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From absolute to affine geometry in terms of point-reflections, midpoints, and collinearity

Jesse Alama
Theory and Logic Group, Institut für Computersprachen, Vienna University of Technology, Favoritenstraße 9/E1852, 1040 Vienna, Austria
alama@logic.at

Victor Pambuccian
School of Mathematical and Natural Sciences (MC 2352), Arizona State University - West campus, P. O. Box 37100, Phoenix, AZ 85069-7100, U.S.A.
pamb@asu.edu

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Abstract. We investigate equational theories expressed in terms of the point-reflection operation σ and the midpoint operation μ that lie strictly between the absolute and the affine theory, proving a number of dependencies and independencies in the process. Several universal theories enlarged with the collinearity predicate also lie strictly between the absolute and the affine theory. The independence models and several proofs were obtained by Tipi, an aggregate of automatic theorem provers. To show that no set of equations with at most three variables can axiomatize the affine theory is left as an open problem.

Keywords: point-reflections, midpoints, collinearity, affine geometry, axioms, independence, automatic theorem proving

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Introduction

The first axiomatization of the universal geometry of the Euclidean operations of point-reflection and midpoint (to be referred to in the sequel as $\mathcal{V}$) was provided in [12]. There, Vakarelov presented two variants of an axiom system expressed in a language with one sort of individual variables, to be interpreted as *points*, and two binary operation symbols, σ, with $\sigma(ab)$ standing for 'the reflection of b in a' and μ, with $\mu(ab)$ standing for 'the midpoint of segment ab'. One consists of {A1-A4} and the other of {A1-A3, A5, A6}, where the axioms A1, . . . , A6 are:

A 1. $\sigma(aa) = a$

A 2. $\sigma(xa) = \sigma(ya) \rightarrow x = y$

A 3. $\sigma(\mu(ab)a) = b$

A 4. $\sigma(d\sigma(c\sigma(ba))) = \sigma(b\sigma(c\sigma(da)))$

A 5. $\sigma(a\sigma(ax)) = x$

A 6. $\sigma(\sigma(dc)\sigma(ba)) = \sigma(\sigma(db)\sigma(ca))$

If $\mathfrak{M}$ is a model of {A1-A4} (or {A1-A3, A5, A6}), then, for points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in $\mathfrak{M}$, the pair $(\mathbf{a}, \mathbf{b})$ is said to be *equivalent* to the pair $(\mathbf{c}, \mathbf{d})$ (in symbols $(\mathbf{a}, \mathbf{b}) \sim (\mathbf{c}, \mathbf{d})$), if there exists a point $\mathbf{x}$ such that $\sigma(\mathbf{xa}) = \mathbf{d}$ and $\sigma(\mathbf{xb}) = \mathbf{c}$. The relation $\sim$ is an equivalence relation, and an addition can be defined on the set $S := \mathfrak{M}/\sim$ of equivalence classes, turning S into an abelian group in which, for all $a \in S$, the equation $x + x = a$ has a unique solution x. Fixing a point $\mathbf{o}$ in $\mathfrak{M}$, and writing $[\mathbf{a}, \mathbf{b}]$ for the equivalence class under $\sim$ of $(\mathbf{a}, \mathbf{b})$, we obtain that $[\mathbf{o}, \sigma(\mathbf{ba})] = 2[\mathbf{o}, \mathbf{b}] - [\mathbf{o}, \mathbf{a}]$, and $[\mathbf{o}, \mu(\mathbf{ab}] = \frac{1}{2}([\mathbf{o}, \mathbf{a}] + [\mathbf{o}, \mathbf{b}])$. To simplify the notation, we may think of $[\mathbf{o}, \mathbf{a}]$ as the affix of point $\mathbf{a}$, and write instead of $[\mathbf{o}, \mathbf{a}]$ simply $\mathbf{a}$. Our formulas for the operations of reflection in a point and midpoint thus become the familiar ones $\sigma(\mathbf{ba}) = 2\mathbf{b} - \mathbf{a}$ and $\mu(\mathbf{ab}) = \frac{\mathbf{a}+\mathbf{b}}{2}$.

Notice that A4 states that the square of the composition of three point-reflections (namely the reflections in b, c, and d) is the identity.

The subject received a new, more *absolute* treatment in [8], where the starting point is the axiom system {A1-A3, A5} (the theory it defines will be referred to as $\mathcal{A}$), which contains no axiom that would be valid only in an affine setting. Notice that the class of models of $\mathcal{A}$ is a variety, since it is an equational theory. To see this, notice that, if one replaces A2 with

A 7. $\mu(a\sigma(ba)) = b$

then one can prove A2. For, suppose $\sigma(xa) = \sigma(ya)$. Then, by A7, $x = \mu(a\sigma(xa)) = \mu(a\sigma(ya)) = y$.

On the other hand, by A3, we have $\sigma(\mu(a\sigma(ba))a) = \sigma(ba)$, thus, by A2, $\mu(a\sigma(ba)) = b$, which shows that A7 can be derived from A2 and A3.

Thus {A1, A3, A5, A7} is another axiom system for $\mathcal{A}$.

To these, one can add a ternary relation L, with $L(abc)$ to be read as 'points a, b, and c are collinear', and the axioms

A 8. $a \neq b \wedge L(abc) \wedge L(abd) \rightarrow L(acd)$

A 9. $L(abc) \rightarrow L(bac)$

A 10. $L(ab\sigma(ab))$

A 11. $L(abc) \rightarrow L(\sigma(xa)\sigma(xb)\sigma(xc))$

to get an even richer absolute axiom system for point-reflection, midpoint, and collinearity (the theory axiomatized by {A1-A3, A5, A8-A11} will be denoted by $\mathcal{A}'$, and the one in which A4 is added to the above list by $\mathcal{V}'$).

A significantly more complicated — in its variant expressed in terms of σ and μ very far from universal — axiom system for point-reflections and midpoint, which is also absolute in nature, was proposed in [7], in group-theoretical terms. It reads as follows:

Let Γ be a group, let $\mathcal{D}$ be a subset of Γ consisting entirely of involutory elements (i. e. of elements g with $g \neq 1$ and $g^2 = 1$; we write $\iota(g)$ to denote the fact that g is involutory) For $(\Gamma, \mathcal{D})$, the authors consider the following three axioms

S 1. If $a, b, x, y, z \in \mathcal{D}$ are such that $a \neq b$ and $\iota(abx), \iota(aby), \iota(abz)$, then $xyz \in \mathcal{D}$.

S 2. For all $a, b \in \mathcal{D}$, there is exactly one $m \in \mathcal{D}$ such that $b = mam$.

S 3. If $a, b, c \in \mathcal{D}$ are such that $acbcbca = bcacacb$, then $\iota(abc)$.

These axioms are expressed inside the language of group theory, with one binary operation. However, the theory axiomatized by them is synonymous with one axiomatized in terms of σ and μ. To see this, given a pair $(\Gamma, \mathcal{D})$ satisfying S1-S3, we define, for every $g \in \mathcal{D}$, a map $\sigma_g : \mathcal{D} \to \mathcal{D}$, with $\sigma_g(h) = ghg$ (that $ghg \in \mathcal{D}$ follows from S1 with $a = x = z = g$, $b = y = h$). If we let $\sigma(ab)$ be defined as $\sigma_a(b)$ and $\mu(ab)$ as the unique m from axiom S2, then, on $\mathcal{D}$, these two operations satisfy A1, A2, A3, A5, as well as the following axioms:

A 12. $(\forall abxyz)(\exists u)\, a \neq b \wedge \iota(abx) \wedge \iota(aby) \wedge \iota(abz) \to xyz = u$

A 13. $\mu(ab)a\mu(ab) = b$

A 14. $mam = nan \to m = n$

A 15. $acbcbca = bcacacb \to \iota(abc)$

where $\prod_{i=1}^{n} g_i = \prod_{j=1}^{m} h_j$ stands for

$$(\forall x)\, \sigma(g_1(\sigma(g_2 \ldots (\sigma(g_n x) \ldots) = \sigma(h_1(\sigma(h_2 \ldots (\sigma(h_m x) \ldots),$$

and thus A12-A15 are, when written in prenex form, $\forall\exists\forall$ sentences.

Let now $\mathfrak{M}$ be a model of {A1-A3, A5, A12-A15}. If we define, for all $a \in \mathfrak{M}$, $\sigma_a : \mathfrak{M} \to \mathfrak{M}$ by $\sigma_a(b) = \sigma(ab)$, and let $\mathcal{D} = \{\sigma_a \,|\, a \in \mathfrak{M}\}$, and Γ be the group generated by $\mathcal{D}$, then $(\Gamma, \mathcal{D})$ satisfies S1-S3. This means that we may consider {A1-A3, A5, A12-A15} as an axiom system for the geometry axiomatized in [7], whose theory will be denoted by $\mathcal{K}$. There is a logically equivalent theory, $\mathcal{K}'$, expressed in a one-sorted language, with variables to be interpreted as *motions*, with an individual constant 1, a unary predicate symbol P, with $P(a)$ to be read as 'a is a point-reflection', two binary operations, $\cdot$, with $a \cdot b$ to be read as 'the composition of a and b', and ν, with $\nu(ab)$ to be read, in case a and b are point-reflections, as 'the point reflection for which $(\nu(ab) \cdot a) \cdot \nu(ab) = b$'. With $\iota(x)$ standing for $x \cdot x = 1$, its axioms are:

K 1. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

K 2. $x \cdot 1 = x \wedge 1 \cdot x = x$

K 3. $P(x) \rightarrow (x \neq 1 \wedge \iota(x))$

K 4. $P(a) \wedge P(b) \wedge a \neq b \wedge \bigwedge_{i=1}^{3} P(x_i) \wedge \iota(a \cdot (b \cdot x_i)) \rightarrow P(x_1 \cdot (x_2 \cdot x_3))$

K 5. $P(a) \wedge P(b) \rightarrow \nu(ab) \cdot (a \cdot \nu(ab)) = b$

K 6. $P(a) \wedge P(m) \wedge P(n) \wedge m \cdot (a \cdot m) = n \cdot (a \cdot n) \rightarrow m = n$

K 7. $P(a) \wedge P(b) \wedge P(c) \wedge (a \cdot c) \cdot ((b \cdot c) \cdot (b \cdot (c \cdot a))) = (b \cdot c) \cdot ((a \cdot c) \cdot (a \cdot (c \cdot b)))$
$\rightarrow \iota(a \cdot (b \cdot c))$

To get the affine version of the theory, one needs to add

K 8. $P(a) \wedge P(b) \wedge P(c) \rightarrow P(a \cdot (b \cdot c))$

In its presence, the axioms K4, K6, and K7 become redundant (to see that this is the case for K6, notice that, if $mam = nan$, then $nmam = an$, and thus — given that, by K8, we have $nma = amn$ — $amnm = an$, so $nm = mn$, leading, as in the proof of [7, (2.5)], to $m = n$).

The question whether K7 is independent of {K1-K6} was raised but could not be answered in [7]. We do know that K6 is independent of K1–K5. The independence model is given by $(\Gamma, \mathcal{D}) = (D_4, \{s, r^2 s\})$ (i. e. $P(x)$ holds if and only if $x \in \{s, r^2 s\}$), where D_4 denote the dihedral group $\langle r, s \mid r^4 = 1, s^2 = 1, srs = r^3 \rangle$. That K6 does not hold can be seen by setting $(a, m, n) = (s, s, r^2 s)$.

In the non-Euclidean case, i. e., in case the composition of three point-reflections is not always a point-reflection (and it is never the identity in $\mathcal{K}$, as shown in [7, (2.5)]), the incidence structure of the geometry can be defined in terms of σ (or in terms of $\cdot$ and P), and thus L becomes superfluous in that case (as noted in [7, (7.2)], and spelled out in detail in the two-dimensional case in [9]). In the Euclidean case however (i. e., when the composition of three point-reflections is always a point-reflection), the incidence structure cannot be read off from the σ- and μ-structure (or the $\cdot$-, ν-, and P-structure).

σ	1	2	3	4	5	6	7
1	1	7	4	3	6	5	2
2	4	2	5	1	3	7	6
3	7	6	3	5	4	2	1
4	6	3	2	4	7	1	5
5	2	1	7	6	5	4	3
6	3	5	1	7	2	6	4
7	5	4	6	2	1	3	7

μ	1	2	3	4	5	6	7
1	1	5	6	2	7	4	3
2	5	2	4	7	6	3	1
3	6	4	3	1	2	7	5
4	2	7	1	4	3	5	6
5	7	6	2	3	5	1	4
6	4	3	7	5	1	6	2
7	3	1	5	6	4	2	7

Table 1: A model of $\mathcal{A}$ in which A16 fails when $(o, a, b) = (1, 2, 1)$

Our aim in this paper is two-fold:

(i) to put forward the conjecture that there is no axiom system in terms of σ and μ for $\mathcal{V}$ all of whose axioms are prenex statements containing at most three variables (both A4 and A6 require 4 variables; it is quite clear that the set of all at most two-variable universal axioms in σ and μ that hold in $\mathcal{V}$ cannot axiomatize $\mathcal{V}$ — nor can it entail A2)[1];

(ii) to find out the relative strength of several axioms that determine the Euclidean metric if added to the axiom system for metric planes (see [2]), but that are weaker than A4 when added to $\mathcal{A}$ or to $\mathcal{A}'$. We thus determine several intermediate worlds between $\mathcal{A}$ and $\mathcal{V}$ as well as between $\mathcal{A}'$ and $\mathcal{V}'$ that have no counterpart in the metric plane setting.

All the finite independence models and the main steps of the proofs were obtained by `Tipi`, an aggregate of several automatic theorem provers [1].

1 Universal theories between $\mathcal{A}$ and $\mathcal{V}$

An absolute axiom that does not belong to $\mathcal{A}$ is the following axiom, that is equivalent to A13 and K5:

A 16. $\mu(\sigma(oa)\sigma(ob)) = \sigma(o\mu(ab)))$

That it does not belong to $\mathcal{A}$ can be seen from the following 7-element model, with universe $\{1, 2, 3, 4, 5, 6, 7\}$, and with σ and μ given as in Table 1:

A1 can be derived from A3 and A16

[1]Notice that the logically equivalent theory stated in the language of $\mathcal{K}$, axiomatized by {K1-K3, K5, K8} is axiomatized by universal statements with at most three variables.

Proof. The desired result follows from the following equations:

(1) $\mu(\sigma(\mu(uv)w)\sigma(\mu(uv)u)) = \sigma(\mu(uv)\mu(wu))$ [by setting $o \leftarrow \mu(uv)$, $a \leftarrow w$, and $b \leftarrow u$ in A16]

(2) $\mu(\sigma(\mu(uv)w)v) = \sigma(\mu(uv)\mu(wu))$ [by (1) and A3 with $a \leftarrow u$ and $b \leftarrow v$]

(3) $\mu(\sigma(\mu(yx)y)x) = \sigma(\mu(yx)\mu(yy))$ [by (2) with $u \leftarrow y$, $v \leftarrow x$, and $w \leftarrow y$]

(4) $\mu(xx) = \sigma(\mu(yx)\mu(yy))$ [by (3) and A3 with $a \leftarrow y$ and $b \leftarrow x$]

(5) $\mu(\sigma(\mu(yx)\mu(yy))x) = \sigma(\mu(yx)\mu(\mu(yy)y))$ [by (2) with $u \leftarrow y$, $v \leftarrow x$, and $w \leftarrow \mu(yy)$]

(6) $\mu(\mu(xx)x) = \sigma(\mu(yx)\mu(\mu(yy)y))$ [by (5) and (4)]

(7) $\sigma(\mu(v\sigma(ou))\mu(vv)) = \mu(\sigma(\mu(v\sigma(ou))v)\sigma(\mu(v\sigma(ou))v))$ [by A16 with $o \leftarrow \mu(v\sigma(ou))$, $a \leftarrow v$, and $b \leftarrow v$]

(8) $\sigma(\mu(v\sigma(ou))\mu(vv)) = \mu(\sigma(ou)\sigma(ou))$ [by (7) and A3 with $a \leftarrow v$ and $b \leftarrow \sigma(ou)$]

(9) $\sigma(\mu(v\sigma(ou))\mu(vv)) = \sigma(o\mu(uu))$ [by (8) and A16 with $o \leftarrow o$, $a \leftarrow u$, and $b \leftarrow u$]

(10) $\mu(xx) = \sigma(\mu(xx)\mu(xx))$ [by (4) with $x \leftarrow x$ and $y \leftarrow x$]

(11) $\mu(xx) = \sigma(\sigma(\mu(yx)\mu(yy))\mu(xx))$ [by (4)]

(12) $\sigma(\mu(\sigma(\mu(yx)\mu(yy))x)\sigma(\mu(yx)\mu(yy))) = x$ [by A3 with $a \leftarrow \sigma(\mu(yx)\mu(yy))$ and $b \leftarrow x$]

(13) $\mu(\sigma(\mu(yx)\mu(yy))x) = \sigma(\mu(yx)\mu(\mu(yy)y))$ [by (2) with $u \leftarrow y$, $v \leftarrow x$, and $w \leftarrow \mu(yy)$]

(14) $\sigma(\sigma(\mu(yx)\mu(\mu(yy)y)\sigma(\mu(yx)\mu(yy))) = x$ [by (12) and (13)]

(15) $\sigma(\sigma(\mu(yx)\mu(\mu(yy)y)\mu(xx)) = x$ [by (14) and (4)]

(16) $\sigma(\sigma(\mu(y\sigma(ox))\mu(yy))\mu(\sigma(ox)\sigma(ox))) = \mu(\sigma(ox)\sigma(ox))$ [by (11) with $x \leftarrow \sigma(ox)$ and $y \leftarrow y$]

(17) $\sigma(\mu(y\sigma(ox))\mu(yy))) = \sigma(o\mu(xx))$ [by (9) with $o \leftarrow o$, $u \leftarrow x$, and $v \leftarrow y$]

(18) $\sigma(\sigma(o\mu(xx))\mu(\sigma(ox)\sigma(ox))) = \mu(\sigma(ox)\sigma(ox))$ [by (16) and (17)]

(19) $\mu(\sigma(ox)\sigma(ox)) = \sigma(o\mu(xx))$ [by A16 with $o \leftarrow o$, $a \leftarrow x$, and $b \leftarrow x$]

(20) $\sigma(\sigma(o\mu(xx))\sigma(o\mu(xx))) = \sigma(o\mu(xx))$ [by (18) and (19)]

(21) $\sigma(\sigma(\sigma(\mu(yx)\mu(\mu(yy)y))\mu(xx))\sigma(\sigma(\mu(yx)\mu(\mu(yy)y))\mu(xx)))$
$= \sigma(\sigma(\mu(yx)\mu(\mu(yy)y))\mu(xx))$ [by (20) with $o \leftarrow \sigma(\mu(yx)\mu(\mu(yy)y))$ and $x \leftarrow x$]

(22) $\sigma(xx) = x$ (by (21) and (15)]

QED

In light of this result, let $\mathcal{A}_1$ stand for the theory axiomatized by {A2, A3, A5, A16} (which is the theory $\mathcal{A}$+A16).

One of the statements that is equivalent, inside the theory of metric planes, to the axiom of the Euclidean metric ('There exists a rectangle', Axiom R in [2]) is the one stating that the midline is half as long as the basis, or, in the language of μ,

A 17. $\mu(\mu(ab)\mu(cb)) = \mu(b\mu(ac))$.

Lemma 1. From A3 and A17 we get that $\mu(xx) = x$ for all x.

Proof. We have the following equations:

(1) $\sigma(\mu(\mu(vu)\mu(wu))\mu(vu)) = \mu(wu)$ [by A3 with $a \leftarrow \mu(vu)$ and $b \leftarrow \mu(wu)$]

(2) $\mu(\mu(vu)\mu(wu)) = \mu(u\mu(vw))$ [by A17]

(3) $\sigma(\mu(u\mu(vw))\mu(vu)) = \mu(wu)$ [by (1) and (2)]

(4) $\mu(\mu(u\mu(vw))\mu(\mu(xw)\mu(vw))) = \mu(\mu(vw)\mu(u\mu(xw)))$ [by A17 with $a \leftarrow u$, $b \leftarrow \mu(vw)$, and $c \leftarrow \mu(xw)$]

(5) $\mu(\mu(xw)\mu(vw)) = \mu(w\mu(xv))$ [by A17 with $a \leftarrow x$, $b \leftarrow w$, and $c \leftarrow v$]

(6) $\mu(\mu(u\mu(vw))\mu(w\mu(xv))) = \mu(\mu(vw)\mu(u\mu(xw)))$ [by (4) and (5)]

(7) $\mu(\mu(uu)\mu(v\mu(uu))) = \mu(\mu(v\mu(uu))\mu(u\mu(uu)))$ [by (6) with $v \leftarrow u$, $w \leftarrow u$, $u \leftarrow v$, and $x \leftarrow u$]

(8) $\mu(\mu(v\mu(uu))\mu(u\mu(uu))) = \mu(\mu(uu)\mu(vu))$ [by A17 with $a \leftarrow v$, $b \leftarrow \mu(uu)$, and $c \leftarrow u$]

(9) $\mu(\mu(uu)\mu(v\mu(uu))) = \mu(\mu(uu)\mu(vu))$ [by (7) and (8)]

(10) $\mu(\mu(uu)\mu(vu)) = \mu(u\mu(uv))$ [by A17]

(11) $\mu(\mu(uu)\mu(v\mu(uu))) = \mu(u\mu(uv))$ [by (9) and (10)]

(12) $\sigma(\mu(\mu(uu)\mu(v\mu(uu))\mu(uu)) = \mu(v\mu(uu))$ [by A3 with $a \leftarrow \mu(uu)$ and $b \leftarrow \mu(v\mu(uu))$]

(13) $\sigma(\mu(u\mu(uv))\mu(uu)) = \mu(v\mu(uu))$ [by (12) and (11)]

(14) $\sigma(\mu(u\mu(uv))\mu(uu)) = \mu(vu)$ [by (1) with $u \leftarrow u$, $v \leftarrow u$, $w \leftarrow v$ and A17]

(15) $\mu(v\mu(uu)) = \mu(vu)$ [by (13) and (14)]

(16) $\sigma(\mu(v\mu(uu))v) = \mu(uu)$ [by A3 with $a \leftarrow v$ and $b \leftarrow \mu(uu)$]

(17) $\sigma(\mu(vu)v) = \mu(uu)$ [by (16) and (15)]

(18) $\sigma(\mu(vu)v) = u$ [by A3]

(19) $\mu(uu) = u$ [by (17) and (18)]

QED

A1 can now be derived from A3 and A17 (since $\sigma(\mu(aa)a) = a$ by A3, and $\mu(aa) = a$ by Lemma 1).

Lemma 2. A3 and A17 prove that μ is symmetric: $\mu(xy) = \mu(yx)$ for all x, y.

Proof. By A17, $\mu(b\mu(aa)) = \mu(\mu(ab)\mu(ab))$. By Lemma 1, $\mu(\mu(ab)\mu(ab)) = \mu(ab)$, and so $\mu(b\mu(aa)) = \mu(ab)$. However, by Lemma 1, $\mu(aa) = a$, so $\mu(ba) = \mu(ab)$. QED

Also, A5 can be derived from A2, A3 and A17. For, by Lemma 2, $\mu(x\sigma(ax)) = \mu(\sigma(ax)x)$, thus, by A3, $x = \sigma(\mu(\sigma(ax)x)\sigma(ax)) = \sigma(\mu(x\sigma(ax))\sigma(ax))$. However, by A7 (which holds, since A2 and A3 hold), we have $\mu(x\sigma(ax)) = a$. Plugging this in $x = \sigma(\mu(x\sigma(ax))\sigma(ax))$, we get A5.

Notice that

$$\mathcal{A} + \text{A17} \vdash A16 \tag{1.1}$$

Proof. We have the following equations:

(1) $\sigma(\mu(\mu(ba)\mu(ca))\mu(ba)) = \mu(ca)$ [by A3 with $a \leftarrow \mu(ba)$ and $b \leftarrow \mu(ca)$]

(2) $\mu(\mu(ba)\mu(ca)) = \mu(a\mu(bc))$ [by A17]

(3) $\sigma(\mu(a\mu(bc))\mu(ba)) = \mu(ca)$ [by (1) and (2)]

(4) $\sigma(\mu(\sigma(ab)\mu(bc))\mu(b\sigma(ab))) = \mu(c\sigma(ab))$ [by (3) with $a \leftarrow \sigma(ab)$, $b \leftarrow b$, and $c \leftarrow c$]

(5) $\sigma(\mu(\sigma(ab)\mu(bc))a) = \mu(c\sigma(ab))$ [by (4) and $\mu(b\sigma(ab)) = a$ [by A3]]

(6) $\sigma(\mu(a\mu(c\sigma(ab)))a) = \mu(c\sigma(ab))$ [by A3 with $a \leftarrow a$ and $b \leftarrow \mu(c\sigma(ab))$]

(7) $\mu(\sigma(ab)\mu(bc)) = \mu(a\mu(c\sigma(ab)))$ [by (5), (6) and A3 with $a \leftarrow a$, $x \leftarrow \mu(\sigma(ab)\mu(bc))$, and $y \leftarrow \mu(a\mu(c\sigma(ab)))$]

(8) $\sigma(\mu(a\mu(c\sigma(bc)))\mu(ca)) = \mu(\sigma(bc)a)$ [by (3) with $a \leftarrow a$, $b \leftarrow c$, and $c \leftarrow \sigma(bc)$]

(9) $\sigma(\mu(ab)\mu(ca)) = \mu(\sigma(bc)a)$ [by (8), in which $\mu(c\sigma(bc))$ has been replaced by b (by A7)]

(10) $\mu(\sigma(b\sigma(ba))\mu(\sigma(ba)c)) = \mu(b\mu(c\sigma(b\sigma(ba))))$ [by (7) with $a \leftarrow b$, $b \leftarrow \sigma(ba)$, and $c \leftarrow c$]

(11) $\mu(a\mu(\sigma(ba)c)) = \mu(b\mu(ca))$ [by substituting, given A5, a for $\sigma(b\sigma(ba))$ in (10)]

(12) $\mu(a\sigma(\sigma(ba)c)) = \mu(\sigma(\sigma(ba)c)a)$ [by Lemma 2]

(13) $\mu(\sigma(\sigma(ba)c)a) = \sigma(\mu(a\sigma(ba))\mu(ca))$ [by (9) with $a \leftarrow a$, $b \leftarrow \sigma(ba)$, and $c \leftarrow c$]

(14) $\mu(a\sigma(ba)) = b$ [by A7]

(15) $\mu(a\sigma(\sigma(ba)c)) = \sigma(b\mu(ca))$ [by (12), (13), and (14)]

(16) $\mu(ax) = \mu(ay) \rightarrow x = y$ [since, by A3, $\sigma(\mu(ax)a) = x$ and $\sigma(\mu(ay)a) = y$]

(17) $\mu(a\sigma(\mu(b\mu(ca))a)) = \mu(b\mu(ca))$ [by A7 with $a \leftarrow a$ and $b \leftarrow \mu(b\mu(ca))$]

(18) $\mu(a\sigma(\mu(b\mu(ca))a)) = \mu(a\mu(\sigma(ba)c))$ [by (17) and (11)]

(19) $\sigma(\mu(b\mu(ca))a) = \mu(\sigma(ba)c)$ [by applying (16) to (18)]

(20) $\mu(\sigma(oa)\sigma(ob)) = \sigma(\mu(o\mu(\sigma(ob)a))a)$ [by (19) with $a \leftarrow a$, $b \leftarrow o$, and $c \leftarrow \sigma(ob)$

(21) $\mu(o\mu(\sigma(ob)a)) = \mu(\sigma(ob)\mu(ba))$ [by (7) with $a \leftarrow o$, $b \leftarrow b$, and $c \leftarrow a$ and Lemma 2]

(22) $\sigma(\mu(o\mu(\sigma(ob)a))a) = \sigma(\mu(\sigma(ob)\mu(ba))a)$ [by (21)]

(23) $\mu(\sigma(\sigma(ob)a)b) - \sigma(\mu(\sigma(ob)\mu(ba))a)$ [by (19) with $a \leftarrow a$, $b \leftarrow \sigma(ob)$, $c \leftarrow b$]

(24) $\mu(\sigma(\sigma(ob)a)b) = \sigma(o\mu(ab))$ [by (15) with $a \leftarrow b$, $b \leftarrow o$, and $c \leftarrow a$ and Lemma 2]

(25) $\mu(\sigma(oa)\sigma(ob)) = \sigma(o\mu(ab))$ [by (20), (21), (23), and (24)]

$\boxed{QED}$

In light of these two results, let $\mathcal{A}_2$ stand for the theory axiomatized by {A2, A3, A17} (which is an axiom system for $\mathcal{A}$+A17). It is obvious that $\mathcal{A}_1 \subsetneq \mathcal{A}_2$, given that the point-reflection and midpoint operations of the hyperbolic plane do satisfy $\mathcal{A}_1$, but not $\mathcal{A}_2$.

Another axiom that is quite likely to be equivalent to Axiom R inside the theory of metric planes (it is known that it does not hold for any triangle in the hyperbolic plane, as shown in [3]) is the statement that if two medians of a triangle meet in a point, then that point divides each in the ratio 2 : 1 (vertex: midpoint), or, in the language of σ and μ

A 18. $\sigma(\sigma(o\mu(\sigma(\sigma(o\mu(bc))o)c))o) = b$

It states that the medians from b to $\mu(ac)$ and from a to $\mu(bc)$ meet in the point o, which divides $a\mu(bc)$ and $b\mu(ac)$ in the ratio 2 : 1 (vertex: midpoint). Here a stands for $\sigma(\sigma(o\mu(bc))o)$. Of course, the picturesque geometric statement we provided should be taken with a grain of salt, as there is no mention of the fact that the vertices of our triangle are not collinear.

Thus, one question would be whether $\mathcal{A}_2$ is $\mathcal{V}$? We believe the answer to be negative, and, more generally, that

Open Problem . No set of prenex 3-variable statements in terms of σ and μ, true in $\mathcal{V}$, can imply A4.

`Tipi` proved that

$$\text{A2, A3, A17} \vdash \text{A18} \tag{1.2}$$

The proof `Tipi` found is too long to be reproduced here. On the other hand, we have:

$$\mathcal{A} + \text{A18} \nvdash \text{A17} \tag{1.3}$$

and so adding A18 to $\mathcal{A}$ produces a theory $\mathcal{A}_3$ that is strictly included in $\mathcal{A}_2$, but for which it is unknown whether it includes $\mathcal{A}_1$. That (1.3) holds can be seen from the following 7-element model, whose universe is $\{1, 2, 3, 4, 5, 6, 7\}$,

σ	1	2	3	4	5	6	7
1	1	5	4	3	2	7	6
2	5	2	6	7	1	3	4
3	4	6	3	1	7	2	5
4	3	7	1	4	6	5	2
5	2	1	7	6	5	4	3
6	7	3	2	5	4	6	1
7	6	4	5	2	3	1	7

μ	1	2	3	4	5	6	7
1	1	5	4	3	2	7	6
2	5	2	6	7	1	3	4
3	4	6	3	1	7	2	5
4	3	7	1	4	6	5	2
5	2	1	7	6	5	4	3
6	7	3	2	5	4	6	1
7	6	4	5	2	3	1	7

Table 2: A model of $\mathcal{A}$ + A18 in which A17 fails

and with σ and μ interpreted as in Table 2. A counterexample is obtained when the varibles a, b, and c in A17 are interpreted, respectively, as $(2, 3, 1)$.

Another axiom that is equivalent to A4 relative to $\mathcal{A}$ is the σ and μ form of K8, a strengthening of A4, which states that the product of three reflections (in points a, b, and c) is a point-reflection (in point $\mu(a\sigma(c\sigma(ba)))$).

A 19. $\sigma(c\sigma(b\sigma(ax))) = \sigma(\mu(a\sigma(c\sigma(ba)))x)$

In other words, {A1-A3, A5, A19} is another axiom system for $\mathcal{V}$.

To sum up, we have determined that

$\mathcal{A} \subsetneq \mathcal{A}_1 \subsetneq \mathcal{A}_2 \subset \mathcal{V}$ and $\mathcal{A} \subsetneq \mathcal{A}_3 \subsetneq \mathcal{A}_2$.

2 Universal theories between $\mathcal{A}'$ and $\mathcal{V}'$

There is an easy way to get from $\mathcal{A}'$ and $\mathcal{V}'$ without an axiom that would be equivalent to A4. Its only drawback is that it is, from the point of view of first-order logic, a 'hand-waving' way to do so. It is presented in [6]. Just add an axiom stating that there are four points that do not lie on the same plane, and one stating that all planes are exchange planes. Unfortunately, such simple sounding concepts as 'at least three-dimensional' cannot be expressed in first-order logic. Given three non-collinear points a, b, and c, one cannot express the fact that x does not lie in the plane spanned by a, b, and c. The points spanned by these appear in stages, first as points in $U_1 := \{x \mid L(abx) \vee L(bcx) \vee L(cax)\}$, then as points in $U_2 := \{x \mid L(uvx),$ with $u, v \in U_1, u \neq v\}$, and so on recursively, $U_{n+1} := \{x \mid L(uvx),$ with $u, v \in U_n, u \neq v\}$ so the points in the plane spanned by a, b, and c lie in $U = \cup_{i\geq 1} U_i$. Unless there is an n with $U_n = U_{n+1}$, one cannot express the fact that $x \notin U$ by a first-order logic formula.

First, let us note that, just as in the case of $\mathcal{A}$, not all universal sentences true in absolute geometry can be found in $\mathcal{A}'$. And it is not just that A16 is missing. There is more. First, in all metric planes, A19 holds if the three points a, b, and c are collinear, i. e.

A 20. $L(abc) \to \sigma(c\sigma(b\sigma(ax))) = \sigma(\mu(a\sigma(c\sigma(ba)))x)$

is an absolute axiom that quite likely cannot be derived from $\mathcal{A}'$+A16, which is why we list it as an additional axiom.

Also true in all metric planes, as shown in [2], is the fact that if two medians of a triangle meet, then the three medians are concurrent, i. e.

A 21. $L(ao\mu(bc)) \wedge L(bo\mu(ac)) \to L(co\mu(ab))$

Let $\mathcal{A}'^{+}$ denote the theory obtained by adding A16, A20, and A21 to $\mathcal{A}'$. $\mathcal{A}'^{+}$ is the richest absolute L, σ, and μ-based universal theory we consider.

An axiom that can be stated using L and μ, and which is easily seen to follow from {A8-A10, A17} (first considered in [4], and shown to be equivalent to Axiom R inside the theory of metric planes in [10]) states that the vertex a, the midpoint of the opposite side $\mu(bc)$, and the midpoint of the midline $\mu(ab)\mu(ac)$ are collinear, i. e.

A 22. $L(a\mu(bc)\mu(\mu(ab)\mu(ac)))$

First, it is clear that A22 is not in $\mathcal{A}'^{+}$, given that the the hyperbolic plane with the usual point-reflection, midpoint, and collinearity notions is a model of $\mathcal{A}'^{+}$, but A22 holds only for isosceles triangles (i. e., only when ab is congruent to ac).

We also have that

$$\mathcal{A}' + \text{A22} \nvdash \text{A18} \tag{2.1}$$

which can be seen from the following 7-element model in Table 3.

The counterexample is obtained by reading the variables o, b, and c of A18 as, respectively, 2, 2, and 1.

We expect that (2.1) holds with $\mathcal{A}'^{+}$ instead of $\mathcal{A}'$, but `Tipi` has not found an independence model for that.

What's more,

$$\mathcal{A}' + \text{A22} + A18 \nvdash \text{A4} \tag{2.2}$$

which can be seen from the following 7-element model (Table 4).

A counterexample is given by reading for the variables a, b, c, and d in A4 the values 2, 3, 1, and 2, respectively.

We expect that (2.2) also holds with $\mathcal{A}'^{+}$ instead of $\mathcal{A}'$, but `Tipi` has not found a model of independence.

σ	1	2	3	4	5	6	7
1	1	6	4	3	7	2	5
2	5	2	6	7	1	3	4
3	4	5	3	1	2	7	6
4	2	1	7	4	6	5	3
5	3	7	1	6	5	4	2
6	7	3	2	5	4	6	1
7	6	4	5	2	3	1	7

μ	1	2	3	4	5	6	7
1	1	4	5	3	2	7	6
2	4	2	6	7	3	1	5
3	5	6	3	1	7	2	4
4	3	7	1	4	6	5	2
5	2	3	7	6	5	4	1
6	7	1	2	5	4	6	3
7	6	5	4	2	1	3	7

Table 3: A model of $\mathcal{A}' + \text{A22}$ in which A18 fails. The predicate L holds of all triples of points.

σ	1	2	3	4	5	6	7
1	1	5	4	3	2	7	6
2	5	2	7	6	1	4	3
3	4	7	3	1	6	5	2
4	3	6	1	4	7	2	5
5	2	1	6	7	5	3	4
6	7	4	5	2	3	6	1
7	6	3	2	5	4	1	7

μ	1	2	3	4	5	6	7
1	1	5	4	3	2	7	6
2	5	2	7	6	1	4	3
3	4	7	3	1	6	5	2
4	3	6	1	4	7	2	5
5	2	1	6	7	5	3	4
6	7	4	5	2	3	6	1
7	6	3	2	5	4	1	7

Table 4: A model of $\mathcal{A}' + \text{A22} + \text{A18}$ in which A4 fails. The predicate L holds of all triples of points.

If we denote by $\mathcal{A}'_1$ and $\mathcal{A}'_2$ the theories axiomatized by the axioms of $\mathcal{A}'$ together with A22 respectively by the axioms of $\mathcal{A}'$ together with A22 and A18, then we have

$\mathcal{A}' \subsetneq \mathcal{A}'_1 \subsetneq \mathcal{A}'_2 \subsetneq \mathcal{V}'$.

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