Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **35** (2015) no. 2, 115–125. doi:10.1285/i15900932v35n2p115

A result of Strichartz on a generalization of Wiener's characterization of continuous measures revisited

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Received: 19.8.2014; accepted: 18.5.2015.

Abstract. We give the reasons for which the continuous part of a measure that belongs to the Wiener amalgam space M^2 does not contribute to the Bohr mean of its Fourier transform.

Keywords: Fourier transform, Radon measure, Wiener amalgam space

MSC 2010 classification: primary 42B10, secondary 42B35

1 Introduction

In this paper, for $1 < p < \infty$, we denote by p' the real number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. A Radon measure μ on $\mathbb{R}^d$ belongs to the Wiener amalgam space M^p, $(1 \leq p < \infty)$ if ${}_1\|\mu\|_p < \infty$ with

$$ {}_r\|\mu\|_p = \left(\sum_{k \in \mathbb{Z}^d} |\mu|(I_k^r)^p \right)^{\frac{1}{p}}, \quad r > 0 $$

where $I_k^r = \prod_{i=1}^d [k_i r,\ (k_i+1)r)$ for $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $|\mu|$ denotes the total variation of μ.
The Fourier transform on amalgam spaces has been studied by various authors including F. Holland ([8], [9]), J. Stewart [11], J. P. Bertrandias and C. Dupuis [2], J. J. F. Fournier ([6], [7]) and I. Fofana [5].
In [8], the Fourier transform $f \mapsto \widehat{f}$ defined on the usual Lebesgue space L^1 by

$$ \widehat{f}(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y)\, e^{-ixy}\, dy, \qquad x \in \mathbb{R}^d $$

has been extended by F. Holland to the spaces $(L^q,\ l^p)$ defined for $1 \leq q, p \leq \infty$ as follows:

$$(L^q,\ l^p) = \left\{ f \in L^0 \mid {}_1\left\|f\right\|_{q,\,p} < \infty \right\}$$

where L^0 stands for the space of (equivalence classes modulo the equality Lebesgue almost everywhere of) all complex-valued functions defined on $\mathbb{R}^d$ and for $r > 0$,

$${}_r\left\|f\right\|_{q,\,p} = \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}^d} \left(\left\|f\chi_{I_k^r}\right\|_q \right)^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{x\in\mathbb{R}^d} \left\|f\chi_{J_x^r}\right\|_q & \text{if } p = \infty, \end{cases}$$

where $J_x^r = \prod_i^d (x_i - \frac{r}{2},\, x_i + \frac{r}{2})$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\chi_{I_k^r}$ denotes the characteristic function of I_k^r and $\|\cdot\|_q$ is the usual Lebesgue norm. In the same paper, he extended to the spaces M^p $(1 < p \leq 2)$ the Fourier transform $\mu \mapsto \widehat{\mu}$ defined on the space M^1 of finite Radon measures on $\mathbb{R}^d$ by

$$\widehat{\mu}(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ixy}\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

In fact, he proved that if μ belongs to M^p $(1 < p \leq 2)$, then there exists a unique element $\widehat{\mu} \in (L^{p'},\ l^\infty)$ such that for any sequence $(r_n)_{n\geq 1}$ of positive real numbers increasing to ∞, the sequence $(\widehat{\mu\lfloor J_0^{r_n}})_{n\geq 1}$ converges in $(L^{p'},\ l^\infty)$ to $\widehat{\mu}$, where $\mu\lfloor J_0^{r_n}$ is the measure defined by $(\mu\lfloor J_0^{r_n})\,(N) = \mu\,(J_0^{r_n} \cap N)$ for any Borel subset N of $\mathbb{R}^d$. In addition,

$$\int_{\mathbb{R}^d} g\,(x)\,\widehat{\mu}\,(x)\ dx = \int_{\mathbb{R}^d} \widehat{g}(x)\ d\mu\,(x)\,, \quad g \in \left(L^p,\ l^1\right).$$

In [5], I. Fofana has proved the following Hausdorff-Young inequality :

$$r^{-\frac{d}{p'}}\ {}_r\left\|\widehat{\mu}\right\|_{p',\,\infty} \leq C\ {}_{\frac{1}{r}}\left\|\mu\right\|_p\,, \quad r > 0 \tag{1.1}$$

where the real constant C does not depend on μ and r.

The relation between the properties of a Radon measure and the asymptotic behavior of its Fourier transform is an important topic in Harmonic Analysis. A well-known theorem of Wiener [14] states that if $\mu \in M^1$, then

$$\lim_{r\to\infty} r^{-d} \int_{J_0^r} \left|\widehat{\mu}(y)\right|^2 dy = \sum_{a\in D} \left|\mu\left(\{a\}\right)\right|^2, \tag{1.2}$$

where $D = \{ a \in \mathbb{R}^d \mid \mu(\{a\}) \neq 0 \}$ is a countable set.
The left-hand side of (1.2) is the so-called Bohr mean of $\widehat{\mu}$. Wiener's result has found many applications in various areas of mathematics, including ergodic theory and optimal control theory. Since then, Wiener-type characterizations of continuous measures are extensively investigated (see for example [1], [3], [4], [12] and [13, Section 12.5]). Strichartz has established that equality (1.2) continues to hold if the measure μ belongs to M^2 (see Theorem 4.4 in [12]).

The aim of this note is to give, by a new proof, the reasons for which the continuous part of a measure belonging to M^2 does not contribute to formula (1.2). To this end , we prove in Section 2 the following result: if μ is a continuous measure that belongs to M^p, then

$$\lim_{r \to 0} {}_r\|\mu\|_p = 0.$$

In Section 3, we show that if μ is a discrete measure that belongs to M^p, then

$$\lim_{r \to 0} {}_r\|\mu\|_p = \left(\sum_{a \in D} |\mu(\{a\})|^p \right)^{\frac{1}{p}}.$$

Since any Radon measure on $\mathbb{R}^d$ has a decomposition into discrete and continuous parts, we combine these above results with inequality (1.1) to establish, in Section 4, the generalization of Wiener's theorem obtained by Strichartz.

2 M^p-estimate of a continuous measure

Let us recall the definition of a continuous Radon measure.

Definition 1. A Radon measure μ on $\mathbb{R}^d$ is continuous if for any $x \in \mathbb{R}^d$ we have $\mu(\{x\}) = 0$.

Notice that for any Radon measure μ on $\mathbb{R}^d$ and any $x \in \mathbb{R}^d$, we have $|\mu(\{x\})| = |\mu|(\{x\})$. Therefore, a Radon measure on $\mathbb{R}^d$ is continuous if and only if its total variation is continuous.
The following proposition will be useful in the proof of the main result of this section.

Proposition 1. *[10]. Let $1 \leq p < \infty$. If $\mu \in M^p$ and $0 < r < s < \infty$, then*

(i) ${}_s\|\mu\|_p \leq \left(Int\left(\frac{s}{r}\right) + 2\right)^{\frac{d}{p'}} 2^{\frac{d}{p}} \ {}_r\|\mu\|_p$, *where $Int\left(\frac{s}{r}\right)$ denotes the greatest integer not exceeding $\frac{s}{r}$;*

(ii) ${}_r\|\mu\|_p \leq 3^{\frac{d}{p'}} 2^{\frac{d}{p}} \ {}_s\|\mu\|_p$.

We can now prove the following result.

Proposition 2. *Suppose that* $1 \leq p < \infty$ *and* μ *is a continuous measure that belongs to* M^p*. Then we have*

(*i*) $\lim_{r\to 0} \sup_{k\in\mathbb{Z}^d} |\mu|(I_k^r) = 0,$

(*ii*) $p > 1 \Longrightarrow \lim_{r\to 0} {}_r\|\mu\|_p = 0.$

Proof. For each non-negative integer n, set

$$s_n = \sup_{k\in\mathbb{Z}^d} |\mu|(I_k^{2^{-n}}).$$

(i) Notice that $(s_n)_n$ is a decreasing sequence of positive real numbers. Suppose that $\lim_{n\to\infty} s_n = s > 0$. For each non-negative integer n, let

$$K_n = \left\{ k \in \mathbb{Z}^d \mid |\mu|(I_k^{2^{-n}}) > \frac{s}{2} \right\}.$$

Let us notice that for each positive integer n,

$$0 < card\ K_n \leq \left(\frac{2}{s} \quad {}_{2^{-n}}\|\mu\|_p\right)^p \leq \left(\frac{2}{s} \quad {}_1\|\mu\|_p\right)^p < \infty$$

and

$$k \in K_{n+1} \Longrightarrow \exists\, l \in K_n : I_k^{2^{-(n+1)}} \subset I_l^{2^{-n}},$$

where $card\ K_n$ denotes the cardinality of K_n. So there exists a sequence $(k_n)_n$ of elements of $\mathbb{Z}^d$ such that for each non-negative integer n we have

$$k_n \in K_n \quad \text{and} \quad I_{k_{n+1}}^{2^{-(n+1)}} \subset I_{k_n}^{2^{-n}}.$$

It follows that $\cap_n I_{k_n}^{2^{-n}}$ is a one point set and $|\mu|(\cap_n I_{k_n}^{2^{-n}}) > 0$. This is in contradiction with the fact that μ is a continuous measure. So

$$\lim_{n\to\infty} s_n = 0.$$

Now let us consider a real number $\varepsilon > 0$. There exists an integer n_0 such that $s_{n_0} < \frac{\varepsilon}{2^d}$. Let us notice that for each element $(r,\ k)$ of $(0,\ 2^{-n_0}) \times \mathbb{Z}^d$, the set I_k^r intersects at most 2^d elements of $\{ I_l^{2^{-n_0}} \mid l \in \mathbb{Z}^d \}$. So

$$|\mu|(I_k^r) \leq 2^d s_{n_0}, \qquad (r,\ k) \in\ (0,\ 2^{-n_0}) \times \mathbb{Z}^d.$$

It follows that

$$\sup_{k\in\mathbb{Z}^d} |\mu|(I_k^r) < \varepsilon, \qquad r \in\ (0,\ 2^{-n_0}).$$

Therefore,

$$\lim_{r\to 0} \sup_{k\in\mathbb{Z}^d} |\mu|(I_k^r) = 0.$$

(ii) Suppose $p > 1$. Let us consider a real number $t > 0$. There exists a finite subset L of $\mathbb{Z}^d$ such that

$$\sum_{k\in\mathbb{Z}^d\setminus L} |\mu|(I_k^1)^p < t.$$

Let us set

$$E = \bigcup_{k\in L} I_k^1$$

and for each non-negative integer n,

$$L_n = \left\{ k \in \mathbb{Z}^d \mid I_k^{2^{-n}} \subset E \right\},\ \alpha_n = \sum_{k\in L_n} |\mu|(I_k^{2^{-n}})^p \text{ and } t_n = \sum_{k\in\mathbb{Z}^d\setminus L_n} |\mu|(I_k^{2^{-n}})^p.$$

Then, for each non-negative integer n,

$$_{2^{-n}}\|\mu\|_p = (\alpha_n + t_n)^{\frac{1}{p}},\ \ \alpha_n \leq s_n^{p-1}|\mu|(E) \quad \text{and} \quad t_n \leq t.$$

Let us notice that

$$|\mu|(E) \leq card\ L\ _1\|\mu\|_p < \infty$$

(where $card\ L$ denotes the cardinality of L) and $(_{2^{-n}}\|\mu\|_p)_n$ is a decreasing sequence. It follows that

$$\lim_{n\to\infty}\ _{2^{-n}}\|\mu\|_p \leq t^{\frac{1}{p}}.$$

Since the above inequality holds for an arbitrary real number $t > 0$, we deduce that

$$\lim_{n\to\infty}\ _{2^{-n}}\|\mu\|_p = 0.$$

Further, for each non-negative integer n, it follows from Proposition 1 (ii) that

$$0 \leq\ _r\|\mu\|_p \leq 3^{\frac{d}{p'}}\ 2^{\frac{d}{p}}\ _{2^{-n}}\|\mu\|_p, \qquad r \in\ (0,\ 2^{-n}).$$

So

$$\lim_{r\to 0}\ _r\|\mu\|_p = 0.$$

QED

3 M^p-estimate of a discrete measure

This section is devoted to the proof of the following result.

Proposition 3. *Suppose that* $1 \leq p < \infty$ *and* μ *is a discrete measure that belongs to* M^p*. Then*

$$\lim_{r \to 0} {}_r\|\mu\|_p = \left(\sum_{a \in D} |\mu(\{a\})|^p \right)^{\frac{1}{p}},$$

where $D = \{ a \in \mathbb{R}^d \mid \mu(\{a\}) \neq 0 \}$.

Proof. Let us consider a real number $\varepsilon > 0$. There exists a finite subset K of $\mathbb{Z}^d$ such that

$$\left(\sum_{k \in \mathbb{Z}^d \setminus K} |\mu| \left(I_k^1\right)^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2^{\frac{d}{p}}}.$$

Set

$$E = \bigcup_{k \in K} I_k^1, \ \ D_n = \left\{ a \in D \mid |\mu(\{a\})| \geq \frac{1}{n} \right\} \ \text{ and } \ \mu_n(B) = \mu(B \cap D_n)$$

for any positive integer n and any Borel subset B of $\mathbb{R}^d$. Then, for each positive integer n and each element r of $(0,\ 1)$, we have

$$\begin{aligned}
{}_r\|\mu\|_p &\leq \left(\sum_{k \in \mathbb{Z}^d} |\mu| \left(I_k^r \cap E\right)^p \right)^{\frac{1}{p}} + \left(\sum_{k \in \mathbb{Z}^d} |\mu| \left(I_k^r \setminus E\right)^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k \in \mathbb{Z}^d} |\mu| \left(I_k^r \cap E\right)^p \right)^{\frac{1}{p}} + \left(2^d \sum_{k \in \mathbb{Z}^d \setminus K} |\mu| \left(I_k^1\right)^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k \in \mathbb{Z}^d} |\mu| \left(I_k^r \cap E\right)^p \right)^{\frac{1}{p}} + \varepsilon \\
&\leq \left(\sum_{k \in \mathbb{Z}^d} |\mu_n| \left(I_k^r \cap E\right)^p \right)^{\frac{1}{p}} + |\mu| \left(E \cap (D \setminus D_n)\right) + \varepsilon \\
&\leq \left(\sum_{k \in \mathbb{Z}^d} |\mu_n| \left(I_k^r\right)^p \right)^{\frac{1}{p}} + |\mu| \left(E \cap (D \setminus D_n)\right) + \varepsilon.
\end{aligned}$$

Since for each positive integer n the set D_n is finite, then there exists an element r_n of $(0,\ 1)$ such that for all $a,\ b \in D_n$,

$$a \neq b \Longrightarrow \exists\, i \in \{\,1, \ldots, d\,\} \quad : \quad |a_i - b_i| > r_n$$

and for each element r of $(0,\ r_n)$

$$\begin{aligned} \left(\sum_{k\in\mathbb{Z}^d} |\mu_n|\,(I_k^r)^p\right)^{\frac{1}{p}} &= \left(\sum_{a\in D_n} |\mu(\{a\})|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{a\in D} |\mu(\{a\})|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Therefore, for each element r of $(0,\ r_n)$ we have

$$_r\|\mu\|_p \leq \left(\sum_{a\in D} |\mu(\{a\})|^p\right)^{\frac{1}{p}} + |\mu|\,(E \cap (D \setminus D_n)) + \varepsilon.$$

Since $|\mu|(E) < \infty$ and $(D \setminus D_n)_n$ is a decreasing sequence that converges to the empty set, we get

$$\limsup_{r\to 0}\ {}_r\|\mu\|_p \leq \left(\sum_{a\in D} |\mu(\{a\})|^p\right)^{\frac{1}{p}} + \varepsilon.$$

As this inequality holds for an arbitrary real number $\varepsilon > 0$, we deduce that

$$\limsup_{r\to 0}\ {}_r\|\mu\|_p \leq \left(\sum_{a\in D} |\mu(\{a\})|^p\right)^{\frac{1}{p}}.$$

Further, for all real numbers $r > 0$, we have

$$\left(\sum_{a\in D} |\mu(\{a\})|^p\right)^{\frac{1}{p}} = \left[\sum_{k\in\mathbb{Z}^d} \left(\sum_{a\in D\cap I_k^r} |\mu(\{a\})|^p\right)\right]^{\frac{1}{p}} \leq\ {}_r\|\mu\|_p.$$

So

$$\lim_{r\to 0}\ {}_r\|\mu\|_p = \left(\sum_{a\in D} |\mu(\{a\})|^p\right)^{\frac{1}{p}}.$$

QED

4 Generalization of Wiener's theorem

Throughout this section, for $1 < p < \infty$ and $\mu \in M^p$, we set $D = \{ a \in \mathbb{R}^d \mid \mu(\{a\}) \neq 0 \}$.

Proposition 4. *Let $1 < p < \infty$. If $\mu \in M^p$ then*

$$\lim_{r\to 0} {}_r\|\mu\|_p = \left(\sum_{a\in D} |\mu(\{a\})|^p \right)^{\frac{1}{p}}.$$

Proof. Let us write $\mu = \mu_c + \mu_\delta$, where μ_c is a continuous measure and μ_δ is a discrete measure. For each real number $r > 0$, we have

$${}_r\|\mu_\delta\|_p \leq {}_r\|\mu\|_p \leq {}_r\|\mu_c\|_p + {}_r\|\mu_\delta\|_p.$$

The desired result follows from Propositions 2 and 3. QED

As an immediate consequence of (1.1) and Proposition 4, we have the following result.

Corollary 1. *Let $1 < p \leq 2$. There exists a real constant $C > 0$ such that for any $\mu \in M^p$ we have*

$$\limsup_{r\to\infty} r^{-\frac{d}{p'}} {}_r\|\widehat{\mu}\|_{p',\infty} \leq C \left(\sum_{a\in D} |\mu(\{a\})|^p \right)^{\frac{1}{p}}.$$

In the case $p = 2$, we obtain the following generalization of Wiener's theorem.

Proposition 5. *If $\mu \in M^2$ then*

$$\lim_{r\to\infty} r^{-\frac{d}{2}} \left(\int_{J_0^r} |\widehat{\mu}(y)|^2 dy \right)^{\frac{1}{2}} = \left(\sum_{a\in D} |\mu(\{a\})|^2 \right)^{\frac{1}{2}}.$$

Proof. a) Let us write $\mu = \mu_c + \mu_\delta$, where μ_c is a continuous measure and μ_δ is a discrete measure. It follows from Propositions 2 and 3 that

$$\lim_{r\to\infty} {}_{\frac{1}{r}}\|\mu_c\|_2 = 0 \quad \text{and} \quad \lim_{r\to\infty} {}_{\frac{1}{r}}\|\mu_\delta\|_2 = \left(\sum_{a\in D} |\mu(\{a\})|^2 \right)^{\frac{1}{2}} < \infty.$$

Note that for each element $(r,\ x)$ of $(0,\ \infty) \times \mathbb{R}^d$, we have

$$\int_{J_x^r} |\widehat{\mu}(y)|^2\ dy = \int_{J_x^r} |\widehat{\mu_\delta}(y)|^2\ dy + 2Re \int_{J_x^r} \widehat{\mu_\delta}(y)\ \overline{\widehat{\mu_c}(y)}\ dy\ + \int_{J_x^r} |\widehat{\mu_c}(y)|^2\ dy,$$

where $Re \displaystyle\int_{J_x^r} \widehat{\mu_\delta}(y)\ \overline{\widehat{\mu_c}(y)}\ dy$ denotes the real part of $\displaystyle\int_{J_x^r} \widehat{\mu_\delta}(y)\ \overline{\widehat{\mu_c}(y)}\ dy$.
By the Hölder's inequality and (1.1), we have

$$\begin{aligned} \left| \int_{J_x^r} \widehat{\mu_\delta}(y)\overline{\widehat{\mu_c}(y)}\ dy \right| &\leq \left(\int_{J_x^r} |\widehat{\mu_\delta}(y)|^2\ dy \right)^{\frac{1}{2}} \left(\int_{J_x^r} |\widehat{\mu_c}(y)|^2\ dy \right)^{\frac{1}{2}} \\ &\leq\ {}_r\|\widehat{\mu_\delta}\|_{2,\ +\infty}\ \ {}_r\|\widehat{\mu_c}\|_{2,\ +\infty} \\ &\leq\ C^2\ \ {}_{\frac{1}{r}}\parallel \mu_\delta \parallel_2\ \ {}_{\frac{1}{r}}\parallel \mu_c \parallel_2\ r^d \end{aligned}$$

and

$$\int_{J_x^r} |\widehat{\mu_c}(y)|^2\, dy \leq\ {}_r\|\widehat{\mu_c}\|^2_{2,\ +\infty} \leq C^2\ {}_{\frac{1}{r}}\|\mu_c\|_2^2\ r^d.$$

It follows that

$$\limsup_{r\to\infty}\ r^{-\frac{d}{2}}\ {}_r\|\widehat{\mu}\|_{2,\ \infty} = \limsup_{r\to\infty}\ r^{-\frac{d}{2}}\ {}_r\|\widehat{\mu_\delta}\|_{2,\ \infty}$$

and

$$\liminf_{r\to\infty}\ r^{-\frac{d}{2}}\ {}_r\|\widehat{\mu}\|_{2,\ \infty} = \liminf_{r\to\infty}\ r^{-\frac{d}{2}}\ {}_r\|\widehat{\mu_\delta}\|_{2,\ \infty}.$$

Therefore, it suffices to consider the case where μ is a discrete measure to prove the desired result.
b) Suppose now that μ is a discrete measure.
Let us consider an element ε of $\left(0,\ \frac{1}{3}\left(\sum_{a\in D} |\mu(\{a\})|^2\right)^{\frac{1}{2}}\right)$.
For each positive integer n and each Borel subset B of $\mathbb{R}^d$, set

$$D_n = \left\{ a \in D \mid |\mu(\{a\})| \geq \frac{1}{n} \right\} \ \text{ and }\ \mu_n(B) = \mu(B \cap D_n).$$

There exists a positive integer N such that

$$\left(\sum_{a\in D\setminus D_N} |\mu(\{a\})|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{3(C+1)}, \tag{4.1}$$

where C is a real constant as in Corollary 1. So

$$\limsup_{r\to+\infty} r^{-\frac{d}{2}}\,{}_r\|\widehat{\mu}-\widehat{\mu_N}\|_{2,\,\infty} \le C\left(\sum_{a\in D\setminus D_N}|\mu(\{a\})|^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{3}.$$

Thus, there exists a real number $R_1 > 0$ such that for any $r \ge R_1$ and any $x \in \mathbb{R}^d$,

$$\left| r^{-\frac{d}{2}}\left(\int_{J_x^r}|\widehat{\mu}(y)|^2\,dy\right)^{\frac{1}{2}} - r^{-\frac{d}{2}}\left(\int_{J_x^r}|\widehat{\mu_N}(y)|^2\,dy\right)^{\frac{1}{2}}\right| < \frac{\varepsilon}{3}. \tag{4.2}$$

Since μ_N is a finite measure, it follows from (1.2) that

$$\lim_{r\to\infty} r^{-\frac{d}{2}}\left(\int_{J_0^r}|\widehat{\mu_N}(y)|^2\,dy\right)^{\frac{1}{2}} = \left(\sum_{a\in D_N}|\mu(\{a\})|^2\right)^{\frac{1}{2}}.$$

Thus, there exists a real number $R_2 > 0$ such that for any $r \ge R_2$,

$$\left| r^{-\frac{d}{2}}\left(\int_{J_0^r}|\widehat{\mu_N}(y)|^2 dy\right)^{\frac{1}{2}} - \left(\sum_{a\in D_N}|\mu(\{a\})|^2\right)^{\frac{1}{2}}\right| < \frac{\varepsilon}{3}. \tag{4.3}$$

Let $R = \max\{\, R_1,\ R_2\,\}$. From inequalities (4.1), (4.2) and (4.3), we obtain

$$\left| r^{-\frac{d}{2}}\left(\int_{J_0^r}|\widehat{\mu}(y)|^2\,dy\right)^{\frac{1}{2}} - \left(\sum_{a\in D}|\mu(\{a\})|^2\right)^{\frac{1}{2}}\right| < \varepsilon, \qquad r \ge R.$$

Hence

$$\lim_{r\to+\infty} r^{-\frac{d}{2}}\left(\int_{J_0^r}|\widehat{\mu}(y)|^2\,dy\right)^{\frac{1}{2}} = \left(\sum_{a\in D}|\mu(\{a\})|^2\right)^{\frac{1}{2}}.$$

QED

As an immediate consequence of Proposition 5 and Definition 1, we have the following criterion.

Corollary 2. *Let $\mu \in M^2$. Then μ is continuous if and only if*

$$\lim_{r\to\infty} r^{-d}\int_{J_0^r}|\widehat{\mu}(y)|^2 dy = 0.$$

Acknowledgements. The author would like to thank Professor I. Fofana for his help and guidance in the preparation of this paper.

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