathcal{F}_n$ of $\mathcal{U}_n$. TWO wins if $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$; otherwise ONE wins.

In [62] it is proved that for $\mathfrak{V} \in \{\Lambda, \mathcal{O}^{\omega\text{-}gp}, \mathcal{O}^{\gamma\text{-}gp}\}$, ONE does not have a winning strategy in $\mathsf{G}_{fin}(\{\mathcal{O}_n\}_{n\in\mathbb{N}},\mathfrak{V})$ if, and only if, $\mathsf{S}_{fin}(\Omega,\mathfrak{V})$ holds, and the

analogous result is proved for G_1 and S_1. In the case of G_1 and S_1, the assertion also holds for $\mathfrak{V} \in \{\Omega, \Omega^{gp}\}$.

41 Problem. Assume that $\mathfrak{V} \in \{\Omega, \Omega^{gp}\}$. Is it true that ONE does not have a winning strategy in $\mathsf{G}_{fin}(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathfrak{V})$ if, and only if, $\mathsf{S}_{fin}(\Omega, \mathfrak{V})$ holds?

The most interesting problem with regards to these games seems to be the following.

42 Problem ([62, Problem 5.16]). Is it true that X is a strong γ-set (i.e., satisfies $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \Gamma)$) if, and only if, ONE has no winning strategy in the game $\mathsf{G}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \Gamma)$?

A positive answer would give the first game-theoretic characterization of the strong γ-property.

14 Ramsey Theory

14.1 Luzin sets

Recall that $\mathcal{K}$ is the collection of families $\mathcal{U}$ of open sets such that $\{\overline{U} : U \in \mathcal{U}\}$ is a cover of X. Let $\mathcal{K}_\Omega$ be the collection of all $\mathcal{U} \in \mathcal{K}$ such that no element of $\mathcal{U}$ is dense in X, and for each finite $F \subseteq X$, there exists $U \in \mathcal{U}$ such that $F \subseteq \overline{U}$. In [40] it is proved that X satisfies $\mathcal{K}_\Omega \to (\mathcal{K})^2_2$, then X is a Luzin set.

43 Problem ([40, Problem 4]). Does the partition relation $\mathcal{K}_\Omega \to (\mathcal{K})^2_2$ characterize Luzin sets?

14.2 Polarized partition relations

The symbol

$$\binom{\mathfrak{U}_1}{\mathfrak{U}_2} \to \begin{bmatrix}\mathfrak{V}_1\\ \mathfrak{V}_2\end{bmatrix}_{k/<\ell}$$

denotes the property that for each $\mathcal{U}_1 \in \mathfrak{U}_1, \mathcal{U}_2 \in \mathfrak{U}_2$, and k-coloring $f : \mathcal{U}_1 \times \mathcal{U}_2 \to \{1, \dots, k\}$ there are $\mathcal{V}_1 \subseteq \mathcal{U}_1$, $\mathcal{V}_2 \subseteq \mathcal{U}_2$ such that $\mathcal{V}_1 \in \mathfrak{V}_1$ and $\mathcal{V}_2 \in \mathfrak{V}_2$, and a set of less than ℓ colors J such that $f[\mathcal{V}_1 \times \mathcal{V}_2] \subseteq J$.

$\mathsf{S}_1(\Omega, \Omega)$ implies $\binom{\Omega}{\Omega} \to \left[\begin{smallmatrix}\Omega\\ \Omega\end{smallmatrix}\right]_{k/<3}$, which in turn implies $\mathsf{S}_{fin}(\Omega, \Omega)$ as well as $\mathsf{Split}(\Omega, \Omega)$ (see Section 11 for the definition of the last property). Consequently, the critical cardinality of this partition relation lies between $\mathsf{cov}(\mathcal{M})$ and $\min\{\mathfrak{d}, \mathfrak{u}\}$ [46].

44 Problem ([46, Problem 1]). Is $\binom{\Omega}{\Omega} \to \left[\begin{smallmatrix}\Omega\\ \Omega\end{smallmatrix}\right]_{k/<3}$ equivalent to $\mathsf{S}_1(\Omega, \Omega)$? And if not, is its critical cardinality equal to that of $\mathsf{S}_1(\Omega, \Omega)$ (namely, to $\mathsf{cov}(\mathcal{M})$)?

15 Function spaces and Arkhangel'skiĭ duality theory

The set of all real-valued functions on X, denoted $\mathbb{R}^X$, is considered as a power of the real line and is endowed with the Tychonoff product topology. $C_p(X)$ is the subspace of $\mathbb{R}^X$ consisting of the continuous real-valued functions on X. The topology of $C_p(X)$ is known as the *topology of pointwise convergence*. The constant zero element of $C_p(X)$ is denoted $\mathbf{0}$.

15.1 s_1 spaces and sequence selection properties

In a manner similar to the observation made in Section 3 of [41], a positive solution to Problem 11 should imply a positive solution to the following problem. For subset $A \subset X$ we denote

$$\begin{aligned} s_0(A) &= A, \quad s_\xi(A) = \{\lim_{n\to\infty} x_n : x_n \in \bigcup_{\eta<\xi} s_\eta(A) \text{ for each } n \in \mathbb{N}\}, \\ \sigma(A) &= \min\{\xi : s_\xi(A) = s_{\xi+1}(A)\}. \end{aligned}$$

Let $\Sigma(X) = \sup\{\sigma(A) : A \subseteq X\}$. Fremlin [16] proved that $\Sigma(C_p(X))$ must be 0, 1, or ω_1. If $\Sigma(C_p(X)) = 1$ then we say that X is an *s_1-space*.

45 Problem (Fremlin [16, Problem 15(c)]). Is the union of less than $\mathfrak{b}$ many s_1-spaces an s_1-space?

A sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq C_p(X)$ converges *quasi-normally* to a function f on X [11] if there exists a sequence of positive reals $\{\epsilon_n\}_{n\in\mathbb{N}}$ converging to 0 such that for each $x \in X$ $|f_n(x) - f(x)| < \epsilon_n$ for all but finitely many n. X is a *wQN-space* [13] if each sequence in $C_p(X)$ which converges to $\mathbf{0}$, contains a quasi-normally convergent subsequence.

Finally, $C_p(X)$ has the *sequence selection property (SSP)* if for each sequence $\{\{f_k^n\}_{k\in\mathbb{N}}\}_{n\in\mathbb{N}}$ of sequences in $C_p(X)$, where each of them converges to $\mathbf{0}$, there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ such that $\{f_{k_n}^n\}_{n\in\mathbb{N}}$ converges to $\mathbf{0}$. This is equivalent to Arkhangel'skiĭ's α_2 property of $C_p(X)$.

In [41, 17] it is shown that s_1 (for X), wQN (for X), and SSP (for $C_p(X)$) are all equivalent. This and other reasons lead to suspecting that all these equivalent properties are equivalent to a standard selection hypothesis. In [41], Scheepers shows that $\mathsf{S}_1(\Gamma,\Gamma)$ implies being an wQN-space.

46 Conjecture (Scheepers [41, Conjecture 1]). *For sets of reals,* wQN *implies* $\mathsf{S}_1(\Gamma,\Gamma)$.

If this conjecture is true, then Problems 11 and 45 coincide.

A space has *countable fan tightness* if for each $x \in X$, if $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of subsets of X such that for each n $x \in \overline{A_n}$, then there are finite subsets $F_N \subseteq A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_n F_n}$. This property is due to Arkhangelśkiĭ, who proved in [1] that $C_p(X)$ has countable fan tightness if, and only if, every finite power of X satisfies $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$ (this is the same as $\mathsf{S}_{fin}(\Omega, \Omega)$).

The *weak sequence selection property* for $C_p(X)$ [38] is defined as the SSP with the difference that we only require that $\mathbf{0} \in \overline{\{f^n_{k_n} : n \in \mathbb{N}\}}$.

47 Problem ([38, Problem 1]). Does countable fan tightness of $C_p(X)$ imply the weak sequence selection property?

The *monotonic sequence selection property* is defined like the SSP with the additional assumption that for each n the sequence $\{f^n_k\}_{k\in\mathbb{N}}$ converges pointwise *monotonically* to $\mathbf{0}$.

48 Problem ([38, Problem 2]). Does the monotonic sequence selection property of $C_p(X)$ imply the weak sequence selection property?

16 The weak Fréchet-Urysohn property and Pytkeev spaces

Recall that a topological space Y has the Fréchet-Urysohn property if for each subset A of Y and each $y \in \overline{A}$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of A which converges to y. If $y \notin A$ then we may assume that the elements a_n, $n \in \mathbb{N}$, are distinct. The following natural generalization of this property was introduced by Reznichenko [28]: Y satisfies the *weak Fréchet-Urysohn* property if for each subset A of Y and each element y in $\overline{A} \setminus A$, there exists a countably infinite pairwise disjoint collection $\mathcal{F}$ of finite subsets of A such that for each neighborhood U of y, $U \cap F \neq \emptyset$ for all but finitely many $F \in \mathcal{F}$. In several works [26, 27, 36] this property appears as the *Reznichenko* property.

In [27] it is shown that $C_p(X)$ has countable fan tightness as well as Reznichenko's property if, and only if, each finite power of X has the Hurewicz covering property. Recently, Sakai found an exact dual of the Reznichenko property: An open ω-cover $\mathcal{U}$ of X is *ω-shrinkable* if for each $U \in \mathcal{U}$ there exists a closed subset $C_U \subseteq U$ such that $\{C_U : U \in \mathcal{U}\}$ is an ω-cover of X. Then $C_p(X)$ has the Reznichenko property if, and only if, each ω-shrinkable open ω-cover of X is ω-groupable [36]. Thus if X satisfies $\binom{\Omega}{\Omega^{gp}}$, then $C_p(X)$ has the Reznichenko property. The other direction is not clear.

49 Problem ([36, Question 3.5]). Is it true that $C_p(X)$ has the Reznichenko property if, and only if, X satisfies $\binom{\Omega}{\Omega^{gp}}$?

Another simply stated but still open problem is the following.

50 Problem ([36, Question 3.6], [60]). Does $C_p({}^{\mathbb{N}}\mathbb{N})$ have the Reznichenko property?

For a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$ and a finite-to-one function $f : \mathbb{N} \to \mathbb{N}$, $f(\mathcal{F}) := \{A \subseteq \mathbb{N} : f^{-1}[A] \in \mathcal{F}\}$ is again a nonprincipal filter on $\mathbb{N}$. A filter $\mathcal{F}$ on $\mathbb{N}$ is *feeble* if there exists a finite-to-one function f such that $f(\mathcal{F})$ consists of only the cofinite sets. By Sakai's Theorem, if $C_p(X)$ has the Reznichenko property then X satisfies $\binom{C_\Omega}{C_\Omega^{gp}}$. In [60] it is shown that $\binom{C_\Omega}{C_\Omega^{gp}}$ is equivalent to the property that no continous image of X in the Rothberger space $P_\infty(\mathbb{N})$ is a subbbase for a non-feeble filter. Thus, if a subbase for a non-feeble filter cannot be a continuous image of ${}^{\mathbb{N}}\mathbb{N}$, then the answer to Problem 50 is negative.

A family $\mathcal{P}$ of subsets of of a space Y is a *π-network* at $y \in Y$ if every neighborhood of y contains some element of $\mathcal{P}$. Y is a *Pytkeev space* if for each $y \in Y$ and $A \subseteq Y$ such that $y \in \overline{A} \setminus A$, there exists a countable π-network at y which consists of infinite subsets of A. In [36] it is proved that $C_p(X)$ is a Pytkeev space if, and only if, for each ω-shrinkable open ω-cover $\mathcal{U}$ of X there exist subfamilies $\mathcal{U}_n \subseteq \mathcal{U}$, $n \in \mathbb{N}$, such that $\bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ is an ω-cover of X.

51 Problem ([36, Question 2.8]). Can the term "ω-shrinkable" be removed from Sakai's characterization of the Pytkeev property of $C_p(X)$?

If all finite powers of X satisfy $\mathsf{U}_{fin}(\Gamma, \Gamma)$, then every open ω-cover of X is ω-shrinkable [36], thus a positive solution to the following problem would suffice.

52 Problem ([36, Question 2.9]). Assume that $C_p(X)$ is a Pytkeev space. Is it true that all finite powers of X satisfy $\mathsf{U}_{fin}(\Gamma, \Gamma)$?

Let $I = [0, 1]$ be the closed unit interval in $\mathbb{R}$. As all finite powers of I are compact, and $C_p(I)$ is not a Pytkeev space [36], the converse of Problem 52 is false.

Notes added in proof. Zdomsky, in a series of recent works, settled (or partially settled) some of the problems mentioned in the paper, the answers being: "Yes" for Problem 2, "No" for Problem 10, and "Consistently yes" for Problem 26.

Sakai showed that the answer for Problem 24 is "No", in the following strong sense: in his paper *Weak Fréchet-Urysohn property in function spaces*, it is proved that that every analytic set of reals (and, in particular, the Baire space ${}^{\mathbb{N}}\mathbb{N}$) satisfies $\binom{\mathcal{B}_\Omega}{\mathcal{B}_\Omega^{gp}}$. But we know that ${}^{\mathbb{N}}\mathbb{N}$ does not even satisfy Menger's property $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$. This also answers Problem 50 in the affirmative.

The paper: H. Mildenberger, S. Shelah, and B. Tsaban, *The combinatorics of τ-covers* (`http://arxiv.org/abs/math.GN/0409068`) contains new results simplifying some problems. Project 3 is almost completely settled (4 out of the 6 cardinals are found, the two remaining ones are equal but still unknown).

Consequently, 21 out of the 76 potential implications in Project 4 are ruled out.

Finally, the preset author's paper *Some new directions in infinite-combinatorial topology* (`http://arxiv.org/abs/math.GN/0409069`) contains a light introduction to the field and several problems not appearing in the current survey.

References

[1] A. V. Arkhangel'skiĭ: *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Soviet Mathematical Doklady **33** (1986), 396–399.

[2] L. Babinkostova, Lj. D. R. Kocinac, M. Scheepers: *Combinatorics of open covers (VIII)*, preprint.

[3] T. Bartoszynski, H. Judah: *On cofinality of the smallest covering of the real line by meager sets II*, Proceedings of the American Mathematical Society **123** (1995), 1879–1885.

[4] T. Bartoszyński, H. Judah: Set Theory: On the structure of the real line, A. K. Peters, Massachusetts: 1995.

[5] T. Bartoszyński, S. Shelah, B. Tsaban: *Additivity properties of topological diagonalizations*, Journal of Symbolic Logic, **68**, (2003), 1254–1260. (Full version: `http://arxiv.org/abs/math.LO/0112262`)

[6] T. Bartoszyński, B. Tsaban: *Hereditary topological diagonalizations and the Menger-Hurewicz Conjectures*, Proceedings of the American Mathematical Society, to appear. `http://arxiv.org/abs/math.LO/0208224`

[7] V. Bergelson: *Ergodic Ramsey Theory – an update*, in: Ergodic Theory of $\mathbb{Z}^d$-actions (ed. M. Pollicott and K. Schmidt), London Math. Soc. Lecture Note Series **228** (1996), 1–61.

[8] A. Berner, I. Juhász: *Point-picking games and HFD's*, in: Models and Sets, Proceedings of the Logic Colloquium 1983, Springer Verlag LNM **1103** (1984), 53–66.

[9] A. R. Blass: *Combinatorial cardinal characteristics of the continuum*, in: Handbook of Set Theory (eds. Foreman, Kanamori, and Magidor), to appear.

[10] J. Brendle: *Generic constructions of small sets of reals*, Topology and it Applications **71** (1996), 125–147.

[11] Z. Bukovská: *Quasinormal convergence*, Math. Slovaca **41** (1991), 137–146.

[12] L. Bukovský: *Not distinguishing convergences: Open problems*, SPM Bulletin **3**, 2–3.

[13] L. Bukovský, I. Recław, M. Repický: *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology and its Applications **41** (1991), 25–40.

[14] L. Bukovský, J. Haleš: *On Hurewicz Properties*, preprint.

[15] E. K. van Douwen: *The integers and topology*, in: Handbook of Set Theoretic Topology (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam: 1984, 111–167.

[16] D. H. Fremlin: *Sequential convergence in $C_p(X)$*, Commentarii Mathematicae Universitatis Carolinae **35** (1994), 371–382.

[17] D. H. Fremlin: *SSP and WQN*, preprint.

[18] F. Galvin, A. W. Miller *γ-sets and other singular sets of real numbers*, Topology and it Applications **17** (1984), 145–155.

[19] S. García-Ferreira, W. Just: *Some remarks on the γ_p property*, Questions and Answers in General Topology **17** (1999), 1–8.

[20] S. García-Ferreira, A. Tamariz-Mascarúa: *p-Fréchet-Urysohn properties of function spaces*, Topology and its Applications **58** (1994), 157–172.

[21] S. García-Ferreira, A. Tamariz-Mascarúa: *p-sequential like properties in function spaces*, Comment. Math. Univ. Carolinae **35** (1994), 753–771.

[22] J. Gerlits, Zs. Nagy: *Some properties of C(X), I*, Topology and its Applications **14** (1982), 151–161.

[23] W. Hurewicz: *Über Folgen stetiger Funktionen*, Fundamenta Mathematicae **9** (1927), 193–204.

[24] W. Just, A. W. Miller, M. Scheepers, P. J. Szeptycki: *The combinatorics of open covers II*, Topology and its Applications **73** (1996), 241–266.

[25] A. P. Kombarov: *On a theorem of A. H. Stone*, Soveit Math. Dokl. **27** (1983), 544–547.

[26] Lj. D. R. Kočinac, M. Scheepers: *Function spaces and a property of Renichenko*, Topology and it Applications **123** (2002), 135–143.

[27] Lj. D. R. Kočinac, M. Scheepers: *Combinatorics of open covers (VII): groupability*, Fundamenta Mathematicae, to appear.

[28] V. I. Malykhin, G. Tironi: *Weakly Fréchet-Urysohn and Pytkeev spaces*, Topology and its applications **104** (2000), 181–190.

[29] A. W. Miller: *On the length of Borel hierarchies*, Annals of Mathematical Logic **16** (1979), 233–267.

[30] A. W. Miller: *Special subsets of the real line*, in: Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North Holland, Amsterdam: 1984, 201–233.

[31] A. W. Miller: *A Nonhereditary Borel-cover γ-set*, preprint.

[32] A. Nowik, M. Scheepers, T. Weiss: *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic **63** (1998), 301–324.

[33] J. Pawlikowski, I. Recław: *Parametrized Cichoń's diagram and small sets*, Fundamenta Mathematicae **147** (1995), 135–155.

[34] I. Recław: *Every Luzin set is undetermined in point-open game*, Fundamenta Mathematicae **144** (1994), 43–54.

[35] M. Sakai: *Property C'' and function spaces*, Proceedings of the American Mathematical Society **104** (1988), 917–919.

[36] M. Sakai: *The Pytkeev property and the Reznichenko property in function spaces*, Note di Matematica, to appear.

[37] M. Scheepers: *Combinatorics of open covers I: Ramsey Theory*, Topology and its Applications **69** (1996), 31–62.

[38] M. Scheepers: *A sequential property of $\mathsf{C}_p(X)$ and a covering property of Hurewicz*, Proceedings of the American Mathematical Society **125** (1997), 2789–2795.

[39] M. Scheepers: *Combinatorics of open covers (IV): subspaces of the Alexandroff double of the unit interval*, Topology and its Applications **83** (1998), 63–75.

[40] M. Scheepers: *Lusin Sets*, Proceedings of the American Mathematical Society **127** (1999), 251–257.

[41] M. Scheepers: *Sequential convergence in* $\mathsf{C}_p(X)$ *and a covering property*, East-West Journal of Mathematics **1** (1999), 207–214.

[42] M. Scheepers: *The length of some diagonalization games*, Arch. Math. Logic **38** (1999), 103–122.

[43] M. Scheepers: *Combinatorics of open covers (V): Pixley-Roy spaces and sets of reals, and omega-covers*, Topology and its Applications **102** (2000), 13–31.

[44] M. Scheepers: *Combinatorics of open covers (VI): Selectors for sequences of dense sets*, Quaestiones Mathematicae, to appear.

[45] M. Scheepers: $C_p(X)$ *and Arkhangel'skiĭ's* α_i *spaces*, Topology and its Applications, to appear.

[46] M. Scheepers: *Open covers and the square bracket partition relation*, preprint.

[47] M. Scheepers, B. Tsaban: *The combinatorics of Borel covers*, Topology and its Applications **121** (2002), 357–382. http://arxiv.org/abs/math.GN/0302322

[48] M. Scheepers: *Selection principles and covering properties in topology*, Note di Matematica, this volume.

[49] S. Shelah, B. Tsaban: *Critical cardinalities and additivity properties of combinatorial notions of smallness*, Journal of Applied Analysis **9** (2003), 149–162. http://arxiv.org/abs/math.LO/0304019

[50] V. V. Tkachuk: *Some new versions of an old game*, Commentarii Mathematicae Universitatis Carolinae **36** (1995), 177–196.

[51] B. Tsaban: *A topological interpretation of* $\mathfrak{t}$, Real Analysis Exchange **25** (1999/2000), 391–404. http://arxiv.org/abs/math.LO/9705209

[52] B. Tsaban (ed.), SPM Bulletin **1** (January 2003), http://arxiv.org/abs/math.GN/0301011

[53] B. Tsaban (ed.), SPM Bulletin **2** (February 2003), http://arxiv.org/abs/math.GN/0302062

[54] B. Tsaban (ed.), SPM Bulletin **3** (March 2003), http://arxiv.org/abs/math.GN/0303057

[55] B. Tsaban (ed.), SPM Bulletin **4** (April 2003), http://arxiv.org/abs/math.GN/0304087

[56] B. Tsaban (ed.), SPM Bulletin **5** – special issue (May 2003), http://arxiv.org/abs/math.GN/0305367

[57] B. Tsaban (ed.), SPM Bulletin **6** (November 2003), http://arxiv.org/abs/math.GN/0312140

[58] B. Tsaban: *The combinatorics of splittability*, Annals of Pure and Applied Logic, **129**, (2004), 107–130.

[59] B. Tsaban: *Selection principles and the minimal tower problem*, Note di Matematica, this volume. http://arxiv.org/abs/math.LO/0105045

[60] B. Tsaban: *The minimal cardinality where the Reznichenko property fails*, Israel Journal of Mathematics, **140**, (2004), 367–374.