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Totally Contact Umbilical Screen Transversal Lightlike Submanifolds of an Indefinite Sasakian Manifold

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Abstract. In this paper, we study totally contact umbilical radical screen transversal and screen transversal anti-invariant lightlike submanifolds of an indefinite Sasakian manifold and obtain a geometric condition under which the induced connection ∇ on a totally contact umbilical radical screen transversal lightlike submanifold is a metric connection. We prove a classification theorem for totally contact umbilical screen transversal anti-invariant lightlike submanifolds when the ambient space is an indefinite Sasakian manifold.

Keywords: Indefinite Sasakian manifolds, lightlike submanifolds, totally contact umbilical radical ST-lightlike submanifolds, totally contact umbilical ST-anti-invariant lightlike submanifolds

MSC 2000 classification: primary 53C40, secondary 53C15, 53C50

Introduction

In view of Nash's theorem, one can consider a Riemannian manifold as a submanifold of Euclidean space which provides us a natural motivation for the study of submanifolds of Riemannian or semi-Riemannian manifolds. It is known that the geometry of lightlike submanifolds of semi-Riemannian manifolds is different from the geometry of submanifolds immersed in a Riemannian manifold since the normal vector bundle of lightlike submanifolds intersect with tangent bundle making it more interesting to study. The general theory of lightlike submanifolds of a semi-Riemannian manifold has been developed by Duggal-Bejancu[7] and

Kupeli[5]. On the other hand, transversal lightlike submanifolds of an indefinite Kaehler manifold were introduced by B. Sahin [1] and studied some differential geometric properties of such submanifold. In [2], B.Sahin initiated the study of screen transversal lightlike submanifolds of indefinite Kaehler manifolds and obtained a characterization of screen transversal anti-invariant lightlike submanifolds apart from other properties of this submanifold. Transversal lightlike submanifolds in Sasakian setting were studied by Yildirim and Sahin[3] whereas, screen transversal lightlike submanifolds of indefinite Sasakian manifolds were investigated by [10] and [4]. In this paper, we study totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold.

This paper is arranged as follows. In sections 2 and 3, we give the basic concepts on lightlike submanifolds and indefinite Sasakian manifolds needed for this paper. In section 4, we study the integrability of distributions involved in the definition of totally contact umbilical radical screen transversal lightlike submanifolds and obtain a condition under which the induced connection ∇ on totally contact umbilical radical screen transversal lightlike submanifolds to be metric connection. In section 5, we prove a theorem which shows that the induced connection ∇ on a totally contact umbilical ST-anti-invariant lightlike submanifold is a metric connection under some conditions. We also prove a classification theorem of totally contact umbilical screen transversal anti-invariant lightlike submanifold immersed in an indefinite Sasakian manifold.

1 Preliminaries

We follow [7] for the notation and fundamental equation for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\overline{g}$ and radical distribution Rad TM is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of Rad TM in TM, i.e,

$$TM = \text{Rad } TM \perp S(TM)$$

Consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of Rad TM in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of Rad TM, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$ [[7],pg-144]. Let $tr(TM)$ be complementary(but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\overline{M}|_M = S(TM)\perp[\mathrm{Rad}\,TM \oplus ltr(TM)]\perp S(TM^\perp).$$

Following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^\perp))$.
Case 1: r-lightlike if $r < min\{m, n\}$.
Case 2: Co-isotropic if $r = n < m; S(TM^\perp) = 0$.
Case 3: Isotropic if $r = m < n; S(TM) = 0$.
Case 4: Totally lightlike if $r = n = m, S(TM) = 0 = S(TM^\perp)$.
The Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \forall X, Y \in \Gamma(TM) \tag{1}$$

and

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U,\ \forall X, Y \in \Gamma(TM), U \in \Gamma(tr(TM)). \tag{2}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connection on M and on the vector bundle $tr(TM)$, respectively. Moreover,we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \tag{3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N) \tag{4}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W) \tag{5}$$

$\forall X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. Denote the projection of TM on S(TM) by P. Then, by using (1),(3)-(5) and the fact that $\overline{\nabla}$ is a metric connection, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{6}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X).$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{7}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi,$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\ TM)$.
It is important to note that the induced connection ∇ on M is not a metric connection whereas ∇^* and ∇^{*t} are metric connections on $S(TM)$ and Rad TM respectively.
The Gauss equation of a lightlike submanifold is given by

$$\begin{aligned}\overline{R}(X, Y)Z = R(X, Y)Z &+ A_{h^l(X,Z)}Y + A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y\\ &- A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z)\\ &+ D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z)\\ &- (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z))\end{aligned} \tag{8}$$

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifolds of $(\overline{M}, \overline{g})$. For any vector field X tangent to M, we put

$$\phi X = fX + \omega X \tag{9}$$

where fX and ωX are the tangential and transversal parts of ϕX respectively. For $V \in \Gamma(tr(TM))$

$$\phi V = BV + CV \tag{10}$$

where BV and CV are the tangential and transversal parts of ϕV respectively.

2 Indefinite Sasakian Manifolds

An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called a contact metric manifold [6] if there exists a (1,1) tensor field ϕ, a vector field V, called the characteristic vector field, and its 1-form η satisfying

$$\begin{cases} \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \overline{g}(V, V) = \epsilon \\ \phi^2 X = -X + \eta(X)V, \overline{g}(X, V) = \epsilon\eta(X), \\ d\eta(X, Y) = \overline{g}(X, \phi Y), \forall X, Y \in \Gamma(T\overline{M}), \end{cases} \tag{11}$$

where $\epsilon = \pm 1$. One can easily verify that $\phi V = 0,\ \eta o \phi = 0,\ \eta(V) = \epsilon$.
Then $(\phi, V, \eta, \overline{g})$ is called a contact metric structure of $\overline{M}$. The semi-Riemannian manifold $\overline{M}$ is said to have a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [9]. A normal contact metric manifold is called a Sasakian manifold [[9],[11]] for which we have

$$(\overline{\nabla}_X \phi)Y = \overline{g}(X, Y)V - \epsilon\eta(Y)X. \tag{12}$$

$$\overline{\nabla}_X V = -\phi X, \tag{13}$$

From [11], we recall that the curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z = {} & \frac{c+3}{4}\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\} + \frac{c-1}{4}\{\epsilon\eta(X)\eta(Z)Y - \epsilon\eta(Y)\eta(Z)X \\ & + \overline{g}(X, Z)\eta(Y)V - \overline{g}(Y, Z)\eta(X)V + \overline{g}(\phi Y, Z)\phi X + \overline{g}(\phi Z, X)\phi Y \\ & - 2\overline{g}(\phi X, Y)\phi Z\} \end{aligned} \tag{14}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. Without loss of generality, we take $\epsilon = 1$ throughout this paper.

3 Totally Contact Umbilical Radical ST-Lightlike Submanifolds

In the present section, we study totally contact umbilical radical ST-lightlike submanifolds of an indefinite Sasakian manifold. We need the following definitions from [2] for later use.

Definition 1. A r-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is said to be a screen transversal(ST)lightlike submanifold of $\overline{M}$ if there exists a screen transversal bundle $S(TM^{\perp})$ such that

$$\phi(\text{Rad } TM) \subset S(TM^{\perp}).$$

Definition 2. A ST-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is said to be a radical ST-lightlike submanifold if $S(TM)$ is invariant with respect to ϕ.

Now, we recall the definition of totally contact umbilical lightlike submanifolds of an indefinite Sasakian manifold.

Definition 3. [8] A lightlike submanifold (M, g) of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ is said to be totally contact umbilical if there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$\begin{cases} h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H^l + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \\ h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H^s + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V) \end{cases} \tag{15}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(S(TM))$.

The conditions under which the distributions involved in the definition of totally contact umbilical radical ST-lightlike submanifolds immersed in an indefinite Sasakian manifold is given by the following Theorem.

Theorem 1. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold. Then the screen distribution $S(TM)$ is integrable if and only if H^s has no component in $\phi(\text{Rad } TM)$.*

Proof. Using (3), (11)and (12), for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$, after calculations, we obtain

$$\overline{g}([X,Y],N) = \overline{g}(h^s(X,\phi Y) - h^s(Y,\phi X), \phi N). \tag{16}$$

From (3), (11),(13), (15) and (16), we get

$$\overline{g}([X,Y],N) = 2g(X,\phi Y)\overline{g}(H^s,\phi N),$$

from which our assertion follows.

Theorem 2. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold. Then the distribution* Rad TM *is always integrable.*

Proof. For $Z, W \in \Gamma(\text{Rad } TM)$ and $X \in \Gamma(S(TM))$, from (3),(11) and (12) we have

$$g([Z,W],X) = -\overline{g}(h^s(Z,\phi X),\phi W) + \overline{g}(h^s(W,\phi X),\phi Z). \tag{17}$$

Making use of (15) in (17), we arrive at

$$g([Z,W],X) = 0,$$

which proves our assertion.

Theorem 3. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $h^*(X,Y) = 0$ if and only if H^s has no components in $\phi(\text{Rad } TM)$ for any $X, Y \in \Gamma(S(TM))$.*

Proof. Using (3) and (12), a direct calculation shows that

$$\begin{aligned} \nabla_X \phi Y + h^l(X,\phi Y) + h^s(X,\phi Y) = g(X,Y)V - \eta(Y)X + \phi\nabla_X Y + \phi h^l(X,Y) \\ + \phi h^s(X,Y) \end{aligned} \tag{18}$$

for any $X, Y \in \Gamma(S(TM))$. Taking inner product of (18) with ϕN for any $N \in \Gamma(ltr(TM)))$ and using (11), we obtain

$$\overline{g}(h^s(X,\phi Y),\phi N) = \overline{g}(\nabla_X Y, N). \tag{19}$$

From (7),(15) and (19), we have

$$g(X,\phi Y)\overline{g}(H^s,\phi N) = \overline{g}(h^*(X,Y),N),$$

which proves our assertion.

The necessary and sufficient conditions for the induced connection on a totally contact umbilical radical ST-lightlike submanifold to be a metric connection is given by the following theorem.

Theorem 4. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if H^s has no component in $\phi(ltr(TM))$.*

Proof. For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad } TM)$, from (11) and (12) we obtain

$$\overline{\nabla}_X \xi = -\phi(\overline{\nabla}_X \phi\xi) \tag{20}$$

Using (3),(5) and (15) in (20), we arrive at

$$\nabla_X \xi + h^l(X,\xi) + h^s(X,\xi) = \phi A_{\phi\xi} X - \phi(\nabla^s_X \phi\xi) - \phi(D^l(X,\phi\xi)). \tag{21}$$

Taking inner product of (21) with $Y \in \Gamma(S(TM))$ and using (15), we obtain

$$g(\nabla_X \xi, Y) = -g(X,\phi Y)\overline{g}(H^s,\phi\xi). \tag{22}$$

Thus, our assertion follows from (22).

Corollary 1. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the distribution* Rad TM *is always parallel.*

Proof. Using (11) and (12), we obtain

$$\overline{\nabla}_{\xi_1}\xi_2 = -\phi(\overline{\nabla}_{\xi_1}\phi\xi_2) \tag{23}$$

for any $\xi_1, \xi_2 \in \Gamma(\text{Rad } TM)$. From (3),(5), (15) and (23), we have

$$\nabla_{\xi_1}\xi_2 + h^l(\xi_1,\xi_2) + h^s(\xi_1,\xi_2) = \phi A_{\phi\xi_1}\xi_2 - \phi(\nabla^s_{\xi_1}\phi\xi_2) - \phi(D^l(\xi_1,\phi\xi_2)). \tag{24}$$

Taking inner product of (24)with $Y \in \Gamma(S(TM))$ and using (15), a direct calculation shows that

$$g(\nabla_{\xi_1}\xi_2, Y) = 0,$$

which proves our assertion.

Now, we prove a lemma which we shall use in the study of totally contact umbilical radical ST-lightlike submanifolds.

Lemma 1. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then*
(i) $A_{\phi N}X = \{X - \eta(X)V\}\overline{g}(H^s,\phi N) - \eta(X)N + D^l(X,\phi N)$
for any $X \in \Gamma(S(TM))$ *and* $N \in \Gamma(ltr(TM))$.
(ii) $A_{\phi N}X = -g(X,N)V + D^l(X,\phi N)$
for any $X \in \Gamma(\text{Rad}\, TM)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Replacing W by ϕN in (6) and using (15), we get

$$\begin{aligned} g(A_{\phi N}X,Y) = {} & \{g(X,Y) - \eta(X)\eta(Y)\}\overline{g}(H^s,\phi N) + \eta(X)\overline{g}(h^s(Y,V),\phi N) \\ & + \eta(Y)\overline{g}(h^s(X,V),\phi N) + \overline{g}(D^l(X,\phi N),Y) \end{aligned} \tag{25}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$.
If we restrict X to $(S(TM))$, then (i) follows form (3),(13) and (25) but if $X \in \Gamma(\text{Rad } TM)$, then assertion (ii) follows from (3),(13) and (25).

It is known that the induced connection ∇^t on a lightlike submanifold of a semi-Riemannian manifold is not a metric connection. The condition under which the induced connection ∇^t on a totally contact umbilical radical ST-lightlike submanifold immersed in indefinite Sasakian manifolds to be a metric connection on $tr(TM)$ is given by the following theorem:

Theorem 5. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then ∇^t is a metric connection on $tr(TM)$ if and only if $\nabla^s_X \phi N$ has no component in μ and $D^l(X, \phi N) = 0$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$.*

Proof. Using (2),(6),(10),(11) and (12), for $X \in \Gamma(TM), W \in \Gamma(S(TM^\perp))$ and $N \in \Gamma(ltr(TM))$, we arrive at

$$\overline{g}(\nabla^t_X N, W) = \overline{g}(-A_{\phi N}X + \nabla^s_X \phi N + D^l(X, \phi N), BW + C_1W + C_2W) \quad (26)$$

where $BW \in \Gamma(\text{Rad } TM), C_1W \in \Gamma(ltr(TM))$ and $C_2W \in \Gamma(\mu)$. Taking $X \in \Gamma(\text{Rad } TM)$ and using part (ii) of Lemma 1 and (26) we get

$$\overline{g}(\nabla^t_X N, W) = \overline{g}(\nabla^s_X \phi N, C_2W) + \overline{g}(D^l(X, \phi N), BW). \quad (27)$$

On the other hand, if we take $X \in \Gamma(S(TM))$, then from part (i) of Lemma 1 and (26) we conclude that

$$\overline{g}(\nabla^t_X N, W) = \overline{g}(\nabla^s_X \phi N, C_2W) + \overline{g}(D^l(X, \phi N), BW). \quad (28)$$

Thus, our assertion follows from (27) and (28).

Theorem 6. *Let M be a totally contact umbilical radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then*

$$A_{\phi\xi_1}\xi_2 = A_{\phi\xi_2}\xi_1$$

for all $\xi_1, \xi_2 \in \Gamma(\text{Rad } TM)$.

Proof. For $\xi_1, \xi_2 \in \Gamma(\text{Rad } TM)$, from (12) we have

$$\phi\overline{\nabla}_{\xi_1}\xi_2 = \overline{\nabla}_{\xi_1}\phi\xi_2.$$

Using (3)and(5) in the above equation, a direct calculation shows that

$$\phi\nabla_{\xi_1}\xi_2 + \phi h^l(\xi_1, \xi_2) + \phi h^s(\xi_1, \xi_2) = -A_{\phi\xi_2}\xi_1 + \nabla^s_{\xi_1}\phi\xi_2 + D^l(\xi_1, \phi\xi_2). \quad (29)$$

Interchanging ξ_1 and ξ_2 in (29) and then subtracting the resulting equation from (29), we obtain

$$\phi\nabla_{\xi_1}\xi_2 - \phi\nabla_{\xi_2}\xi_1 = -A_{\phi\xi_2}\xi_1 + A_{\phi\xi_1}\xi_2 + \nabla^s_{\xi_1}\phi\xi_2 - \nabla^s_{\xi_2}\phi\xi_1 + D^l(\xi_1, \phi\xi_2) - D^l(\xi_2, \phi\xi_1). \quad (30)$$

Taking inner product of (30) with $X \in \Gamma(S(TM))$ and taking into account that the ambient space $\overline{M}$ is Sasakian, we get

$$g(\nabla_{\xi_1}\xi_2, \phi X) - g(\nabla_{\xi_2}\xi_1, \phi X) = g(A_{\phi\xi_1}\xi_2 - A_{\phi\xi_2}\xi_1, X). \tag{31}$$

On the other hand, from (3) and (15), after calculations we obtain

$$g(\nabla_{\xi_1}\xi_2, \phi X) = 0, \ \ g(\nabla_{\xi_2}\xi_1, \phi X) = 0. \tag{32}$$

Using (32) in (31), we get

$$g(A_{\phi\xi_1}\xi_2 - A_{\phi\xi_2}\xi_1, X) = 0. \tag{33}$$

Thus our assertion follows from (33) and the fact that $S(TM)$ is non-degenerate.

4 Totally Contact Umbilical ST Anti-Invariant Lightlike Submanifolds

For the study of totally contact umbilical ST anti-invariant lightlike submanifolds of an indefinite Sasakian manifold, we need the following definition from [10].

Definition 4. A ST-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is said to be a ST anti-invariant lightlike submanifold of $\overline{M}$ if $S(TM)$ is screen transversal with respect to ϕ, i.e,

$$\phi(S(TM)) \subset S(TM^{\perp}).$$

For the induced connection ∇ on a totally contact umbilical ST anti-invariant lightlike submanifold M to be a metric connection, we have:

Theorem 7. *Let M be a totally contact umbilical ST anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $\nabla^s_X\phi\xi \in \Gamma(\phi(\text{Rad } TM))$ for all $X \in \Gamma(TM), \xi \in \Gamma(\text{Rad } TM)$.*

Proof. Using (12), for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad } TM)$, we obtain

$$\overline{\nabla}_X\phi\xi = \phi(\overline{\nabla}_X\xi). \tag{34}$$

Apply ϕ to (34) and using (3),(5), (11), (13), (15) and (34), a direct calculation shows that

$$\phi(-A_{\phi\xi}X + \nabla^s_X\phi\xi + D^l(X, \phi\xi)) = -\nabla_X\xi - 2\eta(X)\phi\xi,$$

from which we have

$$\omega A_{\phi\xi}X + B\nabla^s_X\phi\xi + C\nabla^s_X\phi\xi + CD^l(X,\phi\xi) = -\nabla_X\xi - 2\eta(X)\phi\xi, \tag{35}$$

where we have used (9) and (10). Taking the tangential components of (35), we get

$$B\nabla^s_X\phi\xi = -\nabla_X\xi. \tag{36}$$

Thus, our assertion follows from (36).

For the distribution Rad TM to be parallel, we have:

Theorem 8. *Let M be a totally contact umbilical ST anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then* Rad TM *is parallel if and only if $\nabla^s_{\xi_1}\phi\xi_2$ has no component in $\phi(S(TM))$ for any $\xi_1,\xi_2 \in \Gamma(\text{Rad } TM)$.*

Proof. Using (3),(5),(9),(10),(12) and (15), we arrive at

$$-A_{\phi\xi_2}\xi_1+\nabla^s_{\xi_1}\phi\xi_2+D^l(\xi_1,\phi\xi_2) = \omega\nabla_{\xi_1}\xi_2+Ch^l(\xi_1,\xi_2)+Bh^s(\xi_1,\xi_2)+Ch^s(\xi_1,\xi_2). \tag{37}$$

for any $\xi_1,\xi_2 \in \Gamma(\text{Rad } TM)$. From (15) and (37) we obtain

$$-A_{\phi\xi_2}\xi_1 + \nabla^s_{\xi_1}\phi\xi_2 + D^l(\xi_1,\phi\xi_2) = \omega\nabla_{\xi_1}\xi_2. \tag{38}$$

Taking inner product of (38) with ϕY for $Y \in \Gamma(S(TM))$, we get

$$\overline{g}(\nabla^s_{\xi_1}\phi\xi_2, \phi Y) = g(\nabla_{\xi_1}\xi_2, Y),$$

which proves our assertion.

Theorem 9. *Let M be a totally contact umbilical ST-anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then H^s has no component in $\phi(ltr(TM))$.*

Proof. For $X,Y \in \Gamma(S(TM))$, from (3),(5)and (12) we have

$$\begin{aligned}-A_{\phi Y}X + \nabla^s_X\phi Y + D^l(X,\phi Y) = g(X,Y)V - \eta(Y)X + \phi\nabla_X Y + \phi h^l(X,Y)\\ + \phi h^s(X,Y)\end{aligned} \tag{39}$$

Taking inner product of (39) with $\xi \in \Gamma(\text{Rad } TM)$ and then using (5),(11),(13) and (15), a direct calculation shows that

$$\overline{g}(\omega X, \omega Y)\overline{g}(H^s, \phi\xi) = 0. \tag{40}$$

Thus our assertion follows from (40) together with non-degeneracy of $S(TM^\perp)$.

Theorem 10. *Let M be a totally contact umbilical ST anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $H^l = 0$ if and only if $\nabla^s_X\phi X$ has no component in $\phi(S(TM))$ for all $X \in \Gamma(S(TM) - \{V\})$.*

Proof. From (3),(5),(9), (10), (12) and (15) we get

$$-A_{\phi X}X+\nabla^s_X\phi X+D^l(X,\phi X)=\omega\nabla_X X+Ch^l(X,X)+Bh^s(X,X)+Ch^s(X,X) \tag{41}$$

for any $X \in \Gamma(S(TM)-\{V\})$. Comparing the screen transversal parts on both sides of (41), we obtain

$$\nabla^s_X\phi X=\omega\nabla_X X+Ch^l(X,X)+Ch^s(X,X).$$

Taking inner product of the above equation with $\phi\xi$ for $\xi \in \Gamma(\text{Rad }TM)$ and using (9) and(15), we get

$$\overline{g}(\nabla^s_X\phi X,\phi\xi)=g(X,X)\overline{g}(H^l,\xi),$$

which proves our assertion.

The following theorem provides a classification of totally contact umbilical ST anti-invariant lightlike submanifold immersed in an indefinite Sasakian manifold.

Theorem 11. *Let M be a totally contact umbilical ST anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then either H^s has no components in $\phi(S(TM))$ or dim $(S(TM)-\{V\})=1$.*

Proof. Taking inner product of the tangential components of (41) with $Z \in \Gamma(S(TM)-\{V\})$ and using (11) and (10), we get

$$g(A_{\phi X}X,Z)=\overline{g}(h^s(X,X),\phi Z) \tag{42}$$

for any $X \in \Gamma(S(TM)-\{V\})$. On the other hand, by the use of (6) we obtain

$$g(A_{\phi X}X,Z)=\overline{g}(h^s(X,Z),\phi X). \tag{43}$$

From (42) and (43), we conclude that

$$\overline{g}(h^s(X,X),\phi Z)=\overline{g}(h^s(X,Z),\phi X).$$

Using (15) in the above equation, we get

$$g(X,X)\overline{g}(H^s,\phi Z)=g(X,Z)\overline{g}(H^s,\phi X) \tag{44}$$

Interchanging X and Z in (44) and after rearranging the terms, we get

$$\overline{g}(H^s,\phi X)=\frac{g(X,Z)}{g(Z,Z)}\overline{g}(H^s,\phi Z). \tag{45}$$

From (44) and (45), one can easily have

$$\overline{g}(H^s,\phi X)=\frac{g(X,Z)^2}{g(X,X)g(Z,Z)}\overline{g}(H^s,\phi X). \tag{46}$$

Thus our assertion follows from (46).

References

[1] B. SAHIN: *Transversal Lightlike Submanifolds of Indefinite Kaehler Manifolds*, Analele Universitaii de Vest, Timisoara Seria Matematica-Informatica, **XLIV**, n.1(2006), 119-145.

[2] B. SAHIN: *Screen Transversal Lightlike Submanifolds of Indefinite Kaehler Manifolds*, Chaos, Solitons and Fractals, **38**, (2008), 1439-1448.

[3] C.YILDIRIM AND B. SAHIN: *Transversal Lightlike Submanifolds of Indefinite Sasakian Manifolds*, Turkish Journal of Mathematics, **33**, (2009), 1-23.

[4] C.YILDIRIM AND B. SAHIN: *Screen Transversal Lightlike Submanifolds of Indefinite Sasakian Manifolds*, An.St.Univ.Ovidius Constanta, **18**, n.2(2010), 315-336.

[5] D.N. KUPELI: Singular Semi-Riemannian Geometry, vol. 366 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[6] K.L. DUGGAL: *Spacetime manifolds and contact structures*, Internat. J Math. & Math. Sci., **13**(1990), 545-554.

[7] K.L. DUGGAL AND A. BEJANCU: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, vol. 364 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[8] K.L. DUGGAL AND B. SAHIN: *Lightlike submanifolds of indefinite Sasakian manifolds*, Internat. J. Math. & Math. Sci., **2007**(2007), Article ID 57585, 21 pages.

[9] K.YANO AND M. KON: Structures on Manifolds , vol.3 of series in Pure Mathematics, World Scientific, Singapore, 1984.

[10] S.M.K.HAIDER, M.THAKUR AND ADVIN: *Screen Transversal Lightlike Submanifolds of Indefinite Sasakian Manifolds*, Int.Journal of Math. Analysis, **4**, n.39 (2010), 1899-1911.

[11] T. TAKAHASHI: *Sasakian manifolds with pseudo-Riemannian metric*, Tohoku Math.J., **21**(1969),271-290.