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THE MINIMAL FOCAL DISTANCE OF A COMPLEX HYPERSURFACE IN COMPLEX PROJECTIVE SPACE

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Abstract. *An upper bound for the distance between a complex hypersurface in complex projective space and its nearest focal point is given. The upper bound depends only on the degree of the hypersurface and not on the particular polynomial that defines the hypersurface.*

1. COMPLEX HYPERSURFACES

Let $\mathbb{C}P^{n+1}(\lambda)$ denote $(n+1)$-dimensional complex projective space with the Fubini-Study metric whose sectional curvature varies between λ and 4λ, where λ is a positive constant. If P_d is a homogeneous polynomial of degree d on the complex vector space $\mathbb{C}^{n+1}$, then P_d gives rise to a well defined function $P_d : \mathbb{C}P^{n+1}(\lambda) \to \mathbb{C}$. The set

$$M(P_d, \lambda) = \{[z_0, \ldots, z_{n+1}] \in \mathbb{C}P^{n+1}(\lambda) | P_d(z_0, \ldots, z_{n+1}) = 0\}$$

is called a *complex hypersurface of degree* d.

In general the local Riemannian geometry of $M(P_d, \lambda)$ will depend on the polynomial P_d. However, if often happens that some of the natural functions that arise in Riemannian geometry depend only on the degree of the hypersurface, and not on the particular polynomials that defines the hypersurface. For example, a theorem of Wirtinger [Wir] (see Theorem 6.37 of [Gr]) states that

$$(1) \qquad volume(M(P_d, (\lambda)) = d \cdot volume(\mathbb{C}P^n(\lambda)) = \frac{d}{n!}\left(\frac{\pi}{\lambda}\right)^n .$$

A second example is the volume $V(r)$ of a tube about a hypersurface. According to Theorem 7.22 of [Gr] we have the formula

$$(2) \qquad V(r) = \frac{1}{(n+1)!}\left(\frac{\pi}{\lambda}\right)^{n+1}(1-(1-d\sin^2(r\sqrt{\lambda}))^{n+1};$$

thus $V(r)$ depends only on d, λ and r.

In this note I shall give an upper bound for the distance from $M(P_d, \lambda)$ to its nearest focal point. Just as for $V(r)$, this upper bound depends only on d and λ, and not on the particular polynomial P_d. I wish to thank S. Donaldson for suggesting this problem to me.

2. THE MINIMAL FOCAL DISTANCE

In general let P be a topologically embedded submanifold of a Riemannian manifold M. The *minimal focal distance* of P in M is the number

$minfoc(P, M)$ = the smallest number r_0 such that any geodesic emanating perpendicularly from P and having length $\leq r_0$ minimizes distance from its end point to P.

It turns out that tube volumes can be used to estimate the minimal focal distance of a complex hypersurface.

Theorem. *For any complex hypersurface $M(P_d, \lambda)$ of degree d in $\mathbb{C}P^{n+1}(\lambda)$ we have*

$$minfoc(M(P_d,\lambda), \mathbb{C}P^{n+1}(\lambda)) \leq \frac{1}{\sqrt{\lambda}} \sin^{-1}\left(\frac{1}{\sqrt{d}}\right). \tag{3}$$

Proof. Let $r_0 = minfoc(M(P_d,\lambda), \mathbb{C}P^{n+1}(\lambda))$. Clearly the volume $V(r_0)$ of a tube of radius r_0 about $M(P_d,\lambda)$ is not greater than the volume of $\mathbb{C}P^{n+1}(\lambda)$. On the other hand, we have as a special case of (1) that

$$volume(\mathbb{C}P^{n+1}(\lambda)) = \frac{1}{(n+1)!}\left(\frac{\pi}{\lambda}\right)^{n+1}. \tag{4}$$

Thus from (2) and (4) we have

$$\frac{1}{(n+1)!}\left(\frac{\pi}{\lambda}\right)^{n+1}(1-(1-d\sin^2(r_0\sqrt{\lambda}))^{n+1}) \leq \frac{1}{(n+1)!}\left(\frac{\pi}{\lambda}\right)^{n+1}. \tag{5}$$

But (5) is equivalent to (3). ∎

Let us consider several examples. Put $f(d,\lambda) = \frac{1}{\sqrt{\lambda}}\sin^{-1}\left(\frac{1}{\sqrt{d}}\right)$. Then

$$f(1,\lambda) = \frac{\pi}{2\sqrt{\lambda}},$$

$$f(2,\lambda) = \frac{\pi}{4\sqrt{\lambda}},$$

$$f(4,\lambda) = \frac{\pi}{6\sqrt{\lambda}}.$$

All of the complex hypersurfaces of degree 1 are isometric to $\mathbb{C}P^n(\lambda)$. Moreover, a tube of radius $\frac{\pi}{2\sqrt{\lambda}}$ about $\mathbb{C}P^n(\lambda) \subset \mathbb{C}P^{n+1}(\lambda)$ exactly fills $\mathbb{C}P^{n+1}(\lambda)$. In fact, all geodesics perpendicular to $\mathbb{C}P^n(\lambda)$ meet in a single point.

A similar phenomenon occurs for the quadratic surface defined by the polynomial

$$Q(z_0, \ldots, z_{n+1}) = \sum_{i=0}^{n+1} z_i^2 .$$

The set of points at a distance $\frac{\pi}{4\sqrt{\lambda}}$ from the complex hypersurface $M(Q, \lambda)$ is a real projective space $\mathbb{R}P^n(\lambda)$ embedded as a totally real totally geodesic submanifold of $\mathbb{C}P^{n+1}(\lambda)$. So again a tube of radius $\frac{\pi}{4\sqrt{\lambda}}$ exactly fills $\mathbb{C}P^{n+1}(\lambda)$.

Question. Which complex hypersurfaces have a tube that exactly fill up $\mathbb{C}P^{n+1}(\lambda)$?

REFERENCES

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