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Sharp Maximal Function Inequalities and Boundedness for Commutators of Riesz Transforms of Schrödinger Operators

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Abstract. In this paper, we establish the sharp maximal function estimates for the commutators associated with the Riesz transforms of Schrödinger operators. As an application, we obtain the boundedness of the commutator on Lebesgue, Morrey and Triebel-Lizorkin spaces.

Keywords: Commutator, Riesz transform, Schrödinger operator, Sharp maximal function, Morrey space, Triebel-Lizorkin space, Lipschitz function

MSC 2000 classification: primary 42B20, secondary 42B25

Introduction

As the development of singular integral operators(see [8][19]), their commutators have been well studied. In [5][16][17], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [2][10][13], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [18], some Schrödinger type operators with certain potentials are introduced, and the boundedness for the operators and their commutators generated by BMO functions are obtained(see [9][20]). Our works are motivated by these papers. In this paper, we will study the commutators associated with the Riesz transforms of Schrödinger operators and the Lipschitz functions.

1 Notations and Lemmas

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [8][19])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

A non-negative locally L^q integrable function V on R^n is said to belong to $B_q (1 < q < \infty)$, if

$$\left(\frac{1}{|Q|} \int_Q V(x)^q dx \right)^{1/q} \le C \left(\frac{1}{|Q|} \int_Q V(x) dx \right)$$

holds for every cube Q in R^n.

The A_p weight is defined by (see [8])

$$A_p = \left\{ w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{ w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e. \}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Triebel-Lizorkin space(see [13]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for} \quad t \geq 0.$$

Let f be a locally integrable function on R^n. Set, for $1 \leq p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x\in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [14][15]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue spaces (see [3]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [3][6][7][11][12]).

In this paper, we will study the commutators associated with the Riesz transforms of Schrödinger operator as following(see [9]).

Let $P = -\Delta + V(x)$ be the Schrödinger differential operator on R^n with $n \geq 3$. $V(x)$ is a non-negative potential belongs to B_q for some $q > n/2$. Let $T_j(j = 1,2,3)$ be the Riesz transforms associated to Schrödinger operators, namely, $T_1 = (-\Delta+V)^{-1}V, T_2 = (-\Delta+V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta+V)^{-1/2}\nabla$. We know that T_j is associated with a kernel $K_j(x,y)(j = 1,2,3)$, that is (see [9][18])

$$T_j(f)(x) = \int_{R^n} K_j(x,y)f(y)dy(j = 1,2,3).$$

Let b be a locally integrable function on R^n. The commutators related to $T_j(j = 1,2,3)$ are defined by

$$[b,T_j](f)(x) = b(x)T_j(f)(x) - T_j(bf)(x)(j = 1,2,3).$$

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [16][17]). The main purpose of this paper is to prove the sharp maximal inequalities for the commutators. As application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the commutators.

Lemma 1. *[see [9]]*
Let $V \in B_q$, $q \geq n/2$. Then

(a) T_1 is bounded on $L^p(R^n)$ for $q' \leq p < \infty$.

(b) T_2 is bounded on $L^p(R^n)$ for $(2q)' \leq p < \infty$.

(c) T_3 is bounded on $L^p(R^n)$ for $p_0' \leq p < \infty$ and $1/p_0 = 1/q - 1/n$.

Lemma 2. *[see [13]] For $0 < \beta < 1$, $1 < p < \infty$ and $w \in A_\infty$, we have*

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left|\left|\sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx\right|\right|_{L^p}$$
$$\approx \left|\left|\sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx\right|\right|_{L^p}.$$

Lemma 3. *[see [8]] Let $0 < p < \infty$ and $w \in \cup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x) dx \leq C \int_{R^n} M^{\#}(f)(x)^p w(x) dx.$$

Lemma 4. *[see [1]] Suppose that $0 < \eta < n$, $1 < s < p < n/\eta$ and $1/q = 1/p - \eta/n$. Then*

$$||M_{\eta,s}(f)||_{L^q} \leq C||f||_{L^p}.$$

Lemma 5. *Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}} \leq C||M^{\#}(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [4]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x -$

$x_0| - d)^n$ if $x \in Q^c$, by Lemma 4, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx$$
$$\leq C \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx$$
$$\leq C \int_{R^n} M^{\#}(f)(x)|^p M(\chi_Q)(x) dx$$
$$= C \left(\int_Q M^{\#}(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p M(\chi_Q)(x) dx \right)$$
$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \right)$$
$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p \frac{M(w)(x)}{2^{n(k+1)}} dx \right)$$
$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p 2^{-kn} dy \right)$$
$$\leq C ||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1} d)$$
$$\leq C ||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} (2^{-n} D)^k \varphi(d)$$
$$\leq C ||M^{\#}(f)||^p_{L^{p,\varphi}} \varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)} \int_{Q(x_0,d)} M(f)(x)^p dx \right)^{1/p} \leq C \left(\frac{1}{\varphi(d)} \int_{Q(x_0,d)} M^{\#}(f)(x)^p dx \right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leq C ||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof. $\boxed{QED}$

Lemma 6. *Let $0 < D < 2^n$, $V \in B_q$ and $q \geq n/2$. Then*

(a) If $q' \leq p < \infty$,

$$||T_1(f)||_{L^{p,\varphi}} \leq C ||f||_{L^{p,\varphi}}.$$

(b) If $(2q)' \leq p < \infty$,

$$||T_2(f)||_{L^{p,\varphi}} \leq C ||f||_{L^{p,\varphi}}.$$

(c) If $p_0' \le p < \infty$ *with* $1/p_0 = 1/q - 1/n$,

$$||T_3(f)||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

Lemma 7. *Let* $0 < D < 2^n$, $1 \le s < p < n/\eta$ *and* $1/q = 1/p - \eta/n$. *Then*

$$||M_{\eta,s}(f)||_{L^{q,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

The proofs of two Lemmas are similar to that of Lemma 5 by Lemma 1 and 3, we omit the details.

Lemma 8. *[see [9]] Let* $m(x,V)^{-1} = \sup\{r > 0 : r^{2-n}\int_{B(x,r)} V(y)dy \le 1\}$, $V \in B_q$, $q \ge n/2$, $d > 0$ *and* $x, x_0 \in R^n$ *with* $|x - x_0| \le d$. *Then there exists* $\delta > 0$ *such that for any integer* $k > 0$, $0 < h < |x-y|/16$,

(a) If $q' \le p < \infty$,

$$|K_1(x+h,y) - K_1(x,y)| \le \frac{C}{(1+m(x,V)|x-y|)^k} \cdot \frac{h^\delta}{|x-y|^{n-2+\delta}} V(y),$$

$$\sum_{k=1}^{\infty} (2^k d)^{n/q'} \left(\int_{2^k d \le |x_0 - y| < 2^{k+1} d} |K_1(x,y) - K_1(x_0,y)|^q dy \right)^{1/q} \le C.$$

(b) If $(2q)' \le p < \infty$,

$$|K_2(x+h,y) - K_2(x,y)| \le \frac{C}{(1+m(x,V)|x-y|)^k} \cdot \frac{h^\delta}{|x-y|^{n-1+\delta}} V(y)^{1/2},$$

$$\sum_{k=1}^{\infty} (2^k d)^{n/(2q)'} \left(\int_{2^k d \le |x_0 - y| < 2^{k+1} d} |K_2(x,y) - K_2(x_0,y)|^{2q} dy \right)^{1/2q} \le C.$$

(c) If $p_0' \le p < \infty$ *and* $1/p_0 = 1/q - 1/n$,

$$\begin{aligned} |K_3(x+h,y) - K_3(x,y)| &\le \\ &\le \frac{C}{(1+m(x,V)|x-y|)^k} \cdot \frac{h^\delta}{|x-y|^{n-1+\delta}} \\ &\quad \cdot \left(\int_{B(x,|x-y|)} \frac{V(z)}{|y-z|} dz + |x-y|^{-1} \right), \end{aligned}$$

$$\sum_{k=1}^{\infty} (2^k d)^{n/p_0'} \left(\int_{2^k d \le |x_0 - y| < 2^{k+1} d} |K_3(x,y) - K_3(x_0,y)|^{p_0} dy \right)^{1/p_0} \le C.$$

2 Theorems and Proofs

We shall prove the following theorems.

Theorem 1. *Let $0 < \beta < 1$, $V \in B_q$, $q \geq n/2$ and $b \in Lip_\beta(R^n)$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

(a) If $q' \leq s < \infty$,

$$M^{\#}([b, T_1](f))(\tilde{x}) \leq C||b||_{Lip_\beta} \left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}(T_1(f))(\tilde{x})\right).$$

(b) If $(2q)' \leq s < \infty$,

$$M^{\#}([b, T_2](f))(\tilde{x}) \leq C||b||_{Lip_\beta} \left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}(T_2(f))(\tilde{x})\right).$$

(c) If $p_0' \leq s < \infty$ with $1/p_0 = 1/q - 1/n$,

$$M^{\#}([b, T_3](f))(\tilde{x}) \leq C||b||_{Lip_\beta} \left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}(T_3(f))(\tilde{x})\right).$$

Theorem 2. *Let $0 < \beta < \min(1, \delta)$, $V \in B_q$, $q \geq n/2$ and $b \in Lip_\beta(R^n)$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

(a) If $q' \leq s < \infty$,

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |[b, T_1](f)(x) - C_0|\, dx \leq C||b||_{Lip_\beta} \left(M_s(f)(\tilde{x}) + M_s(T_1(f))(\tilde{x})\right).$$

(b) If $(2q)' \leq s < \infty$,

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |[b, T_2](f)(x) - C_0|\, dx \leq C||b||_{Lip_\beta} \left(M_s(f)(\tilde{x}) + M_s(T_2(f))(\tilde{x})\right).$$

(c) If $p_0' \leq s < \infty$ with $1/p_0 = 1/q - 1/n$,

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |[b, T_3](f)(x) - C_0|\, dx \leq C||b||_{Lip_\beta} \left(M_s(f)(\tilde{x}) + M_s(T_3(f))(\tilde{x})\right).$$

Theorem 3. *Let $0 < \beta < 1$, $V \in B_q$, $q \geq n/2$, $1/r = 1/p - \beta/n$ and $b \in Lip_\beta(R^n)$. Then* *(a)* $[b, T_1]$ *is bounded from* $L^p(R^n)$ *to* $L^r(R^n)$ *for* $q' \le p < n/\beta$.

(b) $[b, T_2]$ *is bounded from* $L^p(R^n)$ *to* $L^r(R^n)$ *for* $(2q)' \le p < n/\beta$.

(c) $[b, T_3]$ *is bounded from* $L^p(R^n)$ *to* $L^r(R^n)$ *for* $p_0' \le p < n/\beta$ *and* $1/p_0 = 1/q - 1/n$.

Theorem 4. *Let* $0 < D < 2^n$, $0 < \beta < 1$, $V \in B_q$, $q \ge n/2$, $1/r = 1/p - \beta/n$ *and* $b \in Lip_\beta(R^n)$. *Then*

(a) $[b, T_1]$ *is bounded from* $L^{p,\varphi}(R^n)$ *to* $L^{r,\varphi}(R^n)$ *for* $q' \le p < n/\beta$.

(b) $[b, T_2]$ *is bounded from* $L^{p,\varphi}(R^n)$ *to* $L^{r,\varphi}(R^n)$ *for* $(2q)' \le p < n/\beta$.

(c) $[b, T_3]$ *is bounded from* $L^{p,\varphi}(R^n)$ *to* $L^{r,\varphi}(R^n)$ *for* $p_0' \le p < n/\beta$ *and* $1/p_0 = 1/q - 1/n$.

Theorem 5. *Let* $0 < \beta < \min(1, \delta)$, $V \in B_q$, $q \ge n/2$ *and* $b \in Lip_\beta(R^n)$. *Then*

(a) $[b, T_1]$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{\beta,\infty}(R^n)$ *for* $q' \le p < n/\beta$.

(b) $[b, T_2]$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{\beta,\infty}(R^n)$ *for* $(2q)' \le p < n/\beta$.

(c) $[b, T_3]$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{\beta,\infty}(R^n)$ *for* $p_0' \le p < n/\beta$ *and* $1/p_0 = 1/q - 1/n$.

To prove the theorems, we need the following lemmas.

Main Lemma 1. Let $m > 1$, $0 < \beta < 1$, $m' \le s < \infty$ and $b \in Lip_\beta(R^n)$. Suppose that the operator $T(f)(x) = \int_{R^n} K(x, y)f(y)dy$ is bounded on $L^p(R^n)$ for every $m' < p < \infty$, and $K \in H(m)$, namely, there exists a constant $C > 0$ such that for any $d > 0$, $x, x_0 \in R^n$ with $|x - x_0| \le d$, there is

$$\sum_{k=1}^{\infty}(2^k d)^{n/m'}\left(\int_{2^k d \le |x_0-y| < 2^{k+1}d} |K(x,y) - K(x_0,y)|^m dy\right)^{1/m} \le C,$$

where $1/m + 1/m' = 1$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^{\#}([b,T](f))(\tilde{x}) \le C||b||_{Lip_\beta}\left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}(T(f))(\tilde{x})\right).$$

Proof. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |[b,T](f)(x) - C_0|\,dx \le C||b||_{Lip_\beta}\left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}(T(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$[b,T](f)(x) = (b(x) - b_{2Q})T(f)(x) - T((b - b_{2Q})f_1)(x) - T((b - b_{2Q})f_2)(x).$$

Then

$$\begin{aligned} & \frac{1}{|Q|}\int_Q |[b,T](f)(x) - T((b_{2Q} - b)f_2)(x_0)|\,dx \\ \le\ & \frac{1}{|Q|}\int_Q |(b(x) - b_{2Q})T(f)(x)|dx + \frac{1}{|Q|}\int_Q |T((b - b_{2Q})f_1)(x)|dx \\ & + \frac{1}{|Q|}\int_Q |T((b - b_{2Q})f_2)(x) - T((b - b_{2Q})f_2)(x_0)|dx \\ =\ & I_1 + I_2 + I_3. \end{aligned}$$

For I_1, by Hölder's inequality and Lemma 2, we obtain

$$\begin{aligned} I_1 \ \le\ & \frac{C}{|Q|}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{1-1/s}\left(\int_Q |T(f)(x)|^s dx\right)^{1/s} \\ \le\ & C||b||_{Lip_\beta}|Q|^{\beta/n}|Q|^{-1/s}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q |T(f)(x)|^s dx\right)^{1/s} \\ \le\ & C||b||_{Lip_\beta}M_{\beta,s}(T(f))(\tilde{x}). \end{aligned}$$

For I_2, by the boundedness of T, we get

$$\begin{aligned} I_2 \ \le\ & \left(\frac{1}{|Q|}\int_{R^n} |T((b - b_{2Q})f_1)(x)|^s dx\right)^{1/s} \\ \le\ & C\left(\frac{1}{|Q|}\int_{R^n} |(b(x) - b_{2Q})f_1(x)|^s dx\right)^{1/s} \\ \le\ & C|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q} |f(x)|^s dx\right)^{1/s} \\ \le\ & C||b||_{Lip_\beta}M_{\beta,s}(f)(\tilde{x}). \end{aligned}$$

For I_3, recalling that $s > m'$, we have

$$\begin{aligned} I_3 \ \le\ & \frac{1}{|Q|}\int_Q\int_{(2Q)^c} |b(y) - b_{2Q}||f(y)||K(x,y) - K(x_0,y)|dydx \\ \le\ & \frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} |K(x,y) - K(x_0,y)||b(y) - b_{2^{k+1}Q}| \\ & |f(y)|dydx \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^kd\le|y-x_0|<2^{k+1}d}|K(x,y)-K(x_0,y)||b_{2^{k+1}Q}-b_{2Q}| \\
& |f(y)|dydx \\
\le\ & \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}|K(x,y)-K(x_0,y)|^m dy\right)^{1/m} \\
& \times||b||_{Lip_\beta}|2^kQ|^{\beta/n}\left(\int_{2^{k+1}Q}|f(y)|^{m'}dy\right)^{1/m'}dx \\
\le\ & \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}(2^kd)^{n/m'}\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}|K(x,y)-K(x_0,y)|^m dy\right)^{1/m}dx \\
& \times||b||_{Lip_\beta}\left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} \\
\le\ & C||b||_{Lip_\beta}M_{\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

These complete the proof of the lemma. QED

Main Lemma 2. Let $m>1$, $0<\beta<1$, $m'\le s<\infty$ and $b\in Lip_\beta(R^n)$. Suppose that the operator $T(f)(x)=\int_{R^n}K(x,y)f(y)dy$ is bounded on $L^p(R^n)$ for every $m'<p<\infty$ and $K\in H(m,\beta)$, namely, there exists a constant $C>0$ such that for any $d>0$, $x,x_0\in R^n$ with $|x-x_0|\le d$, there is

$$\sum_{k=1}^{\infty}2^{k\beta}(2^kd)^{n/m'}\left(\int_{2^kd\le|x_0-y|<2^{k+1}d}|K(x,y)-K(x_0,y)|^m dy\right)^{1/m}\le C,$$

where $1/m+1/m'=1$. Then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,

$$\sup_{Q\ni\tilde{x}}\frac{1}{|Q|^{1+\beta/n}}\int_Q|[b,T](f)(x)-C_0|\,dx\le C||b||_{Lip_\beta}\left(M_s(f)(\tilde{x})+M_s(T(f))(\tilde{x})\right).$$

Proof. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q|[b,T](f)(x)-C_0|\,dx\le C||b||_{Lip_\beta}\left(M_s(f)(\tilde{x})+M_s(T(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$\begin{aligned}
& \frac{1}{|Q|^{1+\beta/n}} \int_Q |[b,T](f)(x) - T((b_{2Q} - b)f_2)(x_0)|\, dx \\
\leq\ & \frac{1}{|Q|^{1+\beta/n}} \int_Q |(b(x) - b_{2Q})T(f)(x)| dx + \frac{1}{|Q|} \int_Q |T((b - b_{2Q})f_1)(x)| dx \\
& + \frac{1}{|Q|^{1+\beta/n}} \int_Q |T((b - b_{2Q})f_2)(x) - T((b - b_{2Q})f_2)(x_0)| dx \\
=\ & I_4 + I_5 + I_6.
\end{aligned}$$

By using the same argument as in the proof of Main Lemma 1, we get

$$\begin{aligned}
I_4 \ \leq\ & \frac{C}{|Q|^{1+\beta/n}} ||b||_{Lip_\beta} |2Q|^{\beta/n} |Q|^{1-1/s} \left(\int_Q |T(f)(x)|^s dx \right)^{1/s} \\
\leq\ & C||b||_{Lip_\beta} \left(\frac{1}{|Q|} \int_Q |T(f)(x)|^s dx \right)^{1/s} \\
\leq\ & C||b||_{Lip_\beta} M_s(T(f))(\tilde{x}), \\
I_5 \ \leq\ & \frac{1}{|Q|^{1+\beta/n}} |Q|^{1-1/s} \left(\int_{R^n} |T((b - b_{2Q})f_1)(x)|^s dx \right)^{1/s} \\
\leq\ & \frac{C}{|Q|^{1+\beta/n}} |Q|^{1-1/s} \left(\int_{R^n} |(b(x) - b_{2Q})f_1(x)|^s dx \right)^{1/s} \\
\leq\ & \frac{C}{|Q|^{1+\beta/n}} |Q|^{1-1/s} ||b||_{Lip_\beta} |2Q|^{\beta/n} |2Q|^{1/s} \left(\frac{1}{|2Q|} \int_{2Q} |f(x)|^s dx \right)^{1/s} \\
\leq\ & C||b||_{Lip_\beta} M_s(f)(\tilde{x}), \\
I_6 \ \leq\ & \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \leq |y - x_0| < 2^{k+1} d} |K(x,y) - K(x_0,y)||b(y) - b_{2^{k+1}Q}| \\
& |f(y)| dy dx \\
& + \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \leq |y - x_0| < 2^{k+1} d} |K(x,y) - K(x_0,y)||b_{2^{k+1}Q} - b_{2Q}| \\
& |f(y)| dy dx \\
\leq\ & \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k d \leq |y - x_0| < 2^{k+1} d} |K(x,y) - K(x_0,y)|^m dy \right)^{1/m} \\
& \times ||b||_{Lip_\beta} |2^k Q|^{\beta/n} \left(\int_{2^{k+1}Q} |f(y)|^{m'} dy \right)^{1/m'} dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} 2^{k\beta}(2^k d)^{n/m'} \\
&\quad \left(\int_{2^k d\leq |y-x_0|<2^{k+1}d} |K(x,y)-K(x_0,y)|^m dy\right)^{1/m} dx \\
&\quad \times ||b||_{Lip_\beta}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} \\
&\leq C||b||_{Lip_\beta} M_s(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of the Lemma. QED

PROOF OF THEOREM 1. By Lemma 10, we know $K_1 \in H(q)$, $K_2 \in H(2q)$ and $K_3 \in H(p_0)$, thus Theorem 1 follows from Main Lemma 1. QED

PROOF OF THEOREM 2. If $q' \leq s < \infty$, by [11], we know

$$\left(\int_{2^k d\leq |x_0-y|<2^{k+1}d} |K_1(x,y)-K_1(x_0,y)|^q dy\right)^{1/q} \leq C\frac{d^\delta}{(2^k d)^{\delta+n/q'}},$$

by Lemma 10 and notice $0<\beta<\delta$, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{k\beta}(2^k d)^{n/q'}\left(\int_{2^k d\leq |x_0-y|<2^{k+1}d} |K(x,y)-K(x_0,y)|^q dy\right)^{1/q} \\
&\leq C\sum_{k=1}^{\infty} 2^{k\beta}(2^k d)^{n/q'}\frac{d^\delta}{(2^k d)^{\delta+n/q'}} \\
&\leq C\sum_{k=1}^{\infty} 2^{k(\beta-\delta)} \leq C,
\end{aligned}$$

thus $K_1 \in H(q,\beta)$. Similarly, $K_2 \in H(2q,\beta)$ and $K_3 \in H(p_0,\beta)$. Theorem 2 follows from Main Lemma 2. QED

PROOF OF THEOREM 3. Choose $q' \leq s < p$ for T_1, $(2q)' \leq s < p$ for T_2, $p_0 \leq s < p$ for T_3 in Theorem 1, we have, by Lemma 1, 3 and 4, for $j = 1, 2, 3$,

$$\begin{aligned}
&||[b,T_j](f)||_{L^r} \leq |M([b,T_j](f))|_{L^r} \\
&\leq C|M^{\#}([b,T_j](f))|_{L^r} \\
&\leq C||b||_{Lip_\beta}\left(|M_{\beta,s}(T(f))|_{L^r} + |M_{\beta,s}(f)|_{L^r}\right) \\
&= C||b||_{Lip_\beta}\left(|M_{\beta,s}(T(f))|_{L^r} + |M_{\beta,s}(f)|_{L^r}\right) \\
&\leq C||b||_{Lip_\beta}(|T(f)|_{L^p} + |f|_{L^p}) \\
&\leq C||b||_{Lip_\beta}|f|_{L^p}.
\end{aligned}$$

This completes the proof of Theorem 3. QED

PROOF OF THEOREM 4. Choose $q' \leq s < p$ for T_1, $(2q)' \leq s < p$ for T_2, $p_0 \leq s < p$ for T_3 in Theorem 1, then, by Lemma 5-7, for $j = 1, 2, 3$,

$$\begin{aligned}
&||[b, T_j](f)||_{L^{r,\varphi}} \\
\leq\ & |M([b, T_j](f))|_{L^{r,\varphi}} \\
\leq\ & C|M^{\#}([b, T_j](f))^{\#}|_{L^{r,\varphi}} \\
\leq\ & C||b||_{Lip_\beta}\,(|M_{\beta,s}(T(f))|_{L^{r,\varphi}} + |M_{\beta,s}(f)|_{L^{r,\varphi}}) \\
=\ & C||b||_{Lip_\beta}\,(|M_{\beta,s}(T(f))|_{L^{r,\varphi}} + |M_{\beta,s}(f)|_{L^{r,\varphi}}) \\
\leq\ & C||b||_{Lip_\beta}(|T(f)|_{L^{p,\varphi}} + |f|_{L^{p,\varphi}}) \\
\leq\ & C||b||_{Lip_\beta}|f|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof of Theorem 4. $\boxed{QED}$

PROOF OF THEOREM 5. Choose $q' \leq s < p$ for T_1, $(2q)' \leq s < p$ for T_2, $p_0 \leq s < p$ for T_3 in Theorem 2, then, by using Lemma 2, we obtain, for $j = 1, 2, 3$,

$$\begin{aligned}
&||[b, T_j](f)||_{\dot{F}_p^{\beta,\infty}} \\
\leq\ & C\left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q |[b, T_j](f)(x) - T((b_{2Q} - b)f_2)(x_0)|\,dx\right|\right|_{L^p} \\
\leq\ & C||b||_{Lip_\beta}\,(|M_s(T(f))|_{L^p} + |M_s(f)|_{L^p}) \\
=\ & C||b||_{Lip_\beta}\,(|M_s(T(f))|_{L^p} + |M_s(f)|_{L^p}) \\
\leq\ & C||b||_{Lip_\beta}(|T(f)|_{L^p} + |f|_{L^p}) \\
\leq\ & C||b||_{Lip_\beta}||f||_{L^p}.
\end{aligned}$$

This completes the proof of the theorem. $\boxed{QED}$

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