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METRIZABILITY OF AFFINE CONNECTIONS ON ANALYTIC MANIFOLDS

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It is well-known (see e.g. [3]) that a torsion-free connection ∇ on a connected smooth manifold M is a Riemannian connection of a Riemannian metric g if and only if its holonomy group $\phi(x)$ (with a fixed reference point $x \in M$) preserves a positive scalar product on the tangent space $T_x M$.

In this paper we are occupied with the corresponding *computational problems*:

a) How to decide effectively whether a given (torsion-free) connection is a Riemannian connection?

b) In the positive case, how to find out effectively all corresponding Riemannian metrics (in the prescribed local coordinates)?

We shall solve both problems under the restriction that the basic manifold M is connected and simply connected, and that both M and the given connection ∇ are analytic.

Let us note that the problems above have been solved for very special cases in [1], [2], and in more general terms by the author in [4] and [5], where some kind of regularity for the curvature tensor was assumed. See also [6] and [7] for related results.

§1. Let M be a connected analytic manifold with a torsion-free connection ∇, and let $x \in M$ be an arbitrary point. According to ([3], Chapter III), the restricted holonomy group $\phi^\circ(x)$ of ∇ with the reference point x is a connected Lie transformation group. The corresponding holonomy algebra $\underline{h}(x)$ is generated, as a vector space, by all endomorphisms of the form $(\nabla^k R)(X, Y; Z_1, \ldots, Z_k)$, where $X, Y, Z_1, \ldots, Z_k \in T_x M$ and $0 \leq k < \infty$.

Let us denote by $\underline{h}^{(r)}(x)$ the subspace of $\underline{h}(x)$ generated by all endomorphisms $(\nabla^k R)$ $(X, Y; Z_1, \ldots, Z_k)$ for $0 \leq k \leq r$. The point $x \in M$ is said to be ϕ-regular if dim $\underline{h}^{(r)}(x)$ attains its maximum in a neighborhood U_x for all r. We can prove easily that the set of all ϕ-regular ponts is an open dense subset of M. At a ϕ-regular point we have $\underline{h}^{(N+1)}(x) = \underline{h}^{(N)}(x)$ for some N, and the same must hold in a neighborhood U_x of x. Hence we obtain (using the covariant differentiation) that $\underline{h}(y) = \underline{h}^{(N)}(y)$ for all $y \in U_x$. We see that

(i) we can decide effectively if a given point x is regular or not, calculating successively the subspaces $\underline{h}^{(r)}(y)$ in a coordinate neighborhood U_x;

(ii) around a regular point, the algebra $\underline{h}(y)$ can be calculated effectively in the prescribed local coordinates.

We can see even more: *the number N is constant on the set of all ϕ-regular points of M*. Indeed, let first $x, y \in M$ be two ϕ-regular points in the same connected coordinate neighborhood $U(x^1, \ldots, x^n)$, and suppose that dim $\underline{h}^{(r)}(x) <$ dim $\underline{h}^{(r)}(y)$.

Denote $k = \dim \underline{h}^{(r)}(x)$. This means that, in a neighborhood $V_x \subset U$, $\dim \underline{h}^{(r)}(z) < k+1$. Consider the matrix $R^{(r)}$ whose entries are the local components $R^v_{iju\,i_1,\dots,i_k}$, the n^2 columns are indexed by the ordered pairs (u, v) and the rows are indexed by the ordered sets $(i, j; i_1, \dots, i_k)$, $k = 0, 1, \dots, r$; $i, j, i_1, \dots, i_k = 1, \dots, n$. Then all determinants of degree $k+1$ of the matrix $R^{(r)}$ are real analytic functions in U which are identically zero on the open subset V_x. Hence these determinants must vanish in U, which is a contradiction to the assumption $\dim \underline{h}^{(r)}(y) \geq k+1$. If $x, y \in M$ are arbitrary ϕ-regular points, we can joint them by a finite chain of intersecting connected coordinate neighborhoods. Hence the assertion follows.

§2. Let (M, ∇) be a connected smooth manifold with a torsion-free affine connection. Let $\underline{h}(x)$ denote again the holonomy algebra of ∇ at x, i.e., the Lie algebra corresponding to the restricted holonomy group $\phi^\circ(x)$ with the reference point x. Further, let $H(x) \subset S^2T_xM$ denote the subspace of all symmetric bilinear forms G_x on T_xM satisfying the condition

$$G_x(Au, v) + G_x(u, Av) = 0 \tag{1}$$

for all $A \in \underline{h}(x)$ and all $u, v \in T_xM$. Now, we have

Proposition 1. *If M is connected and simply connected, and if the subspace $H(x) \subset S^2T_xM$ for some $x \in M$ contains a positive definite form, then ∇ is a Riemannian connection on M. Conversely, if ∇ is Riemannian, then each subspace $H(x)$, $x \in M$, contains a positive definite form.*

Proof is obvious because $G_x \in H(x)$ means that G_x is invariant with respect to the full holonomy group $\phi(x)$. For a positive definite $G_x \in H(x)$ we obtain a unique Riemannian metric on M via the parallel transport.

Remark. We do not know any *direct* decision algorithm based on Proposition 1 and using only linear algebra. Therefore, we shall develop a more «geometrical» algorithm (see §4) based on the de Rham decomposition of a Riemannian manifold. The next paragraph has a preparatory character.

§3. Let (M, g) be a connected and simply connected smooth Riemannian manifold and ∇ the corresponding Riemannian connection. Let $\{G^\alpha, \alpha = 1, \dots, p\}$ be any basis of the subspace $H(x) \subset S^2T_xM$, and let $\hat{g} \in H(x)$ be a *regular* form (not necessarily positive). Define the symmetric endomorphisms S^α, $\alpha = 1, \dots, p$, on $(T_xM, \hat{g})$ by the formula

$$\hat{g}(S^\alpha u, v) = G^\alpha(u, v), \quad u, v \in T_xM. \tag{2}$$

Let C_x denote the *commutant* of the set $\{S^1, \ldots, S^p\}$, i.e., the subspace in $\mathrm{End}(T_xM)$ generated by all commutators $[S^\alpha, S^\beta]$, and let N_x be the (common) *null-space* of C_x in T_xM.

We have

Proposition 2. *If $C_x \neq (0)$, then the orthogonal complement $N_x^\perp$ of N_x with respect to $\hat{g}$ coincides with the maximal subspace $T_0 \subset T_xM$ on which the holonomy group $\phi(x)$ acts trivially. If $C_x = (0)$, then either* $\dim T_0 = 0$, *or* $\dim T_0 = 1$.

The restrictions $G^\alpha/T_0 (\alpha = 1, \ldots, p)$ are generators of the space S^2T_0 (of all symmetric bilinear forms on T_0). Moreover, the restriction $\hat{g}/N_x$ is regular and the subspace $H(x)/N_x \subset S^2N_x$ is generated by positive semi-definite forms $g_1, \ldots, g_s$ whose null-spaces $N_1, \ldots N_s$ in N_x satisfy the following orthogonal decomposition (with respect to $\hat{g}/N_x$):

$$N_x = N_1^\perp + \ldots + N_s^\perp. \tag{3}$$

Proof. According to [3, Chapter IV], we can write $T_xM = T_0 + T_1 + \ldots + T_s$ (direct sum), where the decomposition is orthogonal with respect to the Riemannian inner product g and $\phi(x)$-invariant. Here T_0 is the subspace of all fixed vectors under the action of $\phi(x)$, and $T_1, \ldots, T_s$ are irreducible subspaces under the action of $\phi(x)$. Further, $\phi(x)$ itself decomposes as a direct product $\phi_0(x) \times \phi_1(x) \times \ldots \times \phi_s(x)$, where $\phi_0(x)$ is trivial. Now, $H(x) \subset S^2T_xM$ is just the subspace of all $\phi(x)$-invariant forms.

Let $\hat{T}_i$ $(i = 0, 1, \ldots, s)$ denote the orthogonal complement of T_i in T_xM with respect to g. We shall now prove the following

Lemma. *The subspace $H(x) \subset S^2T_xM$ coincides with the set of all forms*

$$G_x = G_0 + \lambda_1 G_1 + \ldots + \lambda_s G_s \quad (\lambda_1, \ldots, \lambda_s \in R) \tag{4}$$

where G_0 runs over all symmetric bilinear forms whose null-spaces contain $\hat{T}_0$, and $G_1, \ldots, G_s$ are fixed forms which are positive definite on $T_1, \ldots, T_s$ respectively and whose null-spaces are $\hat{T}_1, \ldots, \hat{T}_s$ respectively.

Proof. Obviously, all forms $G \in S^2T_xM$ whose null-spaces contain $\hat{T}_0$ are $\phi(x)$-invariant and hence they belong to $H(x)$. For $i = 1, \ldots, s$, define the form G_i as that with the null-space $\hat{T}_i$ and coinciding with the Riemannian scalar product g on T_i. Obviously, $G_1, \ldots, G_s \in H(x)$. Further, any $\phi(x)$-invariant symmetric bilinear form on T_i must be a multiple of $g|T_i$ (see [3], Appendix 5). Hence each form $G \in H(x)$ whose null-space contains $\hat{T}_i$ must be a multiple of G_i. It remains to check that, if $G_x \in H(x)$, then $G_x(u, v) = 0$

whenever $u \in T_i$, $v \in T_j$, $i \neq j$, $i, j \in \{0, 1, \ldots, s\}$. Then we obtain the decomposition (4) taking first the restrictions of G_x to $T_0, T_1, \ldots, T_s$ respectively and then extending each restriction $g_x|T_i$ to a new form on T_xM in a trivial manner (by taking zero values on the union $(\hat{T}_i \times T_i) \cup (T_i \times \hat{T}_i) \cup (\hat{T}_i \times \hat{T}_i)$).

Fix $i, j \in \{0, 1, \ldots, x\}$ such that $i \neq 0$, and fix $v \in T_j$. Consider the linear form $L \in T_i^*$ given by $L(u) = G_x(u, v)$, $u \in T_i$. Because G_x is $\phi(x)$-invariant, then $L(u)$ must be $\phi_i(x)$-invariant. But then $L = 0$, otherwise the dual representation of $\phi(x)$ on T_i^* would admit a fixed direction, a contradiction to the irreducibility, q.e.d.

Let now $\{G^1, \ldots, G^p\}$ be a basis of $H(x)$, and let $S^1, \ldots, S^p$ be the corresponding symmetric operators given by

$$\hat{g}(S^\alpha u, v) = G^\alpha(u, v) \quad (\alpha = 1, \ldots, p;\ u, v \in T_xM). \tag{5}$$

According to our Lemma, we have

$$\hat{g} = \hat{g}_0 + \alpha_1 G_1 + \ldots + \alpha_s G_s, \tag{6}$$

where $\hat{g}_0$ is regular on T_0 and $\alpha_1, \ldots, \alpha_s \neq 0$. Define the symmetric operators $S_1, \ldots, S_s$ by

$$S_i u = (1/\alpha_1) u \text{ for } u \in T_i, \quad S_i u = 0 \text{ for } u \in \hat{T}_i, \tag{7}$$

and the symmetric operators E_0^α $(\alpha = 1, \ldots, p)$ by

$$\hat{g}(E_0^\alpha u, v) = G^\alpha(u, v) \text{ for } u, v \in T_0; \quad E_0^\alpha u = 0 \text{ for } u \in \hat{T}_0. \tag{8}$$

Further, in accordance with our Lemma, put

$$G^\alpha = G_0^\alpha + \lambda_1^\alpha G_1 + \ldots + \lambda_s^\alpha G_s \quad (\alpha = 1, \ldots, p). \tag{9}$$

Then we see easily that

$$S^\alpha = E_0^\alpha + \lambda_1^\alpha S_1 + \ldots + \lambda_s^\alpha S_s \quad (\alpha = 1, \ldots, p). \tag{10}$$

Because all commutators $[S_i, S_j]$ and $[S_i, E_0^\alpha]$ are zero due to (7), (8), we see from (10) that

$$[S^\alpha, S^\beta] = [E_0^\alpha, E_0^\beta], \quad \alpha, \beta = 1, \ldots, p. \tag{11}$$

Hence the commutant C_x of $\{S^1, \ldots, S^p\}$ contains only forms with the null-space $\supset \hat{T}_0$, and thus

$$N_x \supset \hat{T}_0 = T_0^{\perp}. \tag{12}$$

Let now $\{e_1, \ldots, e_r\}$ be an orthonormal basis of T_0 with respect to $\hat{g}_0$, i.e., with respect to $\hat{g}_0|T_0$. It means that $\hat{g}_0(e_i, e_i) = \epsilon_i = \pm 1$ for $i = 1, \ldots, r$. The corresponding operators E_{ij} defined for $1 \leq i \leq j \leq r$ by

$$E_{ij}(e_i) = \epsilon_j e_j, \quad E_{ij}(e_j) = \epsilon_i e_i, \quad E_{ij}(e_k) = 0 \qquad \text{otherwise}, \tag{13}$$

form a basis for the space of all *symmetric* endomorphisms of T_0 (with the scalar product $\hat{g}_0$). Due to our Lemma, the operators $S^\alpha|T_0 (\alpha = 1, \ldots, p)$ form a set of generators of the same space. Hence all E_{ij} are linear combinations of the operators $S^\alpha|T_0$, and all brackets $[E_{ij}, E_{kl}]$ are contained in the restriction $C_x|T_0$ of the commutant C_x.

On the other hand, these brackets generate the space of all *skew-symmetric* operators of $(T_0, \hat{g}_0)$. Indeed, any skew-symmetric operator J_i^j defined on T_0 by

$$J_i^j(E_i) = \epsilon_j e_j, \quad J_i^j(e_j) = -\epsilon_i e_i, \quad J_i^j(e_k) = 0 \qquad \text{otherwise}, \tag{14}$$

can be expressed as $J_i^j = \epsilon_i[E_{ij}, E_{ii}]$. Hence the space $C_x|T_0$ contains all skew-symmetric operators of $(T_0, \hat{g}_0)$ and thus the null-space N_x of the commutant C_x is contained in $T_0^{\perp}$, *unless* $\dim T_0 \leq 1$. This and (12) implies that $N_x = T_0^{\perp}$ and $T_0 = N_x^{\perp}$ for $C_x \neq (0)$.

We can put $g_i = G_i|N_x$ for $i = 1, \ldots, s$ (in accordance with our Lemma) and this concludes the proof of Proposition 2 for $\dim T_0 \neq 1$.

If $\dim T_0 = 1$, then $N_x = T_x M$, and we attach a new form $g_{s+1} \in H(x)$, which is positive on T_0 with the null-space $\hat{T}_0$. Then Proposition 2 also holds after replacing s with $s+1$.

Let us still keep the assumptions and notations of §3. We shall prove the following

Proposition 3. *Let $N_x = N_1^{\perp} + \ldots + N_s^{\perp}$ be the orthogonal decomposition as in Proposition 2. Let $g^{(1)}, \ldots, g^{(r)}$ be a basis of the subspace $H(x)|N_x \subset S^2 N_x$, and $S^{(1)}, \ldots, S^{(r)}$ be the endomorphisms of N_x defined by*

$$\hat{g}(S^{(\alpha)}u, v) = g^{(\alpha)}(u, v) \quad (\alpha = 1, \ldots, r;\ u, v \in N_x). \tag{15}$$

Let $\{Z_1^{(1)}, \ldots, Z_{p_1}^{(1)}, Z_2^{(2)}, \ldots, Z_{p_2}^{(2)}, \ldots, Z_1^{(r)}, \ldots, Z_{p_r}^{(r)}\}$ *be the set of all eigenspaces of all operators* $S^{(1)}, \ldots, S^{(r)}$ *(written in some order). Then each of the subspaces* $N_1^{\perp}, \ldots, N_s^{\perp}$ *can be written as an intersection of the form* $Z_{\alpha_1}^{(1)} \cap \ldots \cap Z_{\alpha_r}^{(r)}$ $(1 \le \alpha_l \le p_j, j = 1, \ldots, r)$, *and each of the eigenspaces* $Z_{\alpha_i}^{(i)}$ *can be written as a direct sum of some of* $N_1^{\perp}, \ldots, N_s^{\perp}$. *Moreover, we have* $r = s$, *and any restriction* $g^{(\alpha)}|N_j^{\perp}$ *is a multiple of the corresponding form* $g_j|N_j^{\perp}$.

Proof. According to Proposition 2 and its proof, we can write

$$g^{(\alpha)} = \lambda_1^{(\alpha)} g_1 + \ldots + \lambda_s^{(\alpha)} g_s \quad (\alpha = 1, \ldots, r), \tag{16}$$

where $g_i = \hat{g}|N_i$ for $i = 1, \ldots, s$ and hence

$$S^{(\alpha)} = \lambda_1^{(\alpha)} E_1 + \ldots + \lambda_s^{(\alpha)} E_s \tag{17}$$

where E_i denotes the operators with the null-space N_i acting as the identity operator on $N_i^{\perp}$. Thus the eigenspaces of $S^{(\alpha)}$ are just direct sums of some $N_i^{\perp}$. Conversely, because $g^{(1)}, \ldots, g^{(r)}$ form a basis of $H(x)|N_x$, we have $r = s$, and the operators $S^{(1)}, \ldots, S^{(s)}$ are linearly independent. Thus, each E_i is a linear combination of $S^{(1)}, \ldots, S^{(s)}$. Now, the intersections of the form $Z_\alpha^{(1)} \cap \ldots \cap Z_\gamma^{(s)}$ (involving eigenspaces of all operators $S^{(1)}, \ldots, S^{(s)}$, respectively) generate the set $\{(0), N_1^{\perp}, \ldots, N_s^{\perp}\}$. Indeed, let $Z_\alpha^{(1)} \cap \ldots \cap Z_\gamma^{(s)} \neq (0)$. Then we have, after a renumeration of the subspaces $N_1^{\perp}, \ldots, N_s^{\perp}$, $Z_\alpha^{(1)} \cap \ldots \cap Z_\gamma^{(s)} = N_1^{\perp} + \ldots + N_p^{\perp}$. We have to prove that $p = 1$. If $p > 1$, then (17) implies

$$S^{(\alpha)} = \lambda_1^{(\alpha)}(E_1 + \ldots + E_p) + \lambda_{p+1}^{(\alpha)} E_{p+1} + \ldots + \lambda_s^{(\alpha)} E_s \quad (\alpha = 1, \ldots, s) \tag{18}$$

and this means that the operators $S^{(1)}, \ldots, S^{(s)}$ are not linearly independent, a contradiction. On the other hand, each $N_i^{\perp}$ is included in some eigenspace of each $S^{(\alpha)}$, $\alpha = 1, \ldots, s$. Hence our last argument follows.

§4. Now we describe an *effective algorithm* for deciding whether a torsion-free connection ∇ on a manifold M is Riemannian, or not. Hence (M, ∇) is supposed to be connected, simply connected and *analytic*.

Step 1. Choose a system of local coordinates in an open set $U \subset M$. Calculate the curvature tensor and its covariant derivatives at a ϕ-regular (i.e., generic) point up to the least order N for which we get $\underline{h}^{(N+1)}(x) = \underline{h}^{(N)}(x)$.

Step 2. Calculate the space $H(x) \subset S^2 T_x M$ at a ϕ-regular point $x \in U$ from the Frobenius theorem. Then a general $G \in N(x)$ is expressed in the form

$$G = \mu_1 G^1 + \ldots + \mu_p G^p, \tag{19}$$

where $G^\alpha (\alpha = 1, \ldots, p)$ are some basis elements of $H(x)$ whose local components G^α_{kl} are expressed as rational functions of the components $R^v_{iju;\, i_1, \ldots, i_k}$. If $p = 0$, ∇ is not Riemannian. Otherwise, we go to Step 3.

Step 3. Check whether $H(x)$ contains a regular form. This is done by writing down the determinant $\det \left\| \sum_\alpha \mu_\alpha G^\alpha_{kl} \right\|$ $(k, l = 1, \ldots, n)$ with independent variables μ_α. If the resulting polynomial is non-zero, then a regular form in $H(x)$ exists. (If this polynomial is identically zero, then the connection is not Riemannian). We can choose $\mu_1, \ldots, \mu_p$ step by step as some *integers* to obtain a concrete regular form $\hat{g} \in H(x)$.

Step 4. Calculate the operators $S^1, \ldots, S^p$ corresponding to $G^1, \ldots, G^p$ respectively via the regular form $\hat{g}$ (fr. Formula (5)). Further, calculate the commutant C_x of the set $\{S^1, \ldots, S^p\}$ and its null-space N_x in our local coordinates. If $\hat{g}|N_x$ is not regular, or if N_x is not invariant with respect to $S^1, \ldots, S^p$, then ∇ is not Riemannian. Otherwise go to the next step.

Step 5. If $C_x \neq (0)$, then calculate the restrictions of $G^1, \ldots, G^p$ to $N_x^\perp = T_0$. If these restrictions do not generate $S^2 T_0$, then the connection ∇ is not Riemannian. If $G^\alpha | T_0$ generate $S^2 T_0$, then go to Step 6.

If $C_x = (0)$, then go to Step 6, directly.

Step 6. Find a set of independent generators $S^{(1)}, \ldots, S^{(s)}$ for the space $H(x)|N_x$ among the restrictions of $S^1, \ldots, S^p$ to N_x.

Further, calculate all eigenspaces of $S^{(1)}, \ldots, S^{(s)}$ and all intersections $Z^{(1)}_\alpha \cap \ldots \cap Z^{(s)}_\gamma$ of the various eigenspaces belonging to $S^{(1)}, \ldots, S^{(s)}$, respcetivlely. Let $\{(0), L_1, \ldots, L_r\}$ be the set of all intersections. Then the necessary condition for ∇ to be Riemannian is that $r = s$ and $N_x = L_1 + \ldots + L_s$ (the orthogonal decomposition with respect to $\hat{g}$). If this necessary condition is satisfied, go to Step 7.

Step 7. Finally, due to Proposition 1, we see that ∇ is a Riemannian connection if, and only if, each restriction $\hat{g}|L_j$ is either positive definite, or negative definite.

§5. Next, we shall show how to calculate, in a neighborhood U_x of a regular pont $x \in M$, all Riemannian metrics g whose Riemannian connection is the prescribed connection ∇. The practical point is that these metrics are to be described through the given local coordinates. Thus we suppose that some local coordinates are fixed in the neighborhood U_x.

Looking at the previous «decision algorithm» we see that, if the given connection ∇ is Riemannian, we can calculate all the objects involved *for the whole neighborhood* U_x assuming that this neighborhood is small enough. In particular, through Step 2 of the previous algorithm we obtain analytic tensor fields G^α of type $(0, 2)$, $\alpha = 1, \ldots, p$, such that the formula (19) holds at all points $y \in U_x$. Further, the regular form $\hat{g} \in H(x)$ constructed in Step 3 can be extended to a tensor field on U_x as a linear combination of $G^1, \ldots, G^p$ with constant coefficients.

The computational procedure for our second problem will be now clear from the following Propositions 4-6.

Proposition 4. *Let ∇ be an analytic connection on a connected and simply connected analytic manifold M, and let U_x be a neighborhood consisting of ϕ-regular points. Suppose that ∇ is Riemannian, and let $\hat{g}$ be a regular form on U_x as above.*

Let $H^{(1)}, \ldots, H^{t)}$ be analytic tensor fields on U_x such that:

a) $H^{(1)}, \ldots, H^{(t)}$ are linearly independent symmetric bilinear forms on T_yM for each $y \in U_x$,

b) the null-space of each $H^{(i)}$ at y is N_y,

c) the restrictions of $H^{(1)}, \ldots, H^{(t)}$ to $N_y^\perp$ at y generate the space $S^2 N_Y^\perp$.

Then

$$\nabla H^{(i)} = \sum_{j=1}^{t} \omega_j^i \otimes H^{(j)} \quad (i = 1, \ldots, t) \tag{20}$$

holds, where ω_j^i are some Pfaffian forms on U. Moreover, the system of linear homogeneous partial differential equations

$$\mathrm{d}\lambda_i + \sum_{k=1}^{t} \lambda_k \omega_i^k = 0 \quad (i = 1, \ldots, t) \tag{21}$$

is completely integrable.

Proof. The first part is obvious because the distributions $\{N_y\}$ and $\{N_y^\perp\}$ on U_x are invariant with respect to the parallel transport, and hence the system of vector spaces Span $(H^{(1)}, \ldots, H^{(t)})_{y \in U_x}$ is also invariant with respect to the parallel transport. More specifically, let (M_0, g_0) be the Euclidean part of (M, g) (where g is arbitrary Riemannian metric belonging to ∇) and $\pi_0 : M \to M_0$ be the canonical projection.

Let $(u^1, \ldots, u^r)$ be the Cartesian coordinates in $(M_0, g_0) \equiv R^r$; then the corresponding differentials $\mathrm{d}u^i$ are parallel with respect to the corresponding Euclidean connection ∇_0 on M_0, and hence the induced forms $G^{(ij)} = \pi_0^*(\mathrm{d}u^i \mathrm{d}u^j)$ on M are ∇-parallel.

Obviously, for each $y \in U_x$, the forms $G_y^{(ij)}$ have the common null-space N_y by construction, and the restrictions of $G_y^{(ij)}$ to $N_y^{\perp} (i, j = 1, \ldots, r; i \leq j)$ form a basis of $S^2 N_y^{\perp}$. Because $\{H^{(k)}\}$ and $\{G^{(ij)}\}$ form two bases of the same vector space at each $y \in U_x$, we have $t = \dfrac{r(r+1)}{2}$ and we can numerate $G^{(ij)}$ as $G^{(k)}$, $k = 1, \ldots, t$, as well. Put

$$G^{(k)} = \sum_{i=1}^{t} \lambda_i^k H^{(i)} \quad (k = 1, \ldots, t) \tag{22}$$

on U_x. Then the matrix $||\lambda_i^k||$ is non-singular on U_x. Because $\nabla G^{(k)} = 0$ for $k = 1, \ldots, t$, we see from (20) and (22) that $(\lambda_1^k, \ldots, \lambda_t^k)$, $k = 1, \ldots, t$, form t independent solutions of the system (21). Hence (21) is completely integrable, q.e.d.

Proposition 5. *Let M, ∇, U, $\hat{g}$ be as in Proposition 4, and let, for each $y \in U_x$, $N_y = L_{1,y} + \ldots + L_{s,y}$ be the orthogonal decomposition constructed in Step 6 of the decision algorithm. Let h_i $(i = 1, \ldots, s)$ denote the tensor field on U which coincides with $\hat{g}$ on each subspace $L_{i,y}$, $y \in U$, and whose null-space at each $y \in U$ is the orthogonal complement of $L_{i,y}$ in $T_y M$ with respect to $\hat{g}$. Then*

$$\nabla h_k = \omega_k \otimes h_k \quad (k = 1, \ldots, s), \tag{23}$$

where ω_k are exact differentials on U, $\omega_k = \mathrm{d} f_k$.

Proof. As we know from the theory of the de Rham decomposition, each distribution $\{L_{i,y}\}_{y \in U}$ is ∇-parallel. Let g be any Riemannian metric on U belonging to ∇. Then denote by g_i the tensor field on U which coincides with g on each $L_{i,y}$ and whose null-space at each $y \in U$ is the orthogonal complement of $L_{i,y}$ in $T_y M$ with respect to g. Then we get obviously $\nabla g_i = 0$. On the other hand, Step 7 of our decision algorithm says that $h_i = \pm e^{f_i} g_i$ on U, where the same sign holds in the whole neighborhood and f_i is an analytic function $(i = 1, \ldots, s)$. Hence $\nabla h_i = \mathrm{d} f_i \otimes h_i$, q.e.d.

Proposition 6. *Let M, ∇, U be the same as in Proposition 4, and let $H^{(1)}, \ldots, H^{(t)}$, $h_1, \ldots, h_s$ be analytic tensor fields on U satisfying the conditions of Proposition 4 and 5, respectively. Then all admissible Riemannian metrics g on U are of the form*

$$g = \sum_{i,k=1}^{t} b_i \lambda_k^i H^{(k)} + \sum_{k=1}^{s} c_k e^{-f_k} h_k, \tag{24}$$

where $f_1, \ldots, f_s$ *are some primitive functions for the exact differentials* $\omega_1, \ldots, \omega_s$ *respectively,* $(\lambda_1^i, \ldots, \lambda_t^i)$ *for* $i = 1, \ldots, t$ *form a basis of the space of solutions of the completely integrable system (21), and the constants* $b_1, \ldots, b_t, c_1, \ldots, c_s$ *are arbitrary but such that the resulting forms are positive definite.*

Proof . Each Riemannian metric g belonging to ∇ is of the form $g = g_0 + d_1 g_1 + \ldots + d_r g_r$, where $g_0 = \sum_{i=1}^{t} b_i G^{(i)}$ and $g_k = \pm e^{-f_k} h_k$ for $k = 1, \ldots, r$. The result now follows Proposition 4 and (22). (Recall that $s = r$ if $\dim N_y \neq 1$, and $s = r + 1$, $t = 0$ holds for $\dim N_y = 1$).

Remark 1. The following observation is obvious for a *Riemannian* connection ∇: if the commutant C_x is zero (i.e., if the Euclidean part is at most 1-dimensional), then the set of all admissible Riemannian metrics g for the connection ∇ can be found only by algebraic operations, differentations and by integrations of exact differentials.

In the general case $C_x \neq (0)$, the integration of a completely integrable system of linear homogeneous 1-st order PDE is needed to express the Euclidean part explicitly in the given coordinates.

Remark 2. For the low-dimensional manifolds, the decision procedure is essentially simplified.

If $\dim M = 2$, then the connection ∇ is Riemannian just in two cases: either $p = \dim H(x) = 1$ and $H(x)$ is generated by a positive definite form at the given ϕ-regular point x. Or $p = 3$, and then ∇ is a Euclidean connection.

If $\dim M = 3$, then ∇ is Riemannian only if either $p = 1$, or $p = 2$, or $p = 6$. For $p = 1$, ∇ is Riemannian iff $H(x)$ is generated by a positive definite form. The corresponding Riemannian manifold is then irreducible. For $p = 2$, if ∇ is Riemannian, then $(M, g) = R \times (N, h)$, where R is the real line and (N, h) is irreducible. For $p = 6$, ∇ is automatically Euclidean.

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