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SOME IDEMPOTENT-SEPARATING CONGRUENCES ON A ¶-REGULAR SEMIGROUP

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1. **INTRODUCTION.** Edwards describes in [4] the maximum idempotent-separating congruence on a eventually regular (equivalently, ¶-regular) semigroup which is given by

$\mu = \{(a,b) \in S \times S$: if $x \in S$ is regular then each of $x\mathcal{R}xa$, $x\mathcal{R}xb$ implies $xa\mathcal{H}xb$, and each $x\mathcal{L}ax$, $x\mathcal{L}bx$ imples $ax\mathcal{H}bx\}$.

In this paper we describe the maximum idempotent-separating congruence and their kernels on some subclasses of ¶-regular semigroups. Also, we describe the minimum idempotent-separating r-semiprime congruence and its kernel on an r-semigroup. In this way we obtain a generalization of results of Meakin [10],[11], Feigenbaum [5],[6] and Howie [8].

2. **PRELIMINARIES.** A semigroup S is *¶-regular* if for every $a \in S$ there exists a positive integer m such that $a^m \in a^m S a^m$. We shall denote by RegS the set of all regular elements of S. An element a' is an inverse for a if a=aa'a and a'=a'aa'. As usually we shall denote by V(a) the set of all inverses of a. A semigroup S is *¶-orthodox* if S is ¶-regular and the set E(S) of all idempotents of S is a subsemigroup of S [2]. A semigroup S is *strongly ¶-inverse* if it is ¶-regular and idempotents commute [2]. Define on a ¶-regular semigroup an equivalence $\mathcal{L}^*(\mathcal{R}^*)$ by

$$a\mathcal{L}^*b \iff Sa^m = sb^n \qquad (a\mathcal{R}^*b \iff a^mS = b^nS)$$

where m and n are the smallest positive integers such that $a^m, b^n \in RegS$ [7]. In [14] we define a mapping $r : S \to RegS$ with $r(a)=a^m$ where m is the smallest positive integer such that $a^m \in RegS$. Recall that we can define a partial ordering on the $\mathcal{L}^*$-($\mathcal{R}^*$-) classes of a ¶-regular semigroup S by

$$L_a^* \leq L_b^* \iff Sr(a) \subseteq Sr(b) \qquad (R_a^* \leq R_b^* \iff r(a)S \subseteq r(b)S).$$

On a ¶-regular semigroup S we define an equivalence $\mathcal{H}^*$ by $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$, then each $\mathcal{H}^*$-class contains at most one idempotent [7]. Then

PROPOSITION 2.1. [7]. Let S be a ¶-regular semigroup, then $a\mathcal{H}^*b$ if and only if there are $a' \in V(r(a))$ and $b' \in V(r(b))$ such that $a'r(a) = b'r(b)$ and $r(a)a' = r(b)b'$. Also,

$$(a,b)\in\mathcal{H}^* \implies (\forall a'\in V(r(a)))(\exists b'\in V(r(b)))$$
$$a'r(a) = b'r(b),\ r(a)a' = r(b)b'.$$

3. IDEMPOTENT-SEPARATING CONGRUENCES

The first part of next lemma, which generalizes the known Lallement's lemma for regular semigroups, is proved in [1], [3] and [4].

LEMMA 3.1. Let ρ be a congruence on a ¶-regular semigroup S. If $a\rho$ is an idempotent in S/ρ then there exists an idempotent e in S such that $a\rho = e\rho$. Moreover, e can be chosen so that $R_e^* \leq R_a^*$, $L_e^* \leq L_a^*$.

Proof. If $a\rho$ is an idempotent in S/ρ then $a\rho = a^k\rho$ for every positive integer k, so $a\rho = r(a)^p\rho = r(a)^{2p}\rho$ and p is positive integer such that $r(a)^{2p}\in \mathrm{Reg}S$. Let $y\in V(r(a)^{2p})$ and $e=r(a)^p y r(a)^p$. Since $y=yr(a)^{2p}y$ we have

$$e^2 = (r(a)^p y r(a)^p)(r(a)^p y r(a)^p) = r(a)^p(yr(a)^{2p}y)r(a)$$

$$= r(a)^p y r(a)^p = e$$

and so e is an idempotent. Also

$$e = r(a)^p y r(a)^p \rho r(a)^{2p} y r(a)^{2p} = r(a)^{2p}\rho a$$

and so $a\rho = e\rho$. From $e = r(a)^p y r(a)^p$ it follows that

$$Se \subseteq Sr(a) \iff L^*_e \le L^*_a \quad (eS \subseteq r(a)S \iff R^*_e \le R^*_a).$$

A congruence ρ on a semigroup will be called idempotent-separating if each ρ-class contains at most one idempotent.

DEFINITION 3.1. [14] A relation ρ on a ¶-regular semigroup S is r-semiprime if

$$(\forall a\in S)\ a\rho r(a).$$

The following theorem generalizes a known result which holds for regular semigroups.

THEOREM 3.1. *If* S *is* *¶-regular semigroup, then an* r*-semiprime congruence* ρ *on* S *is idempotent-separating if and only if* $\rho \subseteq \mathcal{H}^*$. *Hence* $\mathcal{H}^{*b} = \{(a,b)\in S\times S : (\forall x,y\in S^1)(xay,xby)\in\mathcal{H}^*\}$ *is the maximum idempotent-separating congruence on* S.

Proof. Since each $\mathcal{H}^*$-class on a ¶-regular semigroup S contains at most one idempotent, it follows that each congruence $\rho \subseteq \mathcal{H}^*$ is idempotent-separating. Conversely, let ρ be an r-semiprime idempotent-separating congruence on S, $a,b \in S$ and suppose that $a\rho b$. Then $r(a)\rho a\rho b\rho r(b)$. If $a' \in V(r(a))$, then $r(a)a'\rho = r(b)a'\rho$ is an idempotent in S/ρ. By Lemma 3.1 there exists $e \in E(S)$ such that $e\rho = r(b)a'\rho$ and $R^*_e \leq R^*_{r(b)a'}$. Since ρ is an idempotent-separating congruence we have $e = r(a)a'$. From $r(a) = r(a)a'r(a)$ it follows that $r(a)S = r(a)a'S$ and so $R^*_a = R^*_{r(a)a'}$. Let $r(r(b)a') = (r(b)a')^p$. Now we have

$$r(r(b)a')S = r(b)a'(r(b)a')^{p-1}S \subseteq r(b)S$$

whence $R^*_{r(b)a'} \leq R^*_b$. Hence,

$$R^*_e = R^*_a = R^*_{r(a)a'} \leq R^*_{r(b)a'} \leq R^*_b .$$

Similarly $R^*_b \leq R^*_a$ and so

$$R^*_a = R^*_b \iff r(a)S = r(b)S \iff a\mathcal{R}^*b.$$

But then one can use a closely similar argument to show that $a\mathcal{L}^*b$. Hence $a\mathcal{H}^*b$ as required.

By Proposition I 5.13 [19]

$$\mathcal{H}^{*b} = \{(a,b) \in S\times S : (\forall x,y \in S^1)(xay,xby) \in \mathcal{H}^*\}$$

is the largest congruence on S contained in $\mathcal{H}^*$. Since $\mathcal{H}^*$ is an idempotent-separating equivalence, then $\mathcal{H}^{*b}$ is also idempotent-separating. Let ρ be an idempotent-separating congruence on S such that $\mathcal{H}^{*b} \subseteq \rho$, then $\mathcal{H}^* \subseteq \rho$. Since $\mathcal{H}^*$ is an r-semiprime equivalence, then ρ is an r-semiprime congruence on S and by preliminary consideration $\rho \subseteq \mathcal{H}^*$, which is a contradiction. Hence, $\mathcal{H}^{*b}$ is the maximum idempotent-separating congruence on S and this completes the proof of theorem.

We now introduce the following notation: if a is an element of a ¶-regular semigroup S then we define

$$EL^*(a) = \{e \in E(S) : L_e^* \leq L_a^*\} \text{ and } ER^*(a) = \{e \in E(S) : R_e^* \leq R_a^*\}.$$

We remark that for any $a \in S$, $EL^*(a) \neq \emptyset$ and $ER^*(a) \neq \emptyset$. From $a\mathcal{L}^*b \iff L_a^* = L_b^*$ ($a\mathcal{R}^*b \iff R_a^* = R_b^*$) we have $EL^*(a) = EL^*(b)$ ($ER^*(a) = ER^*(b)$). Hence, if $a\mathcal{H}^*b$ then $EL^*(a) = EL^*(b)$ and $ER^*(a) = ER^*(b)$. Also $EL^*(a) = EL^*(r(a))$ and $ER^*(a) = ER^*(r(a))$. If e and f are idempotents of S then

$$L_e^* \leq L_f^* \iff ef = e \text{ and } R_e^* \leq R_f^* \iff fe = e.$$

THEOREM 3.2. *Let S be a ¶-regular semigroup, then the following relation*

(1) $$\mu = \{(a,b) \in S\times S : (\exists a' \in V(r(a)))(\exists b' \in V(r(b)))$$
$$[(\forall e \in EL^*(a) \cup EL^*(b))\ r(a)ea' = r(b)eb' \quad \text{and}$$
$$(\forall f \in ER^*(a) \cup ER^*(b))\ a'fr(a) = b'fr(b)]\}$$

is an r-semiprime idempotent-separating equivalence relation containing every r-semiprime idempotent-separating congruence on S. The maximum idempotent-separating congruence on S is given by:

$$\mu^b = \{(a,b) \in S\times S : (\forall x,y \in S^1)(xay, xby) \in \mu\}.$$

Proof. It is obvious that μ is reflexive, symmetric and r-semiprime relation. To show that μ is transitive, we first show that μ is contained in the equivalence $\mathcal{H}^*$.

Let $(a,b)\in\mu$, then $(r(a), r(b))\in\mu$ and let a', b' be the inverses of $r(a)$ and $r(b)$, respectively, as in the definition (1) of μ. Since $r(a)S = r(a)a'S \iff a\,\mathcal{R}^*r(a)a'$ we have $R^*_a = R^*_{r(a)a'}$ so $r(a)a'\in ER^*(a) = ER^*(r(a)a')$. By (1) it follows that

$$a'r(a) = a'(r(a)a')r(a) = b'(r(a)a')r(b)$$

and since $r(b)b' \in ER^*(b) = ER^*(r(b)b')$

$$b'r(b) = b'(r(b)b')r(b) = a'(r(b)b')r(a).$$

Similarly, $a'r(a)\in EL^*(a) = EL^*(a'r(a))$ and

$$r(a)a' = r(b)(a'r(a))b', \; r(b)b' = r(a)(b'r(b))a'.$$

Since $b'r(a)a'r(b) \in EL^*(a)$, it follows that

$$r(a)a' = r(a)(a'r(a))a' = r(a)(b'r(a)a'r(b))a'$$

$$= r(b)(b'r(a)a'r(b))b' = (r(b)b')(r(a)a')(r(b)b').$$

Hence

$$(r(b)b')(r(a)a') = (r(b)b')(r(a)a')(r(b)b') = r(a)a'$$

and

$$(r(a)a')(r(b)b') = (r(b)b')(r(a)a')(r(b)b') = r(a)a'.$$

By symmetry, $r(b)b' = (r(b)b')(r(a)a') = (r(a)a')(r(b)b')$ and so $r(a)a' = r(b)b'$. Similarly, $a'r(a) = b'r(b)$, and by Proposition 2.1 it follows that $(a,b)\in\mathcal{H}^*$, and so $\mu\subseteq\mathcal{H}^*$.

We now prove that the relation μ defined by (1) is transitive. Let $(a,b)\in\mu$ and $(b,c)\in\mu$. Then $a\mathcal{H}^*b\mathcal{H}^*c$ and there are $a'\in V(r(a))$, $b',b^*\in V(r(b))$ and $c^*\in V(r(c))$ such that $r(a)ea' = r(b)eb'$, $r(b)eb^* = r(c)ec^*$ for each $e\in EL^*(a) = EL^*(a'r(a)) = EL^*(b'r(b)) = EL^*(b) = EL^*(b^*r(b)) = EL^*(c) = EL^*(c^*r(c))$, and similarly $a'fr(a)=b'fr(b)$, $b^*fr(b) = c^*fr(c)$ for each $f\in ER^*(a)=ER^*(b)=ER^*(c)$. Hence $r(a)a' = r(b)b'$, $a'r(a) = b'r(b)$, $r(b)b^* = r(c)c^*$, $b^*r(b) = c^*r(c)$, and by Proposition 2.1 there exists $a^*\in V(r(a))$ and $c'\in V(r(c))$ such that $r(a)a' = r(c)c'$, $a'r(a) = c'r(c)$, $r(a)a^*=r(c)c^*$ and $a^*r(a) = c^*r(c)$. Then for each $e\in EL^*(a) = EL^*(b) = EL^*(c)$ we have

$$r(a)ea^* = r(a)(ea'r(a))a^* = (r(a)ea')(r(a)a^*)$$

$$= (r(a)ea')(r(b)b^*) = (r(b)eb')(r(b)b^*)$$

$$= r(b)(eb'r(b))b^* = r(b)eb^* = r(c)ec^*,$$

and for each $f\in ER^*(a) = ER^*(r(a)a') = ER^*(b) = ER^*(r(b)b') = ER^*(c)$, we have

$$a^*fr(a) = a^*(r(a)a'f)r(a) = (a^*r(a))(a'fr(a))$$

$$= (b^*r(b))(b'fr(b)) = b^*(r(b)b'f)r(b)$$

$$= b^*fr(b) = c^*fr(c).$$

Hence, $(a,c)\in\mu$, and so μ is transitive.

Since $\mu \subseteq \mathcal{H}^*$, μ is an r-semiprime idempotent-separating equivalence.

Let ρ be an r-semiprime idempotent-separating congruence on S, and let $(a,b)\in\rho$. Then $(a,b)\in\mathcal{H}^*$ and so there are $a'\in V(r(a))$ and $b'\in V(r(b))$ such that $r(a)a' = r(b)b'$ and $a'r(a)=b'r(b)$. Let $e\in EL^*(a)=EL^*(r(a))=EL^*(b)=EL^*(r(b))=EL^*(b'r(b))$. Then $ea'r(a)=e= = eb'r(b)$ and

$$(r(a)ea')(r(a)ea') = r(a)(ea'r(a)ea' = r(a)eea'$$

$$= r(a)ea'\in E(S),$$

and similarly $r(b)eb'\in E(S)$. Since ρ is an r-semiprime congruence we have $(r(a),r(b))\in\rho$, so

$$b' = b'r(b)b' = b'r(a)a'\rho b'r(b)a' = a'r(a)a' = a',$$

i.e. $(a',b')\in\rho$, and hence $(r(a)ea',\ r(b)eb')\in\rho$. Since $r(a)ea'$, $r(b)eb'\in E(S)$, this implies that $r(a)ea' = r(b)eb'$. Similarly, $a'fr(a) = b'fr(b)$ for each $f\in ER^*(a) = ER^*(b)$, and so $(a,b)\in\mu$. Hence, $\rho\subseteq\mu$.

The proof that $\mu^b = \{(a,b)\in S\times S : (\forall x,y\in S^1)(xay,xby)\in\mu\}$ is the maximum idempotent-separating congruence on S is analogous to corresponding part of Theorem 3.1.

DEFINITION 3.2. [14]. A π-regular semigroup is an r-semigroup if

$$(\forall a,b\in S)(r(ab) = r(a)r(b)).$$

LEMMA 3.2 [14]. Let S be an r-semigroup. Then RegS is a subsemigroup of S and the mapping $r : S \to \mathrm{Reg}\, S$ is a homomorphism.

The following corollary generalizes a known result of Meakin [11].

COROLLARY 3.1. Let S be an r-semigroup, then the relation defined by (1) is the maximum idempotent-separating congruence on S.

Proof. By Theorem 3.2 μ is an r-semiprime idempotent-separating equivalence on S. Let us prove that this equivalence is compatible. Since RegS is subsemigroup of S, we have that $\bar{\mu}=\mu_{|RegS}$ is the maximum idempotent-separating congruence on RegS (Theorem 3.1 [11]). Let $a\mu b$, $c\mu d$, then $r(a)\bar{\mu}r(b)$, $r(c)\bar{\mu}r(d)$. Hence,

$$r(a)r(b)\bar{\mu}r(c)r(d) \Rightarrow r(ab)\mu r(cd) \Longleftrightarrow ab\mu cd$$

and so μ is a congruence on S. By Theorem 3.2 μ is the maximum (r-semiprime) idempotent-separating congruence on S.

LEMMA 3.3. Let S be an r-semigroup, then the equivalence $\mathscr{L}^*$ ($\mathscr{R}^*$) is right (left) congruence.

Proof. Let $a,b,\in S$ and $a\mathscr{L}^*b \Longleftrightarrow Sr(a) = Sr(b)$. Then

$$Sr(ac) = Sr(a)r(c) = Sr(b)r(c) = Sr(bc) \Longleftrightarrow ac\mathscr{L}^*bc$$

and so $\mathscr{L}^*$ is a right congruence. Similarly, $\mathscr{R}^*$ is a left congruence.

If ρ is an congruence on semigroup S, then

$$\ker\rho = \{a\in S : (\exists e\in E(S))a\rho e\} .$$

THEOREM 3.3. *Let* S *be an* r-*semigroup. If* τ *is the relation given by*

$$\tau = \{(a,b)\in S\times S : (\exists a'\in V(r(a)))(\exists b'\in V(r(b)))$$
$$r(a)a' = r(b)b',\ a'r(a) = b'r(b),\ r(a)b'\in\ker\mu\},$$

then $\tau = \mu$.

Proof. Let $a\tau b$, then $r(a)a' = r(b)b'$ and $a'r(a)=b'r(b)$ imply $a\mathscr{H}^*b$ and $a'\mathscr{H}^*b'$. By Lemma 3.3 $a'\mathscr{R}^*b'$ gives $r(a)a'\mathscr{R}^*r(a)b'$ and $a\mathscr{L}^*b \iff r(a)\mathscr{L}^*r(b)$ gives $r(a)b'\mathscr{L}^*r(b)b'$. Hence, $r(a)b'\mathscr{H}^*r(b)b'$. Since $r(a)b'\in\ker\mu$, then there exists $e\in E(S)$ such that $r(a)b'\mu e$. From $\mu\subseteq\mathscr{H}^*$ it follows that $r(a)b'\mathscr{H}^*e$. Now $r(b)b'=e$, since each $\mathscr{H}^*$-class contains at most one idempotent. Hence, $r(a)b'\mu r(b)b'$, so

$$a\mu = r(a)\mu = r(a)\mu(a'r(a))\mu = r(a)\mu(b'r(b))\mu$$

$$=(r(a)b')\mu r(b)\mu = (r(b)b')\mu r(b)\mu = r(b)\mu = b\mu .$$

Conversely, if $a\mu b$, then there are $a'\in V(r(a))$ and $b'\in V(r(b))$ such that $r(a)a' = r(b)b'$ and $a'r(a) = b'r(b)$ (by the proof of Theorem 3.2). By Corollary 3.1, $a\mu b \iff r(a)\mu r(b)$ gives $r(a)b'\mu r(b)b'$. Hence, $r(a)b'\in\ker\mu$ and $\mu=\tau$.

Let S be an r-semigroup. Then RegS is a subsemigroup of S and the relation

$$\bar{\mu} = \mu_{|RegS} = \{(a,b)\in RegS\times RegS : (\exists a'\in V(a))(\exists b'\in V(b))$$
$$|(\forall e\in EL^*(a)\cup EL^*(b))aea' = beb' \quad \text{and}$$
$$(\forall f\in ER^*(a)\cup ER^*(b)a'fa = b'fb|\}$$

is the maximum idempotent-separating congruence on RegS (Meakin [11], Theorem 3.1). By [6] of it follows that

$$\ker\bar{\mu} = \{a\in RegS : (\exists a'\in V(a))\ (\forall e\in EL^*(a))aea'e = a'ae$$
$$\text{and } (\forall f\in ER^*(a))\ fa'fa = faa'\}.$$

THEOREM 3.4. *Let* S *be an* r*-semigroup, then*

$$\ker\mu = \{x\in S : r(x)\in\ker\bar{\mu}\}$$

$$= \{x\in S: (\exists x'\in V(r(x)))(\forall e\in EL^*(x))\, r(x)ex'e$$

$$= x'r(x)e \text{ and } (\forall f\in ER^*(x))\ fx'fr(x)=fr(x)x'\}.$$

Proof. Let $A = \{x\in S : r(x)\in\ker\bar{\mu}\}$, then $a\in A$ implies $r(a)\in\ker\bar{\mu}$. Now there exists $e\in E(S)$ such that $r(a)\bar{\mu}e$. Since μ is an r-semiprime congruence, we have $a\mu r(a)\mu e$. Hence, $a\in\ker\mu$. Conversely, $a\in\ker\mu$ implies $a\mu e$ for some $a\in E(S)$. Now $a\mu r(a)\mu e$ implies $r(a)\bar{\mu}e$ and it follows that $r(a)\in\ker\bar{\mu}$, so $a\in A$. Hence, $\ker\mu=A$.

DEFINITION 3.3 [14]. Let S be a ¶-regular semigroup. If A is a subset of S, then $\mathrm{reg}A = \{a\in A : a\in \mathrm{Reg}S\} = A\cap \mathrm{Reg}S$. A subset A of S is self-conjugate if $a'(\mathrm{reg}A)a \subseteq \mathrm{reg}A$ for every $a\in\mathrm{Reg}S$ and $a'\in V(a)$.

DEFINITION 3.4. A semigroup S is ¶-conventional if S is ¶-regular and the set E(S) of all idempotents of S is self-conjugate.

THEOREM 3.5. *Let* S *be a* ¶*-conventional semigroup and define the relation on* S *by*

(2) $$\mu_1 = \{(a,b)\in S\times S : (\exists a'\in V(r(a)))(\exists b'\in V(r(b)))(\forall e\in E(S))$$

$$r(a)ea' = r(b)eb' \text{ and } a'er(a) = b'er(b)\}.$$

Then relation μ_1 *is an* r*-semiprime idempotent-separating equivalence relation containing every* r*-semiprime idempotent-separating congruence on* S*. The maximum idempotent-separating congruence on* S *is given by:*

$$\mu_1^b = \{(a,b)\in S\times S\ :\ (\forall x,y\in S^1)(xay,xby)\in\mu_1\}.$$

Proof. It is evident that μ_1 is reflexive, symmetric and r-semiprime relation. To show that μ_1 is transitive, we first show that μ_1 is contained in equivalence $\mathscr{H}^*$. Let $a\mu_1 b$ and let a',b' be the inverses of $r(a)$ and $r(b)$, respectively, as in the definition (2) of μ_1. Since S is ¶-conventional semigroup, we have that $r(a)a'r(b)b'r(a)a'\in E(S)$. By definition (2) it follows that

$$a'(r(a)a'r(b)b'r(a)a')r(a) = b'(r(a)a'r(b)b'r(a)a')r(b).$$

Hence, since $b'r(a)a'r(b)\in E(S)$ we have

(3) $a'r(b)b'r(a) = (b'r(a)a'r(b))(b'r(a)a'r(b)) = b'r(a)a'r(b).$

But $r(b)b'\in E(S)$, so

$$a'(r(b)b')r(a) = b'(r(b)b')r(b) = b'r(b)$$

and similarly $b'(r(a)a')r(b) = a'r(a)$. From (3) we have $a'r(a) = b'r(b)$. It is not difficult to see that $r(a)a' = r(b)b'$ also holds. From these two results and by Proposition 2.1 we deduce that $\mu_1 \subseteq \mathscr{H}^*$. We now proced to the proof of the transitivity of μ_1.

Suppose that $a\mu_1 b$ and $b\mu_1 c$. Then there are $a'\in V(r(a))$, $b',b^*\in V(b)$ and $c^*\in V(r(c))$ such that

$$a'er(a) = b'er(b),\ r(a)ea' = r(b)eb',\ b^*er(b) = c^*er(c)$$

$$\text{and}\quad r(b)eb^* = r(c)ec^*$$

for all $e\in E(S)$. In particular, we have seen that this implies that $r(a)a' = r(b)b'$, $a'r(a) = b'r(b)$, $r(b)b^* = r(c)c^*$, $b^*r(b)$

$= c^*r(c)$, and hence that a,b and c are $\mathcal{H}^*$equivalent elements of S. By Proposition 2.1 there are $a^*\epsilon V(r(a))$ and $c'\epsilon V(r(c))$ such that

$$r(a)a' = r(b)b' = r(c)c',\ a'r(a) = b'r(b) = c'r(c)$$

and

$$r(a)a^* = r(b)b^* = r(c)c^*,\ a^*r(a) = b^*r(b) = c^*r(c).$$

Now, $a^*r(a)a'\epsilon V(r(a))$ and $c^*r(c)c'\epsilon V(r(c))$, and for all $e\epsilon E(S)$

$$(a^*r(a)a')er(a) = (a^*r(a))(a'er(a)) = (b^*r(b))(b'er(b))$$

$$= (b^*r(b))(b'er(b)b'r(b)) = b^*(r(b)b'er(b)b')r(b)$$

$$= c^*(r(c)c'er(c)c')r(c) = (c^*r(c)c')er(c),$$

and

$$r(a)e(a^*r(a)a') = r(a)(a^*r(a)ea^*r(a))a'=r(b)(b^*r(b)eb^*r(b))b'$$

$$= (r(b)eb^*)(r(b)b')=(r(c)ec^*)(r(c)c')=r(c)e(c^*r(c)c').$$

Hence $(a,c)\epsilon\mu_1$ and μ_1 is transitive. That μ_1 separates idempotents is obvious since we have already proved that $\mu_1 \subseteq \mathcal{H}^*$

Now let ρ be an r-semiprime idempotent-separating congruence on S. Then if $(a,b)\epsilon\rho$, we have that $(a,b)\epsilon\mathcal{H}^*$, and hence there are $a'\epsilon V(r(a))$ and $b'\epsilon V(r(b))$ such that $r(a)a'=r(b)b'$ and $a'r(a)= b'r(b)$. Then, since $a\rho b \Leftrightarrow r(a)\rho r(b)$, we have $r(b)b'=r(a)a'\rho r(b)a'$ an hence

$$b' = b'r(b)b'\rho b'r(b)a' = a'r(a)a' = a',$$

i.e. $(b',a')\in\rho$. Hence, for all $e\in E(S)$ we have $r(a)ea'\rho r(b)eb'$, and so $r(a)ea'=r(b)eb'$, since both $r(a)ea'$ and $r(b)eb'$ are idempotents, and ρ separates idempotents. Also $(b'er(b)),a'er(a))\in\rho$, and so $a'er(a)=b'er(b)$. From this it follows that $(a,b)\in\mu_1$, and consequently that $\rho\subseteq\mu_1$.

The proof that $\mu_1^b=\{(a,b)\in S\times S : (\forall x,y\in S^1)(xay,xby)\in\mu_1\}$ is the maximum idempotent-separating congruence on S is analogous to corresponding part of Theorem 3.1.

PROPOSITION 3.1. If S is a ¶-conventional semigroup and $(x,y)\in\mu_1$, and if x^* is an arbitrary inverse of $r(x)$, then there exists an inverse y^* of $r(y)$ such that $r(x)ex^*=r(y)ey^*$ and $x^*er(x)=y^*er(y)$ for all $e\in E(S)$.

Proof. If $(x,y)\in\mu_1$ and x^* is an arbitrary inverse of $r(x)$, then there are $x'\in V(r(x))$ and $y'\in V(r(y))$ such that $r(x)ex'=r(y)ey'$ and $x'er(x)=y'er(y)$ for all $e\in E(S)$. Also, since $(x,y)\in\mathcal{H}^*$, there is an inverse $y^*\in V(r(y))$ such that $r(x)x^*=r(y)y^*$ and $x^*r(x)=y^*r(y)$. Thus, for all $e\in E(S)$

$$\begin{aligned} r(x)ex^* &= r(x)(x^*r(x)ex^*r(x)x'r(x)x^* \\ &= r(x)(x^*r(x)ex^*r(x))x'(r(x)x^*) \\ &= r(y)(x^*r(x)ex^*r(x))y'(r(y)y^*) \\ &= r(y)(y^*r(y)ey^*r(y))y'(r(y)y^*) = r(y)ey^*, \end{aligned}$$

and dually, $x^*er(x)=y^*er(y)$.

The following theorem generalizes a result of Meakin [10].

THEOREM 3.6. *Let* S *be a ¶-orthodox r-semigroup, then relation*

μ_1 *defined by* (2) *is the maximum idempotent-separating congruence on* S.

Proof. We shall prove that $\mu_1 = \mu$, where by Corollary 3.1 μ is the maximum idempotent-separating congruence on S.

Obviously $\mu_1 \subseteq \mu$.

Conversely, let $(a,b)\in\mu$. By the proof of the Theorem 3.2 we have $r(a)a' = r(b)b'$ and $a'r(a) = b'r(b)$ where are $a'\in V(r(a))$ and $b'\in V(r(b))$ as in the definition of μ. Also $EL^*(a)=EL^*(r(a))= = EL^*(a'r(a))$. If $h\in E(S)$ then $Sha'r(a) \subseteq Sr(a) \iff L^*_{ha'r(a)} \leq L^*_{r(a)}$ and so $ha'r(a)\in EL^*(a)$ ($ha'r(a)\in E(S)$ since S is ¶-orthodox). Now

$$\begin{aligned} r(a)ha' &= r(a)h(a'r(a)a') = r(a)(ha'r(a))a' \\ &= r(b)(ha'r(a))b' = r(b)(hb'r(b))b' \\ &= r(b)h(b'r(b)b') = r(b)hb'. \end{aligned}$$

Similarly, $ER^*(a) = ER^*(r(a)) = ER^*(r(a)a'), r(a)a'hS \subseteq r(a)S \iff R^*_{r(a)a'h} \leq R^*_a$ and so $r(a)a'h\in ER^*(a)$. Now

$$\begin{aligned} a'hr(a) &= a'(r(a)a'h)r(a) = b'(r(a)a'h)r(b) \\ &= b'(r(b)b'h)r(b) = b'hr(b). \end{aligned}$$

Hence, $\mu \subseteq \mu_1$ and so $\mu = \mu_1$.

Let S be a ¶-orthodox r-semigroup. Then RegS is an orthodox subsemigroup of S and the relation

$$\bar{\mu}_1 = \mu_{1|RegS} = \{(a,b)\in RegS\times RegS : (\exists a'\in V(a))(\exists b'\in V(b))\} \\ (\forall e\in E(S))\ r(a)ea' = r(b)eb',\ a'er(a)=b'er(b)\}$$

is the maximum idempotent-separating congruence on RegS (Meakin [10], Theorem 4.4). In [5] it is proved that

$$\ker\bar{\mu}_1 = \{a\in \mathrm{RegS} : (\exists a'\in V(a))(\forall e\in E(S))a'eae = aa'e \text{ and } eaea' = ea'a\}.$$

THEOREM 3.7. *Let* S *be a* ¶*-orthodox* r*-semigroup, then*

$$\ker\mu_1 = \{x\in S : r(x)\in\ker\bar{\mu}_1\}$$
$$= \{x\in S : (\exists x'\in V(r(x)))(\forall e\in E(S))x'er(x)e$$
$$= r(x)x'e \text{ and } er(x)ex' = ex'r(x)\}.$$

Proof. Analogously to proof of Theorem 3.4.

Let S be a strongly ¶-inverse semigroup and $a\in S$, then unique inverse for r(a) we denote by $r(a)^{-1}$.

THEOREM 3.8. *Let* S *be a strongly* ¶*-inverse semigroup, then the relation*

(4) $\mu_2 = \{(a,b)\in S\times S : (\forall e\in E(S))\ r(a)^{-1}er(a) = r(b)^{-1}er(b)\}$

is an r*-semiprime idempotent-separating equivalence relation containing every* r*-semiprime idempotent-separating congruence on* S *. The maximum idempotent-separating congruence on* S *is given by:*

$$\mu_2^b = \{(a,b)\in S\times S : (\forall x,y\in S^1)(xay,xby)\in\mu_2\}.$$

Proof. If is obvious that μ_2 is an r-semiprime equivalence relation. Since RegS is inverse semigroup and $\bar{\mu}_2 = \mu_{2|\mathrm{RegS}} = \{(a,b)\in \mathrm{RegS}\times\mathrm{RegS} : (\forall e\in E(S))a^{-1}ea = b^{-1}eb\}$ is the maximum idempotent-separating congruence on RegS (Howie [8], Theorem 2.4), we have that μ_2 is idempotent-separating. The proof that μ_2 contains each r-semiprime idempotent-separating congruence on S is analogous to the proof of the corresponding part of Theorem 3.5. The proof that μ_2^b is the maximum idempotent-separating congruence on S is analogous to the corresponding part of Theorem 3.1.

The following corollary generalizes a result of Howie [8].

COROLLARY 3.2. Let S be a strongly ¶-inverse r-semigroup, then relation μ_2 defined by (4) is the maximum (r-semiprime) idempotent-separating congruence on S.

proof. Similarly to the proof of Corollary 3.1.

If S is a strongly ¶-inverse semigroup, then by [5] we have that

$$\ker\bar{\mu}_2 = \{a\in RegS : (\forall e\in E(S))ea = ae\}.$$

THEOREM 3.9. *Let* S *be a strongly ¶-inverse r-semigroup, then*

$$\ker\mu_2 = \{x\in S : r(x)\in\ker\bar{\mu}_2\}$$
$$= \{x\in S : (\exists e\in E(S))\ er(a) = r(a)e\}.$$

Proof. Analogous to the proof of Theorem 3.4.

THEOREM 3.10. *Let* S *be a ¶-regular semigroup, then the relation*

$$\nu = \{(a,b)\in S\times S : r(a) = r(b)\}$$

is an r-semiprime idempotent-separating equivalence relation which is contained in each r-semiprime equivalence relation on S. *The congruence*

$$\nu^b = \{(a,b)\in S\times S : (\forall x,y\in S^1)(xay,xby)\in\nu\}$$

is an idempotent-separating congruence on S.

Proof. It is obvious that ν is an r-semiprime equivalence. let $e,f\in E(S)$, then $e\nu f \iff e = f$ and so ν is idempotent-separating. Further, suppose that ρ is an r-semiprime equivalence on S and

$$a\nu b \iff r(a)\nu r(b) \iff r(a) = r(b).$$

Then $r(a)\rho r(b)$ and since ρ is a r-semiprime equivalence we have $a\rho b$, so $\nu\subseteq\rho$.

COROLLARY 3.3. Let S be an r-semigroup. Then ν is the minimum r-semiprime idempotent-separating congruence on S.

Proof. Let $a,b,c,d\in S$, then

$$\begin{aligned} a\nu b,\ c\nu d &\iff r(a) = r(b),\ r(c) = r(d) \\ &\implies r(ac) = r(a)r(c) = r(b)r(d) = r(bd) \\ &\iff ac\nu bd. \end{aligned}$$

THEOREM 3.11. *Let* S *be an* r*-semigroup, then*

$$\ker\nu = \{x\in S : r(x) \in E(S)\}.$$

Proof. Let $A = \{x\in S : r(x)\in E(S)\}$, then $a\in A$ implies $r(a)\in E(S)$ and since $a\nu r(a)$ we have $a\in\ker\nu$. Conversely, if $a\in\ker\nu$ then there exists $e\in E(S)$ such that $a\nu e$ and so $r(a) = e$ and $a\in A$. Hence, $\ker\nu = A$.

If S is a regular semigroup, then the relation ν is the equality relation.

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