Note di Matematica **20,** n. 2, 2000/2001, 77–88.

Some new classes of finite parallelisms

Norman L. Johnson
Mathematics Dept., University of Iowa
Iowa City, IA 52246 USA
njohnson@math.uiowa.edu

Received: 16 February 2001; accepted: 16 February 2001.

Abstract. Some new constructions of parallelisms in $PG(3, q)$ are given that produce parallelisms consisting of one Desarguesian spread and $q^2 + q$ derived Knuth semifield spreads and other types consisting of one Knuth semifield spread, one Hall spread and the remaining spreads are derived Knuth semifield spreads.

Keywords: parallelisms, Knuth spreads, semifields

MSC 2000 classification: 51E23, 51A40

1 Introduction

A construction of parallelisms in $PG(3, K)$ which is valid both for finite and infinite fields is given by the author in [5]. These parallelisms may be constructed over any finite or infinite field that admits a quadratic extension. The construction of these parallelisms uses a collineation group G of an associated Pappian affine plane Σ with spread in $PG(3, K)$, for K a field, that ultimately acts transitively on all but one of the spreads involved in the parallelism. Actually, G is the full central collineation group with fixed axis of the Pappian affine plane Σ and it turns out that the parallelisms are uniquely determined by such a collineation group. In these parallelisms, there is a unique Pappian spread and the remaining spreads are Hall spreads.

Actually, all known infinite classes of finite parallelisms only have the Desarguesian or Hall planes within the spread class of the parallelism. The main problem inherent in the study of parallelisms is whether any spread in $PG(3, q)$ can be a spread of some parallelism of $PG(3, q)$. Perhaps, we should merely ask if it is possible to have an infinite class of parallelisms where the spreads are not either Hall or Desarguesian.

In this article, we provide a construction technique which produces some new examples of finite parallelisms for every odd strict prime power $q = p^r$ such that r has a proper odd factor. There are two types of parallelisms and the spreads in the parallelisms are Desarguesian, Knuth semifield, Hall and derived Knuth semifield spreads.

2 The Construction

As mentioned, the construction in Johnson [5] for the parallelisms in $PG(3,q)$ uses the full central collineation group G with fixed axis ℓ of a Desarguesian affine plane acting on the parallelism. In this case, the other spreads are forced to be Hall spreads.

Theorem 1 (Johnson and Pomareda [7]). *Let K be a skewfield, Σ a spread in $PG(3,K)$ and $\mathcal{P}$ a partial parallelism of $PG(3,K)$ containing Σ.*

If $\mathcal{P}$ admits as a collineation group the full central collineation group G of Σ with a given axis ℓ that acts two-transitive on the remaining spread lines then

(1) Σ is Pappian,

(2) $\mathcal{P}$ is a parallelism,

(3) the spreads of $\mathcal{P}-\Sigma$ are Hall, and

(4) G acts transitively on the spreads of $\mathcal{P}-\Sigma$.

(5) Moreover, $\mathcal{P}$ is one of the parallelisms of the construction of Johnson.

However, if a proper subgroup G^- of G is taken so that G^- is transitive on the reguli of Σ incident with the axis of G then it may be possible to construct parallelisms which admit one Pappian spread and the remaining spreads are non-Hall and in a transitive class under G^-.

We consider the possibility that G^- contains the full elation group with axis ℓ and is transitive on the spreads not equal to the given Pappian spread Σ. Using this technique, we are able to construct several new infinite classes of parallelisms. We also consider the associated isomorphism classes of our constructed parallelisms.

2. 1 A Construction Technique

Let Σ be any Pappian spread in $PG(3,K)$ and let Σ' any spread which shares a regulus R with Σ such that Σ' is derivable with respect to R. Assume that there exists a subgroup G^- of the central collineation group G with fixed axis L with the following properties:

(0) : Σ and Σ' share exactly R,

(i) : Every line skew to L and not in Σ is in $\Sigma' G^-$,

(ii) : G^- is transitive on the reguli that share L and

(iii) : a collineation g of G^- such that for $L' \in \Sigma'$ then $L'g \in \Sigma'$ implies that g is a collineation of Σ'.

Let $(Rg)^*$ denote the opposite regulus to Rg.

Theorem 2. *Under the above assumptions,* $\Sigma \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ *for all* $g \in G^-\}$ *is a parallelism of* $PG(3, K)$ *consisting of one Pappian spread* Σ *and the remaining spreads derived* Σ'*-spreads.*

Proof. Assume that there exists two spreads $\Sigma' g$ and $\Sigma' h$ for $g, h \in G^-$ that share a common line. Then, $\Sigma' gh^{-1}$ and Σ' share a common line ℓ. But, ℓ in Σ' and $\Sigma' gh^{-1}$ implies that ℓhg^{-1} is in Σ' which implies by (iii) that $\Sigma' hg^{-1} = \Sigma'$ if and only if $\Sigma' g = \Sigma' h$. Hence, every line skew to L and not in Σ is in some unique spread $\Sigma' g$ for some $g \in G$. Any other line M is either in Σ or nontrivially intersects Σ. In this case, M is a Baer subplane of the affine plane defined by Σ and hence lies in a unique regulus R_1.

The question becomes given any regulus R_1 of Σ containing the axis of G^-, does there exist a collineation of G^- which maps R onto R_1? However, this is guaranteed by (ii).

Hence, any line which is not skew to $x = 0$ lies in some unique regulus of Σ which is then in some Rg for $g \in G^-$. Specifically, if an Baer subplane of the regulus net R and the regulus net Rg are equal then $R = Rg$ (the defined regulus net of the associated affine plane defined by Σ is unique) so that by (iii), g leaves Σ' invariant. Hence, every line of $PG(3, K)$ is uniquely covered by a spread so that a parallelism is obtained. This completes the proof of the theorem. QED

Theorem 3. *Assume that* $\Sigma \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ *for all* $g \in G^-\}$ *is a parallelism. Then* $\{\Sigma - R\} \cup R^* \cup \Sigma' \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ *for all* $g \in G^- - \{1\}\}$ *is a parallelism. In this case, the spreads are Hall,* Σ' *(undetermined) and derived* Σ' *type spreads.*

Proof. The lines covered by $\Sigma \cup \{\Sigma' - R\} \cup R^*$ are the same as covered by $\{\Sigma - R\} \cup R^* \cup \Sigma'$. Since the remaining lines are covered as in the original parallelism, it follows that we have an associated parallelism. QED

Remark 1. (1) Note that in the finite case, there are exactly $q(q+1)$ reguli that share $x = 0$ since the number is $\binom{q^2}{2} / \binom{q}{1} = q(q+1)$.Hence, the group G^- is divisible by $q(q+1)$ and $G^-_{\Sigma'}$ is a collineation group of Σ'.

(2) Let G^- be any central collineation group and G the full central collineation group with axis $x = 0$. Let G_R denote the subgroup which leaves invariant the regulus R and acts sharply doubly transitively on the remaining components.

If $(G^-/G^- \cap G_R) \simeq G/G_R$ by the mapping which takes $g(G^- \cap G_R)$ to gG_R then G^- acts transitively on the reguli of Σ that share $x = 0$.

Thus, the idea is to take G^- with the properties above and such that $G^- \cap G_R$ is as small as possible.

3 New Examples

Assume that Σ is a Desarguesian affine plane of order $p^{2r} = q^2$ where p is odd. Let γ_1 denote a nonsquare in $GF(q)$. Represent Σ with components of the following form:

$$x = 0, y = x \begin{bmatrix} u & \gamma_1 t \\ t & u \end{bmatrix} \ \forall\ u, t \in GF(q).$$

Let Σ' have spread

$$x = 0, y = x \begin{bmatrix} u & \gamma_2 t^{\sigma} \\ t & u \end{bmatrix} \ \forall\ u, t \in GF(q)$$

and σ an automorphism of $GF(q)$, where γ_2 is a nonsquare in $GF(q)$ such that

$$\gamma_2 t^{\sigma - 1} \neq \gamma_1$$

for all nonzero $t \in GF(q)$.

For $q = p^r$, let $r = 2^b z$, where it is assumed that z is odd and > 1 and assume that 2^a is the largest 2-power dividing $q-1$, written $2^a \parallel (q-1)$. In this setting, we consider those automorphisms σ defined as follows:

$$\sigma : x \longmapsto x^{p^{2^b s}}$$

where s is any factor of z, including 1.

We assert that

$$(q-1, 2^a(q+1)) = 2^a.$$

To see this, note that $(q-1, 2^a(q+1)) = 2^a((q-1)/2^a, (q+1)$ which is $2^a(z, (q+1)) = 2^a$ since $(z, q+1) = 1$ as z divides $q-1$ and is odd.

Furthermore, we assert that 2^a divides $p^{2^b s} - 1$ for any integer s. Since 2^a divides

$$\begin{aligned} q - 1 &= p^{2^b z} - 1 = (p^{2^b} - 1)(\sum_{i=o}^{z-1} p^{2^b i}) \\ &= (p^{2^b} - 1)(p^{2^a(z-1)} + \sum_{j=0}^{\frac{(z-3)}{2}} p^{2^{b+1} j}(1 + p^{2^b})) \end{aligned}$$

and since $(1 + p^{2^b})$ is even and p is odd, it follows that 2^a divides $p^{2^b} - 1$ which, in turn divides $p^{2^b s} - 1$ for any integer s.

Since $2^a \mid p^{2^b s} - 1$, we also shall use the notation that

$$2^a \mid (\sigma - 1).$$

Let E^+ denote the full elation group of Σ with axis $x = 0$, and let H denote the homology group with axis $x = 0$ and coaxis $y = 0$ of order $2^a(q + 1)$ where $2^a \parallel (q - 1)$, as above. We note that since Σ is a Desarguesian affine plane of order q^2, and $2^a(q + 1)$ divides $q^2 - 1$ if and only if 2^a divides $q - 1$, we have a cyclic homology group of order $2^a(q + 1)$ in the Desarguesian plane. We let $G^- = EH$.

Hence, the elements of E^+ have the following form

$$: \begin{bmatrix} 1 & 0 & u & \gamma_1 t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the elements of H have the form:

$$\tau_{w,s} = \begin{bmatrix} w & \gamma_1 s & 0 & 0 \\ s & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the upper 2×2 matrix is nonzero of order dividing $2^a(q + 1)$.

Hence, $s = 0$ if and only if the element has order dividing $q-1$ and $2^a(q+1)$. Since

$$(q - 1, 2^a(q + 1)) = 2^a,$$

it follows that the order of this element divides both $q - 1$ and $2^a(q + 1)$ if and only if $w^{2^a} = 1$.

Now note that when this occurs, then $\tau_{w,0}$ is a collineation of the Knuth semifield plane listed above, since

$$\begin{aligned} y &= x \begin{bmatrix} u & \gamma_2 t^\sigma \\ t & u \end{bmatrix} \longmapsto y = x \begin{bmatrix} w^{-1} & 0 \\ 0 & w^{-1} \end{bmatrix} \begin{bmatrix} u & \gamma_2 t^\sigma \\ t & u \end{bmatrix} \\ &= (y = x \begin{bmatrix} uw^{-1} & \gamma_2 t^\sigma w^{-1} \\ tw^{-1} & uw^{-1} \end{bmatrix}) = (y = x \begin{bmatrix} uw^{-1} & \gamma_2 (tw^{-1})^\sigma \\ tw^{-1} & uw^{-1} \end{bmatrix}) \end{aligned}$$

since $w^{-1} = (w^{-1})^\sigma$ if and only if $w^{\sigma-1} = 1$ which is valid since 2^a divides $\sigma - 1$.

Note that Σ and Σ' share $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ but since $\gamma_2 t^{\sigma-1} \neq \gamma_1$ for non-zero t in $GF(q)$, they share exactly these components.

Now assume that $\Sigma' g$ and Σ' share a line ℓ not in Σ.

Let $g = \begin{bmatrix} w & \gamma_1 s & 0 & 0 \\ s & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & m & \gamma_1 r \\ 0 & 1 & r & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which maps the spread Σ':

$$x = 0, y = x \begin{bmatrix} u & \gamma_2 t^\sigma \\ t & u \end{bmatrix} \ \forall\ u, t \in GF(q)$$

onto the spread $\Sigma' g$:

$$x = 0, y = x \begin{bmatrix} w & \gamma_1 s \\ s & w \end{bmatrix}^{-1} \left(\begin{bmatrix} u & \gamma_2 t^\sigma \\ t & u \end{bmatrix} + \begin{bmatrix} m & \gamma_1 r \\ r & m \end{bmatrix} \right).$$

The component (line) ℓ is common to Σ' and $\Sigma' g$, which is not in Σ, if and only there exists a nonzero t^* and u^* in $GF(q)$ such that ℓ has the equation, for nonzero t^*:

$y = x \begin{bmatrix} u^* & \gamma_2 t^{*\sigma} \\ t^* & u^* \end{bmatrix}$ is in $\Sigma' g$ of the above form.

Hence, we have the following implications:

$$\begin{bmatrix} u^* & \gamma_2 t^{*\sigma} \\ t^* & u^* \end{bmatrix} = \begin{bmatrix} w & \gamma_1 s \\ s & w \end{bmatrix}^{-1} \begin{bmatrix} u+m & \gamma_2 t^\sigma + \gamma_1 r \\ t+r & u+m \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} w & \gamma_1 s \\ s & w \end{bmatrix} \begin{bmatrix} u^* & \gamma_2 t^{*\sigma} \\ t^* & u^* \end{bmatrix} = \begin{bmatrix} u+m & \gamma_2 t^\sigma + \gamma_1 r \\ t+r & u+m \end{bmatrix}.$$

Thus, we have:

$$\begin{bmatrix} wu^* + \gamma_1 s t^* & w\gamma_2 t^{*\sigma} + \gamma_1 s u^* \\ su^* + wt^* & s\gamma_2 t^{*\sigma} + wu^* \end{bmatrix} = \begin{bmatrix} u+m & \gamma_2 t^\sigma + \gamma_1 r \\ t+r & u+m \end{bmatrix}.$$

Hence, we must have the $(1,1)$ and $(2,2)$ entries on the left hand side matrix equal which implies that $\gamma_1 s t^* = s\gamma_2 t^{*\sigma}$. Hence, we must have that $s = 0$. Then $w^{2^a} = 1$.

Thus, using the $(1,2)$-entries, we have

$$t + r = wt^* \text{ and } \gamma_2 t^\sigma + \gamma_1 r = w\gamma_2 t^{*\sigma} = w\gamma_2 (\frac{t+r}{w})^\sigma = w^{1-\sigma}\gamma_2 (t+r)^\sigma.$$

Since $w^{1-\sigma} = 1$, it then follows that $\gamma_1 r = \gamma_2 r^\sigma$ which implies that $r = 0$. When $r = 0$ and $s = 0$, we have the group element

$$\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & m & 0 \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and since both of the elements of the product are collineations of Σ', it follows that g is a collineation of Σ'.

In this case, we see that g must leave invariant Σ' and act as a collineation. Hence, if Σ' and $\Sigma' g$ share a common component not in Σ, then $\Sigma' = \Sigma' g$. So, assume that $\Sigma' h$ and $\Sigma' j$ for $h, j \in E^+H$, share a line M not in Σ.

Hence, Σ' and $\Sigma' jh^{-1}$ for $jh^{-1} \in E^+H$ share a line Mh^{-1} which is not in Σ since E^+H is a collineation group of Σ. Thus, $\Sigma' = \Sigma' jh^{-1}$ if and only if $\Sigma' h = \Sigma' j$ if and only if $\Sigma' h$ and $\Sigma' j$ share a common line not in Σ.

Now we derive each plane $\Sigma' g$ by Rg where R is the regulus

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \ \forall\, u \in GF(q).$$

There are exactly $q(q+1)$ reguli in the spread for Σ which share $x = 0$. Suppose some element in E^+H fixes one of these. Without loss of generality, we may assume that it is R. Then, clearly, $g \in E\langle \rho \rangle$ where E has elements $\begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\rho = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for a primitive 2^a-th. element w. In other words, the stabilizer of R has order $2^a q$. Hence, the $q(q+1)$ reguli are in an orbit under E^+H.

Thus, we have q^2+q distinct spreads $\Sigma' g$ for $g \in E^+H$ and each such spread shares the regulus Rg with Σ. Since the reguli all share $x = 0$ and are in the spread for Σ, the regulus nets can share no common Baer subplane incident with the zero vector.

Hence, the $q(q+1)^2$ lines which lie in the opposite reguli lie exactly in the $q(q+1)$ derived spreads so that the derived spreads share no line which is not in Σ and share no line since if so such a line would be a Baer subplane of some opposite reguli which lies in exactly one derived plane.

Hence, we have a parallelism of q^2+q planes of derived Knuth type and one Desarguesian plane.

Theorem 4. *Let q be odd equal to $p^{2^b z}$ where z is an odd integer > 1. Assume that $2^a \parallel (q-1)$ then there exists a nonidentity automorphism σ of $GF(q)$ such that $2^a \mid (\sigma - 1)$.*

Let γ_2 and γ_1 be nonsquares of $GF(q)$ such that the equation $\gamma_2 t^\sigma = \gamma_1 t$ implies that $t = 0$.

(1) Then, there exists a parallelism $\mathcal{P}_{\gamma_2,\sigma}$ of derived Knuth type with q^2+q derived Knuth planes and one Desarguesian plane.

(2) The collineation group of this parallelism contains the central collineation group of the Desarguesian plane with fixed axis ℓ of order $q^2 2^a (q+1)$.

3. 1 The Isomorphisms

The full collineation group of a Knuth plane is completely determined in Johnson and Liu [6]. Furthermore, the collineation group of any derived semifield plane is known to be the group inherited from the semifield plane provided the order is not 9 or 16 (see Johnson [4]).

A Knuth spread is covered by a set of q reguli that mutually share a line and it is known that the collineation group acts 2-transitively on this set of q reguli.

Hence, we may assume that any isomorphism between Knuth spreads maps the standard regulus to the standard regulus, maps any given regulus of the first Knuth spread to any given regulus of the second Knuth spread and fixes a component corresponding to the elation axis.

notation 1. We represent the Desarguesian spread by

$$x = 0, y = x \begin{bmatrix} u & \gamma_0 t \\ t & u \end{bmatrix} \forall u, t \in GF(q)$$

and γ_o is a fixed nonsquare and the Knuth semifield spreads $\pi_1$and π_2 by

$$x = 0, y = x \begin{bmatrix} u & \gamma_i t^{\sigma_i} \\ t & u \end{bmatrix} \forall u, t \in GF(q),$$

for $i = 1, 2$ respectively, where γ_i are nonsquares and σ_i are automorphisms of $GF(q)$.

Theorem 5. *Two such parallelisms $\mathcal{P}_{\gamma_1,\sigma_1}$ and $\mathcal{P}_{\gamma_2,\sigma_2}$ are isomorphic if and only if one of the following two conditions hold:*

$$(\gamma_2, \sigma_2) = (\frac{\gamma_o^2}{\gamma_1^\rho}, \sigma_1^{-1}) \textit{ or } (\gamma_2, \sigma_2) = (\gamma_1^\rho, \sigma_1)$$

for some automorphism ρ of $GF(q)$.

Proof. If the parallelisms are isomorphic by an element τ of $\Gamma L(4, q)$ then this element is a collineation of the unique Desarguesian spread Σ which fixes a regulus R of Σ shared by the Knuth semifield spreads π_1 and π_2 defined as follows:

$$x = 0, y = x \begin{bmatrix} u & \gamma_i t^{\sigma_i} \\ t & u \end{bmatrix} \forall u, t \in GF(q),$$

for $i = 1, 2$ respectively. An isomorphism τ must fix $x = 0$ where we are assuming that R is the standard regulus (net). Since we have an elation group acting transitively on the nonaxis lines (components) of the regulus, it follows that we may assume that τ fixes $x = 0, y = 0$, fixes the standard regulus net and maps the net when $t = 1$ in π_1 to the net for $t = 1$ in π_2.

Note that our construction requires that the Desarguesian spread be taken in a given manner. Hence, we let Σ have spread:

$$x=0, y=x\begin{bmatrix} u & \gamma_0 t \\ t & u \end{bmatrix} \forall u,t\in GF(q)$$

and γ_o is a fixed nonsquare.

Let $A=\begin{bmatrix} v & \gamma_0 s \\ s & v \end{bmatrix}$ and consider that τ may be taken to have the following form:

$$\tau:(x_1,x_2,y_1,y_2)\longmapsto(x_1^\rho,x_2^\rho,y_1^\rho,y_2^\rho)\begin{bmatrix} A & 0 \\ 0 & Au_oI \end{bmatrix}.$$

Then, we must have:

$$\begin{bmatrix} v & \gamma_0 s \\ s & v \end{bmatrix}^{-1}\begin{bmatrix} u^\rho & \gamma_1^\rho \\ 1 & u^\rho \end{bmatrix}\begin{bmatrix} v & \gamma_0 s \\ s & v \end{bmatrix}u_oI$$

$$= \begin{bmatrix} (vu^\rho-\gamma_o s)v+s(v\gamma_1^\rho-\gamma_o su^\rho) & (vu^\rho-\gamma_o s)\gamma_o s+(v\gamma_1^\rho-\gamma_o su^{gr})v \\ v(-su^\rho+v)+s(-s\gamma_1^\rho+vu^\rho) & \gamma_o s(-su^\rho+v)+v(-s\gamma_1^\rho+vu^\rho) \end{bmatrix}$$

must have the general form $\begin{bmatrix} u_2 & \gamma_2^{\sigma_2} \\ 1 & u_2 \end{bmatrix}$ for all $u\in GF(q)$. Equating the $(1,1)$- and $(2,2)$-entries, we obtain

$$sv\gamma_1^\rho=-sv\gamma_1^\rho.$$

Hence, we must have either s or $v=0$.

First assume that $v=0$. Since we have a 1 in the $(1,2)$-entry, we must have:

$$u_o=\frac{\gamma_o}{\gamma_o^\rho}.$$

It then follows immediately that

$$\frac{\gamma_o^2}{\gamma_o^\rho}=\gamma_2.$$

Since $y=x\begin{bmatrix} 0 & \gamma_1 t^{\sigma_1} \\ t & 0 \end{bmatrix}$ then onto $y=x\begin{bmatrix} 0 & \frac{\gamma_o^2}{\gamma_o^\rho}t^\rho \\ t^\rho\sigma_1 & v \end{bmatrix}$, it then follows that $\sigma_2=\sigma_1^{-1}$. Hence, in this case, we must have

$$(\gamma_2,\sigma_2)=(\frac{\gamma_o^2}{\gamma_o^\rho},\sigma_1^{-1}).$$

Now assume that $s=0$. It then follows similarly that

$$\gamma_2=\gamma_1^\rho \text{ and } \sigma_1=\sigma_2.$$

QED

The above result allows the enumeration of the isomorphism classes albeit a bit cumbersome. For example, the following provides the idea.

Corollary 1. *Assume that* $q \equiv -1 \bmod 4$. *Let* $q = p^{st}$, *choose any proper automorphism* $\sigma = p^s$ *and nonsquares* γ_o *and* γ *such that*

$$\gamma_o t^{\sigma-1} \neq \gamma \ \forall t \in GF(q).$$

Then, there exist at least

$$[\frac{(q-1) - (\frac{(q-1)}{(p^s-1)} - 1)}{2st}]$$

mutually nonisomorphic parallelisms $\mathcal{P}_{\gamma,\sigma}$.

PROOF. We note that we obtain a parallelism as in our main theorem since $2 \parallel (q-1)$ and $2 \mid (\sigma-1)$ for any automorphism σ. There are $(q-1)/2$ nonsquare elements γ and there are $(q-1)/2(p^s-1)$ nonsquare elements among the set of elements $\{t^{\sigma-1}; t \in GF(q) - \{0\}\}$. We may use the same automorphism and obtain at least the number of parallelisms as orbits under the Galois group. The number indicated in the result is the minimum number of such orbits. QED

4 Further constructions

Let $\mathcal{P}$ be any Knuth type parallelism as above. Take the regulus R within the planes Σ and Σ'. Let R^* be the opposite regulus within the derived plane of Σ', Σ'^*. Now derive R within Σ and R^* within Σ'^* to produce the Hall plane Σ^* and the Knuth plane Σ' leaving the remaining derived Knuth planes.

Theorem 6. *For each parallelism of type* $\mathcal{P}_{\gamma,\sigma}$, *there is a parallelism consisting of one Hall spread, one Knuth semifield spread, and* $q^2 + q - 1$ *derived Knuth spreads. We shall call such parallelisms the 'derived' parallelisms of the* $\mathcal{P}_{\gamma,\sigma}$*-parallelisms.*

We may discuss isomorphisms in this setting as well. We note that any isomorphism is a collineation of the Hall plane and for $q > 3$, it is known that the full collineation group of the Hall plane is a collineation group of the associated Desarguesian plane from which it was derived. Furthermore, the unique Knuth semifield plane must map to the corresponding Knuth semifield plane so we have the same situation as before.

Theorem 7. *Given two parallelisms* $\mathcal{P}_{\gamma_i,\sigma_i}$ *for* $i = 1, 2$. *Then, the derived parallelisms are isomorphic if and only if the original parallelisms are isomorphic.*

5 A Partial Characterization

We have given some examples of parallelisms using Knuth semifield planes of what is called 'flock type'. By this, it is meant that the translation planes correspond to a flock of a quadratic cone. In the following theorem, we show that in the circumstances of our construction technique, all of the non-Desarguesian spreads are derived conical flock spreads.

Theorem 8. *Let $\mathcal{P}$ be a parallelism in $PG(3,K)$, for K a field, that admits a Pappian spread Σ and a central collineation group G^- with axis ℓ of Σ which acts transitively on the remaining spreads of $\mathcal{P}$.*

(1) If K is finite and if G^- contains the full elation group with axis ℓ then the spreads of $\mathcal{P}-\Sigma$ are derived conical flock spreads.

*(2) If G^- contains the full elation group with axis ℓ and for ρ a spread of $\mathcal{P}-\Sigma$, G^-_ρ contains a non-trivial homology (*i. e. *homology in Σ) then the spreads of $\mathcal{P}-\Sigma$ are derived conical flock spreads.*

Proof. First assume that K is finite. If G^- contains the full elation group of order q^2 and G^- is transitive then given any spread ρ, there is an elation group of order q fixing ρ. This elation group acts on the associated translation plane π_ρ as a collineation group which fixes a Baer subplane pointwise. It follows by applying the results of Johnson [3] and Payne and Thas [8] that the net defined by the Baer subplane ℓ is a regulus net. It furthermore follows from Johnson [3] that the derived plane corresponds to a flock of a quadratic cone in $PG(3,q)$.

If K is not necessarily finite, the above results would follow if we could show that the net defined by ℓ in π_ρ is, actually, a regulus net. If G^-_ρ admits a homology group then there is a Baer group which fixes a second Baer subplane that is forced to lie within the net defined by ℓ (see the analysis of Jha–Johnson [2]). The elation subgroup in G^-_ρ is then transitive on non-fixed 1-dimensional K-subspaces of components that lie on ℓ in π_ρ. That is, this elation subgroup as a Baer group maps a second Baer subplane onto Baer subplanes that cover the net in question. That is, the net is forced into being a regulus net. The plane is derivable and the derived plane must correspond to a flock of a quadratic cone just as in the finite case. QED

5. 1 The Algebraic Construction

Under the assumptions given, the spreads of the parallelisms constructed via groups G^- are derived conical flock spreads. Hence, we may represent the associated conical flock spreads via two functions f and g on K as follows:

Theorem 9. *(Jha and Johnson [2] and Gevaert and Johnson [1] and Johnson and Liu [6])*

Any spread in $PG(3,K)$, for K a field, which corresponds to a conical flock may be represented as follows:
$x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}$ *for all $u,t \in K$ where f and g are functions on K such that the function ϕ_s defined by*

$$\phi_s(t) = ts^2 + g(t)s + f(t)$$

is bijective on K for each s.

Suppose that $(g(-t), f(-t)) \neq (-g(t), -f(t))$. That is, the homology group of order 2: $(x,y) \longmapsto (-x,y)$ then 'moves' the spread but fixes the regulus and this produces problems. Since we need a homology group of order divisible by $q+1$, then, when q is odd, this procedure would not produce parallelisms as $(x,y) \longmapsto (-x,y)$ would not be a collineation of the spread in question. Hence, we can only use functions g and f such that $(g(-t), f(-t)) = (-g(t), -f(t))$.

More generally, if $2^a \parallel (q-1)$ we would require that

$$(g(w), f(w)) = (wg(1), wf(1)) \ \forall w \in GF(q), \text{ for } w^{2^a} = 1.$$

However it might be possible to use other functions when $(q+1)/2$ is odd by using two pairs of functions. For example, one might use two Desarguesian spreads and a Knuth spread initially with a group of order $q^2(q+1)/2$ ultimately producing a parallelisms with one Desarguesian spread, $(q(q+1)/2)$ Hall spreads and $q(q+1)/2$ derived Knuth spreads.

References

[1] H. Gevaert, N. L. Johnson: *Flocks of quadratic cones, generalized quadrangles, and translation planes*, Geom. Dedicata **27** (1988), 301–317.

[2] V. Jha, N. L. Johnson: *Conical, ruled, and deficiency one translation planes*, Bull. Belgian Math. Soc. **6** (1999), 187–218.

[3] N. L. Johnson: *Translation planes admitting Baer groups and partial flocks of quadric sets*, Simon Stevin **63** (1989), 163–187.

[4] N. L. Johnson: *The derivation of dual translation planes*, J. Geom. 36 (1989), 63–90.

[5] N. L. Johnson: Subplane Covered Nets, Monographs/Textbooks in Pure and Applied Mathematics, vol. 222, Marcel Dekker, New York 2000.

[6] N. L. Johnson, X. Liu: *The generalized Kantor-Knuth flocks*, Mostly finite geometries, Lecture Notes in Pure and Applied Math., Vol. 190, Marcel Dekker. New York-Basil-Hong Kong, (1997), 305–314.

[7] N. L. Johnson, R. Pomareda: *Transitive Partial Parallelisms of Deficiency One*, preprint.

[8] S. E. Payne, J.A. Thas: *Conical flocks, partial flocks, derivation and generalized quadrangles*, Geom. Dedicata **38** (1991), 229–243.