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On Mutually Orthogonal Disjoint Copies of Graph Squares

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Abstract. A family of decompositions $\{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_{k-1}\}$ of a complete bipartite graph $K_{n,n}$ is a set of k *mutually orthogonal graph squares* ($MOGS$) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{0, 1, ..., k-1\}$ and $i \neq j$. For any subgraph G of $K_{n,n}$ with n edges, $N(n, G)$ denotes the maximum number k in a largest possible set $\{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_{k-1}\}$ of ($MOGS$) of $K_{n,n}$ by G. Our objective of this paper is to compute $N(n, G) = k \geq 3$ where G represents disjoint copies of certain subgraphs of $K_{n,n}$.

Keywords: Orthogonal graph squares; Orthogonal double cover; Mutually orthogonal Latin squares.

MSC 2000 classification: 05C70, 05B30.

1 Introduction

In this paper, $K_{m,n}$ denotes to the complete bipartite graph with partition sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle on n vertices, $s\ G$ for s disjoint copies of G and K_n for the complete graph on n vertices.

An edge decomposition $\mathcal{G} = \{G_0, G_1, \ldots, G_{s-1}\}$ of a graph H is a partition of the edge set of H into edge-disjoint subgraphs (*pages*) $G_0, G_1, \ldots, G_{s-1}$. If $G_i \cong G$ for all $i \in \{0, 1, ..., s-1\}$, then $\mathcal{G}$ is a decomposition of H by G. Two decompositions $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}\}$ and $\mathcal{F} = \{F_0, F_1, \ldots, F_{n-1}\}$ of the complete bipartite graph $K_{n,n}$ are orthogonal if $|E(G_i) \cap E(F_j)| = 1$ for all $i, j \in \{0, 1, ..., n-1\}$. Orthogonality requires that $|E(G_i)| = |E(F_i)| = n$ for all $i \in \{0, 1, ..., n-1\}$. A family of decompositions $\{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_{k-1}\}$ of $K_{n,n}$ is a set of k *mutually orthogonal graph squares* ($MOGS$) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{0, 1, ..., k-1\}$ and $i \neq j$. We use the notation $N(n, G)$ for the maximum number k in a largest possible set $\{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_{k-1}\}$ of (MOGS) of $K_{n,n}$ by G, where G is a bipartite graph with n edges.

If two decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by G are orthogonal, then $\mathcal{G} \cup \mathcal{F}$ is an *orthogonal double cover* (ODC) of $K_{n,n}$ by G. Orthogonal decompositions of graphs were studied by several authors; see the survey articles [1], [4],[5].

It is well-known that *orthogonal Latin squares* exist for every $n \notin \{2, 6\}$. A family of k-orthogonal Latin squares of order n is a set of k Latin squares any two of which are orthogonal. It is customary to denote $N(n) = \max\{k : \exists k \quad MOLS\}$ by the maximal number of squares in the largest possible set of mutually orthogonal Latin squares $(MOLS)$ of side n. An edge decomposition of $K_{n,n}$ by nK_2 is equal to a Latin square of side n; two edge decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by nK_2 are orthogonal if and only if the corresponding Latin squares of side n are orthogonal; thus $N(n, nK_2) = N(n)$. The computation of $N(n)$ is one of the most complicated problems in combinatorial designs; see the survey articles by Abel et al. [2] and Colbourn and Dinitz in [3]. It is clear that $N(n,\ G)$ is a natural generalization of $N(n)$. Many authors studied ODC of $K_{n,n}$ by G, which equal to $N(n,\ G) = 2$ (i.e., El-Shanawany et al. [5]). Here, we have exposed the first results of $N(n,\ G) = k \geq 3$ in the case of $G \neq nK_2$. El-Shanawany [6] could prove that $N(p,\ K_2 + ((p-1)/2)P_3) = p$ such that, $p > 2$, is a prime number and $N(p, (p-2)K_2 + P_3) \geq p - 1$, where p is a prime number. He also, conjectured that if p is a prime number, then $N(p,\ P_{p+1}) = p$. This guess has been proved by Sampathkumar et al. [7]. In [8] El-Shanawany has presented an interesting another proof of that guess. Also, he has given a new result for $N(n,\ G)$, where $G = \mathbb{P}_{d+1}(F)$ is a path of length d with $d+1$ vertices (*i.e.,* every edge of that path is one-to-one corresponding to an isomorphic to a graph F). The two sets $\{0_0, 1_0, ..., (n-1)_0\}$ and $\{0_1, 1_1, ..., (n-1)_1\}$ denote the vertices of the partition sets of $K_{n,n}$. If there is no chance of confusion, we will write (x, y) instead of $\{x_0, y_1\}$ for the edge between the vertices x_0 and y_1.

In the following, we give now the formal basic definitions of a G-square over additive group $\mathbb{Z}_n$.

Definition 1. (see [6]) Let G be a subgraph of $K_{n,n}$. A square matrix $\mathcal{L}$ of order n is called a G-square if every element in $\mathbb{Z}_n$ occur exactly n times and the graphs $G_\gamma,\quad \gamma \in \mathbb{Z}_n$ with $E(G_\gamma) = \{(x, y) : \mathcal{L}(x, y) = \gamma;\ x, y \in \mathbb{Z}_n\}$ are isomorphic to graph G.

For an edge decomposition G_i we may associate bijectively a $n \times n-$square with entries belonging to $\mathbb{Z}_n$ denoted by $\mathcal{L}_i = \mathcal{L}_i(x, y), 0 \leq i \leq k-1; x, y \in \mathbb{Z}_n$ with

$$\mathcal{L}_i(x, y) = \gamma \Leftrightarrow (x, y) \in E(G_{i\gamma}), \gamma \in \mathbb{Z}_n \tag{1}$$

Similar to Definition 1, we define:

Definition 2. Let i, j be different positive integers. Two square matrices $\mathcal{L}_i$ and $\mathcal{L}_j$ of order n are said to be orthogonal if for any ordered pair (a, b), there is exactly one position (x, y) for $\mathcal{L}_i(x, y) = a$ and $\mathcal{L}_j(x, y) = b$.

Theorem 1. *(see [1])There exist a set of $n-1$ pairwise orthogonal Latin squares of order n whenever n is a prime power.*

In [8] El-Shanawany presented an immediate result of the Definition 2 , $N(3, P_4) = 3$. Define the 3 $MOLSs$ of order 4 (3 *mutually orthogonal decompositions* (MOD) of $K_{3,3}$ by P_4) as follows:

$$\mathcal{K}_0 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}, \quad \mathcal{K}_1 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathcal{K}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \tag{2}$$

Applying Theorem 1 which satisfy as a special case of the Definition 2. We immediately get the following result, $N(4) = N(4, 4K_2) = 3$. Define the 3 $MOLSs$ of order 4 (3 *mutually orthogonal decompositions* (MOD) of $K_{4,4}$ by $4K_2$) as follows:

$$\mathcal{L}_0 = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \mathcal{L}_1 = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \tag{3}$$

2 Mutually Orthogonal Disjoint Copies of Graph Squares

In this section, we discuss new constructions for $MOGS$: one of which is direct and the others recursive. The following theorem mentions the building of $MOGS$, using $MOLS$.

Theorem 2. *Let k and $m \neq 2, 6$ be positive integers with $N(m,\ (m-1)K_2) = k$. Suppose that $N(n,\ G) = k$, where G is a subgraph of $K_{n,n}$, then $N(m\ n, m\ G) \geq k$, where $m\ G$ is a subgraph of $K_{m\ n, m\ n}$.*

Proof. Since $m \geq 3$, $m \neq 6$, suppose that there are k mutually orthogonal Latin squares

$$L^s = (a^s_{ij}), \quad s = 1, 2, \cdots, k, \quad 0 \leq i, j \leq m-1$$

of order m on the set $\{0, 1, ..., m-1\}$.

For any $l \in \{0, 1, ..., m-1\}$ and $G_l \cong G$, let

$$L^s_l = (b^{s,l}_{ij}), \quad s = 1, 2, \cdots, k, \quad 0 \leq i, j \leq n-1,$$

be k mutually orthogonal G_l-squares of order n.

Now we construct k mutually orthogonal (mG)-squares $M_s = (c^s_{ij})$, $s = 1, 2, \cdots, k$, and $0 \le i, j \le mn-1$. For given $i, j \in \{0, 1, ..., mn-1\}$, let $\alpha, \beta, \gamma, \delta$ be defined by

$$i = \alpha \cdot n + \beta, \quad 0 \le \alpha \le m-1, \quad 0 \le \beta \le n-1$$

$$j = \gamma \cdot n + \delta, \quad 0 \le \gamma \le m-1, \quad 0 \le \delta \le n-1,$$

then the entries c^s_{ij} of M_s are as follows:

$$c^s_{ij} = c^s_{\alpha \cdot n+\beta, \gamma \cdot n+\delta} = n \cdot a^s_{\alpha,\gamma} + b^{s,\alpha}_{\beta,\delta}. \tag{4}$$

We prove that this construction of M_s has the desired properties. Firstly, we show that M_s is an (mG)-square. Let $i \in \{0, 1, ..., mn-1\}$ be arbitrarily chosen. Then, let $\overline{\alpha}^s$ and $\overline{\beta}^s$ be

$$i = n \cdot \overline{\alpha}^s + \overline{\beta}^s, \quad 0 \le \overline{\alpha}^s \le m-1, \quad 0 \le \overline{\beta}^s \le n-1.$$

We are looking for all edges of the given graph by the entries i in M_s. By construction we have $n \cdot a^s_{\alpha,\gamma} + b^{s,\alpha}_{\beta,\delta} = n \cdot \overline{\alpha}^s + \overline{\beta}^s$. By the ranges of $\alpha, \beta, \overline{\alpha}, \overline{\beta}$, it follows that $a^s_{\alpha,\gamma} = \overline{\alpha}^s$ and $b^{s,\alpha}_{\beta,\delta} = \overline{\beta}^s$. For any $\alpha \in \{0, 1, ..., m-1\}$ there is a unique γ, with $a^s_{\alpha,\gamma} = \overline{\alpha}^s$, since L^s is a Latin square. For fixed α, γ the graph induced by the vertices

$$n \cdot \alpha + \beta, \quad n \cdot \gamma + \delta, \text{ where } \beta, \delta \in \{0, 1, ..., m-1\}$$

is exactly G_α. Since α is running from 0 to $m-1$ the graph given by the entries i in M_s is the vertex disjoint union of mG. Secondly, we show that M_s, $s = 1, 2, \cdots, k$ are mutually orthogonal. Assume the contrary, i.e., there are two equal pairs of entries of M_r and M_t where $1 \le r < t \le k$. That is, there are (x, y) and (x', y') with $x, y, x', y' \in \{0, 1, ..., mn-1\}$, $(x, y) \ne (x', y')$ and $(C^r_{x,y}, C^t_{x,y}) = (C^r_{x',y'}, C^t_{x',y'})$. Hence $C^r_{x,y} = C^r_{x',y'}$ and $C^t_{x,y} = C^t_{x',y'}$ and

$$n \cdot a^r_{\alpha,\gamma} + b^{r,\alpha}_{\beta,\delta} = n \cdot a^r_{\alpha',\gamma'} + b^{r,\alpha'}_{\beta',\delta'}$$

$$n \cdot a^t_{\alpha,\gamma} + b^{t,\alpha}_{\beta,\delta} = n \cdot a^t_{\alpha',\gamma'} + b^{t,\alpha'}_{\beta',\delta'}.$$

According to the range of $a's$ and $b's$ is follows

$$b^{r,\alpha}_{\beta,\delta} = b^{r,\alpha'}_{\beta',\delta'}, \quad a^r_{\alpha,\gamma} = a^r_{\alpha',\gamma'}$$

$$b^{t,\alpha}_{\beta,\delta} = b^{t,\alpha'}_{\beta',\delta'}, \quad a^t_{\alpha,\gamma} = a^t_{\alpha',\gamma'}.$$

Since L^r and L^t are orthogonal, from $(a^r_{\alpha,\gamma}, a^t_{\alpha,\gamma}) = (a^r_{\alpha',\gamma'}, a^t_{\alpha',\gamma'})$ follows that $\alpha = \alpha'$ and $\gamma = \gamma'$.Since L^r_α and L^t_α are orthogonal, from $(b^{r,\alpha}_{\beta,\delta}, b^{t,\alpha}_{\beta,\delta}) = (b^{r,\alpha'}_{\beta',\delta'}, b^{t,\alpha'}_{\beta',\delta'})$ follow that $\beta = \beta'$ and $\delta = \delta'$, i.e. $x = x'$ and $y = y'$ contradicting the assumption. QED

Until now, $N(n,\ C_n) = 2$ still remains open for all n, except for the special cases $n = 6$, and $n = 2^m$ ($m \geq 2$ is a positive integer) have been solved by El-Shanawany [6]. In the following, we give a direct construction of $N(n,\ C_n) \geq 3$ as the first result in this sense for $n = 4$.

Corollary 1. $N(4,\ C_4) \geq 3$.

Proof. Applying Definition 2 with $n = 4$, and for all $0 \leq s \leq 2$, there exist three mutually orthogonal decompositions (MOD) of $K_{4,4}$ by C_4 iff there exist three mutually orthogonal C_4-squares $\mathcal{N}_s$ of order 4 which defined as follows:

$$\mathcal{N}_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{bmatrix},\ \mathcal{N}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix},\ \mathcal{N}_2 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 3 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 0 & 2 & 2 & 0 \end{bmatrix} \quad (5)$$

We prove that the page obtained from the entries in $\mathcal{N}_0$ equal to 0 is isomorphic to C_4. Also, A similar argument applies to the other pages in $\mathcal{N}_0$, $\mathcal{N}_1$, and $\mathcal{N}_2$.There are exactly the two rows (columns) contain two 0-entry. That is, for all $x \in \mathbb{Z}_4$, there are exactly two vertices x_0 (x_1) have degree two and zero, respectively. QED

Conjecture 1. *If $n \geq 3$ a positive integer, then $N(2n,\ C_{2n}) \geq 3$.*

The next two new results follow immediately from Theorem 2 and Equation (3).

Corollary 2. *Let $q^\lambda \geq 4$ be a prime power for $\lambda \in \mathbb{Z}^+$. Then* $N\left(3q^\lambda,\ q^\lambda P_4\right) \geq 3$.

Proof. Since $q^\lambda \geq 4$, applying Theorem 1 to choose arbitrarily 3 mutually orthogonal Latin squares of order q^λ on the set $\{0, 1, ..., q^\lambda - 1\}$, define as follows:

$$L^s = (a^s_{ij}),\ 0 \leq s \leq 2\ ,\ \ 0 \leq i, j \leq q^\lambda - 1$$

For any $l \in \{0, 1, ..., q^\lambda - 1\}$ and $G_l \cong P_4$, let

$$L^s_l = (b^{s,l}_{ij}),\ \ 0 \leq s \leq 2,\ \ 0 \leq i, j \leq 2,$$

be 3 mutually orthogonal P_4-squares of order 3. Now we construct 3 of mutually orthogonal $(q^\lambda P_4)$-squares $M_s = (c^s_{ij})$, $0 \leq s \leq 2$, and $0 \leq i, j \leq 3q^\lambda - 1$. For given $i, j \in \{0, 1, ..., 3q^\lambda - 1\}$, let $\alpha, \beta, \gamma, \delta$ be defined by

$$i = \alpha \cdot n + \beta, \quad 0 \leq \alpha \leq q^\lambda - 1, \quad 0 \leq \beta \leq 2$$

$$j = \gamma \cdot n + \delta, \quad 0 \le \gamma \le q^\lambda - 1, \quad 0 \le \delta \le 2$$

then the entries c_{ij}^s of M_s are as follows:

$$c_{ij}^s = c_{\alpha\cdot n+\beta,\gamma\cdot n+\delta}^s = 3 \cdot a_{\alpha,\gamma}^s + b_{\beta,\delta}^{s,\alpha}. \tag{6}$$

We prove that the page obtained from the entries in M_0 equal to 0 is isomorphic to $q^\lambda P_4$. Also, a similar argument applies to the other pages in M_0, M_1, and M_2. There are exactly q^λ rows (columns) contain two 0-entry and q^λ rows (columns) contain one 0-entry, and q^λ rows (columns) contain no 0-entry. That is, for all $x \in \mathbb{Z}_{3q^\lambda}$, there are exactly q^λ vertices x_0 (x_1) have degree two, one and zero, respectively. $\boxed{QED}$

As a direct construction of this Corollary, for $n = 3$, $m = q^\lambda = 4$. For all $0 \le s \le 2$, applying Equation (6) using the 3 mutually orthogonal Latin squares $\mathcal{L}_s$ of order 4 as in Equation (3) with the corresponding 3 mutually orthogonal P_4-squares $\mathcal{K}_s$ of order 3 as in Equation (2) to define the 3 mutually orthogonal $4P_4-$squares M_s of order 12 as follows.

$$M_0 = \begin{bmatrix} 3 & 5 & 5 & 0 & 2 & 2 & 9 & 11 & 11 & 6 & 8 & 8 \\ 3 & 4 & 3 & 0 & 1 & 0 & 9 & 10 & 9 & 6 & 7 & 6 \\ 4 & 4 & 5 & 1 & 1 & 2 & 10 & 10 & 11 & 7 & 7 & 8 \\ 6 & 8 & 8 & 9 & 11 & 11 & 0 & 2 & 2 & 3 & 5 & 5 \\ 6 & 7 & 6 & 9 & 10 & 9 & 0 & 1 & 0 & 3 & 4 & 3 \\ 7 & 7 & 8 & 10 & 10 & 11 & 1 & 1 & 2 & 4 & 4 & 5 \\ 9 & 11 & 11 & 6 & 8 & 8 & 3 & 5 & 5 & 0 & 2 & 2 \\ 9 & 10 & 9 & 6 & 7 & 6 & 3 & 4 & 3 & 0 & 1 & 0 \\ 10 & 10 & 11 & 7 & 7 & 8 & 4 & 4 & 5 & 1 & 1 & 2 \\ 0 & 2 & 2 & 3 & 5 & 5 & 6 & 8 & 8 & 9 & 11 & 11 \\ 0 & 1 & 0 & 3 & 4 & 3 & 6 & 7 & 6 & 9 & 10 & 9 \\ 1 & 1 & 2 & 4 & 4 & 5 & 7 & 7 & 8 & 10 & 10 & 11 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 6 & 6 & 7 & 9 & 9 & 10 & 0 & 0 & 1 & 3 & 3 & 4 \\ 8 & 7 & 7 & 11 & 10 & 10 & 2 & 1 & 1 & 5 & 4 & 4 \\ 8 & 6 & 8 & 11 & 9 & 11 & 2 & 0 & 2 & 5 & 3 & 5 \\ 9 & 9 & 10 & 6 & 6 & 7 & 3 & 3 & 4 & 0 & 0 & 1 \\ 11 & 10 & 10 & 8 & 7 & 7 & 5 & 4 & 4 & 2 & 1 & 1 \\ 11 & 9 & 11 & 8 & 6 & 8 & 5 & 3 & 5 & 2 & 0 & 2 \\ 3 & 3 & 4 & 0 & 0 & 1 & 9 & 9 & 10 & 6 & 6 & 7 \\ 5 & 4 & 4 & 2 & 1 & 1 & 11 & 10 & 10 & 8 & 7 & 7 \\ 5 & 3 & 5 & 2 & 0 & 2 & 11 & 9 & 11 & 8 & 6 & 8 \\ 0 & 0 & 1 & 3 & 3 & 4 & 6 & 6 & 7 & 9 & 9 & 10 \\ 2 & 1 & 1 & 5 & 4 & 4 & 8 & 7 & 7 & 11 & 10 & 10 \\ 2 & 0 & 2 & 5 & 3 & 5 & 8 & 6 & 8 & 11 & 9 & 11 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 9 & 10 & 9 & 6 & 7 & 6 & 3 & 4 & 3 & 0 & 1 & 0 \\ 10 & 10 & 11 & 7 & 7 & 8 & 4 & 4 & 5 & 1 & 1 & 2 \\ 9 & 11 & 11 & 6 & 8 & 8 & 3 & 5 & 5 & 0 & 2 & 2 \\ 3 & 4 & 3 & 0 & 1 & 0 & 9 & 10 & 9 & 6 & 7 & 6 \\ 4 & 4 & 5 & 1 & 1 & 2 & 10 & 10 & 11 & 7 & 7 & 8 \\ 3 & 5 & 5 & 0 & 2 & 2 & 9 & 11 & 11 & 6 & 8 & 8 \\ 6 & 7 & 6 & 9 & 10 & 9 & 0 & 1 & 0 & 3 & 4 & 3 \\ 7 & 7 & 8 & 10 & 10 & 11 & 1 & 1 & 2 & 4 & 4 & 5 \\ 6 & 8 & 8 & 9 & 11 & 11 & 0 & 2 & 2 & 3 & 5 & 5 \\ 0 & 1 & 0 & 3 & 4 & 3 & 6 & 7 & 6 & 9 & 10 & 9 \\ 1 & 1 & 2 & 4 & 4 & 5 & 7 & 7 & 8 & 10 & 10 & 11 \\ 0 & 2 & 2 & 3 & 5 & 5 & 6 & 8 & 8 & 9 & 11 & 11 \end{bmatrix}.$$

Corollary 3. *Let $\lambda \geq 0$, be an integer number. Then $N\left(2^{2(\lambda+1)}, 2^{2\lambda}C_4\right) \geq 3$.*

Proof. Note that for $\lambda = 0$, see Corollary 1. For $\lambda > 0$, applying Theorem 1 to choose arbitrarily 3 mutually orthogonal Latin squares of order $2^{2\lambda}$ on the set $\{0, 1, ..., 2^{2\lambda} - 1\}$, define as follows:

$$L^s = (a^s_{ij}),\ 0 \leq s \leq 2\ ,\quad 0 \leq i, j \leq 2^{2\lambda} - 1$$

For any $l \in \{0, 1, ..., 2^{2\lambda} - 1\}$ and $G_l \cong C_4$, let

$$L^s_l = (b^{s,l}_{ij}),\quad 0 \leq s \leq 2,\quad 0 \leq i, j \leq 3,$$

be 3 mutually orthogonal C_4-squares of order 4. Now we construct 3 of mutually orthogonal $(2^{2\lambda}C_4)$-squares $M_s = (c^s_{ij})$, $0 \leq s \leq 2$, and $0 \leq i, j \leq 2^{2(\lambda+1)} - 1$. For given $i, j \in \{0, 1, ..., 2^{2(\lambda+1)} - 1\}$, let $\alpha, \beta, \gamma, \delta$ be defined by

$$i = \alpha \cdot n + \beta,\quad 0 \leq \alpha \leq 2^{2\lambda} - 1,\quad 0 \leq \beta \leq 3$$

$$j = \gamma \cdot n + \delta,\quad 0 \leq \gamma \leq 2^{2\lambda} - 1,\quad 0 \leq \delta \leq 3$$

then the entries c^s_{ij} of M_s are as follows:

$$c^s_{ij} = c^s_{\alpha\cdot n+\beta,\gamma\cdot n+\delta} = 4 \cdot a^s_{\alpha,\gamma} + b^{s,\alpha}_{\beta,\delta}. \tag{7}$$

We prove that the page obtained from the entries in M_0 equal to 0 is isomorphic to $2^{2\lambda}C_4$. Also, a similar argument applies to the other pages in M_0, M_1, and M_2.There are exactly $2^{2\lambda+1}$ rows (columns) contain two 0-entry and $2^{2\lambda+1}$ rows (columns) contain no 0-entry. That is, for all $x \in \mathbb{Z}_{2^{2(\lambda+1)}}$, there are exactly $2^{2\lambda+1}$ vertices x_0 (x_1) have degree two and zero, respectively. QED

As a direct construction of this Corollary, for $n = 4$, $m = 2^{2\lambda} = 4$. For all $0 \leq s \leq 2$, applying Equation (7) using the 3 mutually orthogonal Latin squares $\mathcal{L}_s$ of order 4 as in Equation (3) with the corresponding 3 mutually orthogonal C_4-squares $\mathcal{N}_s$ of order 4 as in Equation (5) to define, the 3 mutually orthogonal $4C_4-$squares M_s of order 16 as follows.

$$M_0 = \begin{bmatrix}
4 & 4 & 5 & 5 & 0 & 0 & 1 & 1 & 12 & 12 & 13 & 13 & 8 & 8 & 9 & 9 \\
4 & 4 & 5 & 5 & 0 & 0 & 1 & 1 & 12 & 12 & 13 & 13 & 8 & 8 & 9 & 9 \\
6 & 6 & 7 & 7 & 2 & 2 & 3 & 3 & 14 & 14 & 15 & 15 & 10 & 10 & 11 & 11 \\
6 & 6 & 7 & 7 & 2 & 2 & 3 & 3 & 14 & 14 & 15 & 15 & 10 & 10 & 11 & 11 \\
8 & 8 & 9 & 9 & 12 & 12 & 13 & 13 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\
8 & 8 & 9 & 9 & 12 & 12 & 13 & 13 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\
10 & 10 & 11 & 11 & 14 & 14 & 15 & 15 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\
10 & 10 & 11 & 11 & 14 & 14 & 15 & 15 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\
12 & 12 & 13 & 13 & 8 & 8 & 9 & 9 & 4 & 4 & 5 & 5 & 0 & 0 & 1 & 1 \\
12 & 12 & 13 & 13 & 8 & 8 & 9 & 9 & 4 & 4 & 5 & 5 & 0 & 0 & 1 & 1 \\
14 & 14 & 15 & 15 & 10 & 10 & 11 & 11 & 6 & 6 & 7 & 7 & 2 & 2 & 3 & 3 \\
14 & 14 & 15 & 15 & 10 & 10 & 11 & 11 & 6 & 6 & 7 & 7 & 2 & 2 & 3 & 3 \\
0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 & 8 & 8 & 9 & 9 & 12 & 12 & 13 & 13 \\
0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 & 8 & 8 & 9 & 9 & 12 & 12 & 13 & 13 \\
2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 & 10 & 10 & 11 & 11 & 14 & 14 & 15 & 15 \\
2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 & 10 & 10 & 11 & 11 & 14 & 14 & 15 & 15
\end{bmatrix}$$

$$M_1 = \begin{bmatrix}
8 & 9 & 8 & 9 & 12 & 13 & 12 & 13 & 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 \\
10 & 11 & 10 & 11 & 14 & 15 & 14 & 15 & 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 \\
8 & 9 & 8 & 9 & 12 & 13 & 12 & 13 & 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 \\
10 & 11 & 10 & 11 & 14 & 15 & 14 & 15 & 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 \\
12 & 13 & 12 & 13 & 8 & 9 & 8 & 9 & 4 & 5 & 4 & 5 & 0 & 1 & 0 & 1 \\
14 & 15 & 14 & 15 & 10 & 11 & 10 & 11 & 6 & 7 & 6 & 7 & 2 & 3 & 2 & 3 \\
12 & 13 & 12 & 13 & 8 & 9 & 8 & 9 & 4 & 5 & 4 & 5 & 0 & 1 & 0 & 1 \\
14 & 15 & 14 & 15 & 10 & 11 & 10 & 11 & 6 & 7 & 6 & 7 & 2 & 3 & 2 & 3 \\
4 & 5 & 4 & 5 & 0 & 1 & 0 & 1 & 12 & 13 & 12 & 13 & 8 & 9 & 8 & 9 \\
6 & 7 & 6 & 7 & 2 & 3 & 2 & 3 & 14 & 15 & 14 & 15 & 10 & 11 & 10 & 11 \\
4 & 5 & 4 & 5 & 0 & 1 & 0 & 1 & 12 & 13 & 12 & 13 & 8 & 9 & 8 & 9 \\
6 & 7 & 6 & 7 & 2 & 3 & 2 & 3 & 14 & 15 & 14 & 15 & 10 & 11 & 10 & 11 \\
0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 & 12 & 13 & 12 & 13 \\
2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 & 14 & 15 & 14 & 15 \\
0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 & 12 & 13 & 12 & 13 \\
2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 & 14 & 15 & 14 & 15
\end{bmatrix},$$

$$M_2 = \begin{bmatrix}
12 & 14 & 14 & 12 & 8 & 10 & 10 & 8 & 4 & 6 & 6 & 4 & 0 & 2 & 2 & 0 \\
15 & 13 & 13 & 15 & 11 & 9 & 9 & 11 & 7 & 5 & 5 & 7 & 3 & 1 & 1 & 3 \\
15 & 13 & 13 & 15 & 11 & 9 & 9 & 11 & 7 & 5 & 5 & 7 & 3 & 1 & 1 & 3 \\
12 & 14 & 14 & 12 & 8 & 10 & 10 & 8 & 4 & 6 & 6 & 4 & 0 & 2 & 2 & 0 \\
4 & 6 & 6 & 4 & 0 & 2 & 2 & 0 & 12 & 14 & 14 & 12 & 8 & 10 & 10 & 8 \\
7 & 5 & 5 & 7 & 3 & 1 & 1 & 3 & 15 & 13 & 13 & 15 & 11 & 9 & 9 & 11 \\
7 & 5 & 5 & 7 & 3 & 1 & 1 & 3 & 15 & 13 & 13 & 15 & 11 & 9 & 9 & 11 \\
4 & 6 & 6 & 4 & 0 & 2 & 2 & 0 & 12 & 14 & 14 & 12 & 8 & 10 & 10 & 8 \\
8 & 10 & 10 & 8 & 12 & 14 & 14 & 12 & 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 \\
11 & 9 & 9 & 11 & 15 & 13 & 13 & 15 & 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 \\
11 & 9 & 9 & 11 & 15 & 13 & 13 & 15 & 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 \\
8 & 10 & 10 & 8 & 12 & 14 & 14 & 12 & 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 \\
0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 & 12 & 14 & 14 & 12 \\
3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 & 15 & 13 & 13 & 15 \\
3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 & 15 & 13 & 13 & 15 \\
0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 & 12 & 14 & 14 & 12
\end{bmatrix}.$$

3 Conclusion

This paper is devoted to computing to $N(n, G)$, where G represents disjoint copies of graph squares as in Theorem 2, and introduce constructions of new results as in Corollaries 1,2,3, respectively.

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