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ON A CHARACTERIZATION OF SIMPLE EXTENSIONS OF TOPOLOGIES (°)

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Sunto. *Dato uno spazio topologico* (S,τ) *ed un sottoinsieme* X *di* S, *la topologia* $\tau(X) = \{A\cup(A'\cap X)/A,A'\in\tau\}$ *si dice estensione semplice di* τ *rispetto ad* X *(si veda Levine* [3]).

In questo lavoro stabiliamo un criterio per verificare se due sottoinsiemi X *ed* Y *di* S *individuano la stessa estensione semplice di* τ *e troviamo una caratterizzazione di quei raffinamenti di* τ *che sono estensioni semplici di* τ *rispetto ad opportuni sottoinsiemi di* S.

1. INTRODUCTION.

Levine gave [3] a method to obtain expansions of topologies called simple extensions. Those expansions were studied also by Borges [1] and Reynolds [4] and are usefull to find maximal topologies.

If X is a subset in a topological space (S,τ), the simple extension of τ

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by X is the topology on S $\tau(X) = \{A\cup(A'\cap X)/A,A'\in\tau\}$.

Here we recognize when an expansion τ' of a given topology τ on S is a simple extension of τ by suitable subsets of S; if so, we show how to construct a subset X of S such that $\tau'=\tau(X)$. It follows from proposition 1 that such a subset X is not uniquely determined.

clX (intX) will denote the closure (the interior) of X in the topological space (S,τ);$cl_{\tau'}X$ $(int_{\tau'}X)$ will denote the closure (the interior) of X relative to any other topology τ' on S. We shall write $\tau(x)$ for the filter of open neighbourhoods of x in τ. If $X\subseteq S$, CX will be the complement of X in S and, if $X\subseteq Y\subseteq S$, C_YX or Y-X will denote the complement of X in Y.

2. SUBSETS WICH GIVE THE SAME SIMPLE EXTENSION.

First we recall that a basis of the simple extension $\tau(X)$ of the topology τ on S by $X\subseteq S$ is the family

$$\mathrm{B}(X)=\tau\cup(\tau\cap X)$$

where $\tau\cap X$ denotes the topology on X induced by τ.

If $\tau(X)$ is a simple extension of (S,τ) and $B\subseteq S$, then (see [3] lemma 1)

$$int_{\tau(X)}B = intB\cup int_{\tau\cap X}(B\cap X).$$

In the following we shall alwais denote by (S,τ) a topological space.

LEMMA 1. *Let* $A\in\tau$,$Y\subseteq X\subseteq S$ *such that* X-intX = Y-intY=X' *and* $cl(X-Y)\cap X'=\emptyset$ *and let* $B=A-cl(A\cap X\cap CY)\in\tau$; *then we have*

i) $A\cap intY = B\cap intX$

ii) $A\cap X' = B\cap X'$.

Proof. i) Let $p\in A$ and $p\in \mathrm{int}Y \subseteq \mathrm{int}X$. If $p\in cl(A\cap X\cap CY) \subseteq clA\cap clX\cap clCY$ then we should have $p\in clCY = C\mathrm{int}Y$ whence $p\notin \mathrm{int}Y$, which is absurd. So $p\in B\cap \mathrm{int}X$.

Conversely it is sufficient to prove that $B\cap \mathrm{int}X \subseteq B\cap \mathrm{int}Y$, being $B \subseteq A$ Preliminarly we have $B\cap \mathrm{int}X = A\cap \mathrm{int}(CA\cup CX\cup Y) \cap \mathrm{int}X = A\cap \mathrm{int}((CA\cap X)\cup Y) = A\cap \mathrm{int}(CA\cup Y)\cap \mathrm{int}X$ and similarly $B\cap \mathrm{int}Y = A\cap \mathrm{int}(CA\cup Y)\cap \mathrm{int}Y$.

Now let $p\in A$ and $N\in \tau(p)$ where $N \subseteq A$, $N\subseteq CA\cup Y$ and $N \subseteq X$; trivially $N \subseteq (CA\cup Y)\cap \mathrm{int}X = (CA\cap \mathrm{int}X)\cup \mathrm{int}Y = (CA\cup \mathrm{int}Y) \cap \mathrm{int}X$ and from $N \subseteq A$ it follows that $N \subseteq \mathrm{int}Y$ and $p\in A\cap \mathrm{int}(CA\cup Y)\cap \mathrm{int}Y$.

ii) The assertion follows from $A\cap X' = (A-cl(X-Y)) \cap X' \subseteq (A-cl(A\cap (X-Y)))\cap X' = B\cap X'$.

PROPOSITION 1. *Let* (S,τ) *be a topological space and* X,Y *two subsets of* S. *Then* $\tau(X) = \tau(Y)$ *if and only if*

$$X-\mathrm{int}X = Y-\mathrm{int}Y = X'$$

$$cl(X-Y) \cap X' = cl(Y-X) \cap X' = \emptyset$$

Proof. The given conditions are sufficient.

We suppose first $Y \subseteq X$. Then $\mathbf{B}(X) \subseteq \tau(Y)$ since $U = A\cap X = (A\cap(\mathrm{int}X) \cup (A\cap Y)\in \tau(X)$ for every $A\in \tau$; thus $\tau(X) \subseteq \tau(Y)$.

Moreover $\mathbf{B}(Y) \subseteq \mathbf{B}(X)$; indeed by lemma 1 we have $V = A\cap Y = (A\cap \mathrm{int}Y)\cup (A\cap X')= (B\cap \mathrm{int}X)\cup(B\cap X') = B\cap X \in\mathbf{B}(X)$ for every $A\in\tau$.

In the general case, let $Z = X\cap Y$; we have

$Z - \mathrm{int}Z = (X\cap Y)\cap C(\mathrm{int}X\cap \mathrm{int}Y)=(X\cap Y\cap C\mathrm{int}X)\cup(X\cap Y\cap C\mathrm{int}Y)=(Y\cap X')\cup(X\cap X')=X'$;

$cl(X-Z)\cap X' = cl(X\cap(CX\cup CY))\cap X' = cl((X\cap CX)\cup(X\cap CY))\cap X' = cl(X-Y)\cap X' = \emptyset$;

and similarly $cl(Y-Z)\cap X' = \emptyset$.

It follows that $\tau(X) = \tau(Z) = \tau(Y)$.

The conditions are necessary.

Let $\tau(X) = \tau(Y)$ and let $p\in Y-\mathrm{int}Y$. If $p\notin X$ and $Y = A\cup(B\cap X)$ with $A,B\in\tau$,

then $p \in A \subseteq Y$ and $p \in \mathrm{int}Y$, a contradiction.
If $p \in \mathrm{int}X$ and $p \in N \subseteq X$, $N \in \tau(p)$, we have $p \in \mathrm{int}_{\tau(X)}(N \cap Y) = \mathrm{int}(N \cap Y) \cup \cup\, \mathrm{int}_{\tau \cap X}(N \cap Y \cap X) = ((\mathrm{int}N \cap \mathrm{int}Y) \cup \mathrm{int}_{\tau \cap X}(N \cap Y \cap X)$. Since $p \notin \mathrm{int}Y$ we must have $p \in \mathrm{int}_{\tau \cap X}(N \cap Y \cap X) = \mathrm{int}_{\tau \cap N} N \cap \mathrm{int}_{\tau \cap X} Y \cap \mathrm{int}_{\tau \cap X} X$.
Then $N' \in \tau(p)$ exists such that $p \in N' \cap X \subseteq Y$; but from $p \in N \cap N' \subseteq N' \cap X \subseteq Y$ it follows that $p \in \mathrm{int}\, Y$, a contradiction again.
The inclusion $Y - \mathrm{int}Y \subseteq X - \mathrm{int}X$ is now proved; similarly the converse is shown to hold, so $Y - \mathrm{int}Y = X - \mathrm{int}X$.
Finally we prove $\mathrm{cl}(X-Y) \cap X' = \emptyset$. Suppose, ab absurdo, $p \in \mathrm{cl}(X-Y) \cap X'$ and $p \in N \in \tau$; then for every open set $O = A \cup (B \cap X)$ in $\tau(X)$ that contains p we must have $O \cap (X-Y) \neq \emptyset$. On the other hand the open set $N \cap Y$ in $\tau(Y)$ should not intersect X-Y and consequently $N \cap Y \neq O$ for each $O \in \tau(X)$, which contradicts the assumption $\tau(Y) = \tau(X)$.

An identical proof can be given for the condition $\mathrm{cl}(Y-X) \cap X' = \emptyset$.

COROLLARY 1. *Given a topological space* (S, τ), *two subsets* X, Y *without interior points define the same simple extension of* τ *if and only if they coincide.*

Proof. $X \in \tau(Y) \Rightarrow X = A \cup (B \cap Y)$ with $A, B \in \tau \Rightarrow \emptyset = \mathrm{int}X \supseteq \mathrm{int}A \cup \mathrm{int}(B \cap Y) \Rightarrow \Rightarrow A = \mathrm{int}A = \emptyset \Rightarrow X \subseteq Y$. Similarly $Y \subseteq X$.

COROLLARY 2. *Let* (S, τ) *be a topological space*, $X, Y \subseteq S$ *and* $X' = X - \mathrm{int}X$. *Then*

$$Y = X - F,\ \mathrm{cl}F = F \text{ and } F \cap X' = \emptyset \Rightarrow \tau(X) = \tau(Y).$$

Proof. The conditions of proposition 1 are easily shown to be true;

$$Y - \mathrm{int}Y = Y \cap CF \cap C(\mathrm{int}X \cap CF) = X \cap CF \cap (C\mathrm{int}X \cup F) = (X \cap CF \cap C\mathrm{int}X) \cup (X \cap CF \cap F) = X' \cap CF = X'.$$

$$X - Y \subseteq F \Rightarrow \mathrm{cl}(X-Y) \cap X' \subseteq F \cap X' = \emptyset.$$

COROLLARY 3. *Let* (S, τ) *be a topological space,* X *and* Y *two subsets of* S *and* $X' = X - \text{int}X$. *Then*

$$Y = X \cup 0,\ 0 \in \tau\ ,\ \text{cl}0 \cap X' = \emptyset \Rightarrow \tau(X) = \tau(Y).$$

Proof. As $\text{int}Y = \text{int}X \cup 0$, we have $Y' = Y - \text{int}Y = (X \cup 0) \cap C(\text{int}X \cup 0) = (X \cap C\text{int}X \cap C0) \cup (0 \cap C\text{int}X \cap C0) = X' \cap C0 = X'$.

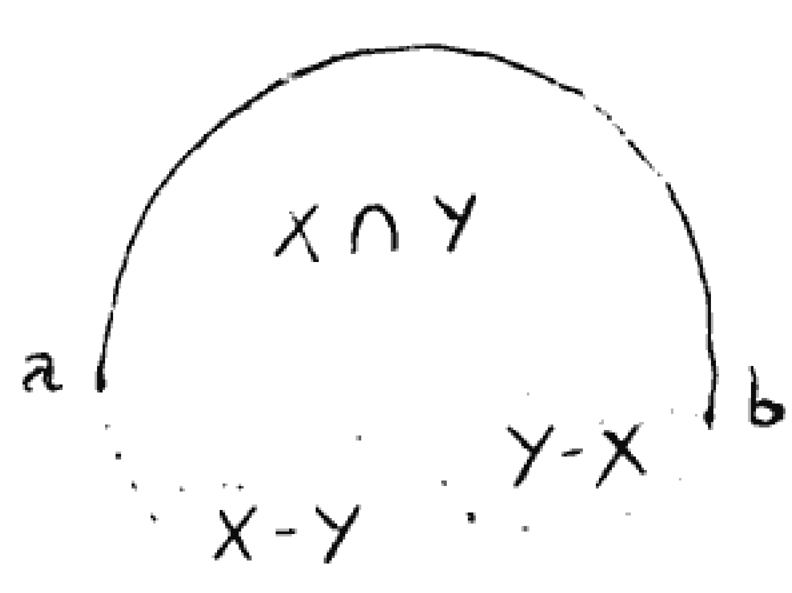
Furthermore we have trivially $\text{cl}(X-Y) \cap X' = \emptyset$ and it follows from $Y-X \subseteq 0$ that $\text{cl}(Y-X) \cap X' \subseteq \text{cl}0 \cap X' = \emptyset$.

REMARK 1. From corollary 2 and 3 it follows that, given a topological space (S, τ) and a subset $X \subseteq S$ with interior points, there is neither a maximal subset $M \subseteq S$ nor a minimal subset $N \subseteq S$ such that $\tau(M) = \tau(X)$ or $\tau(N) = \tau(X)$.

The results given in proposition 1 allow us to provide the following examples in which the underlying topological space is the euclidean plane with the usual topology.

In the figures the continuous lines mean that the boundary points belong to indicated subsets,

Example 1. Trivially the conditions of proposition 1 are verified and $\tau(X) = \tau(Y)$.



$X' = X - \text{int}X = Y - \text{int}Y$

$a \in X'$

$b \notin X'$

Fig. 1

Example 2. The condition X-intX = Y-intY does not hold, so $\tau(X) \neq \tau(Y)$. Indeed the open set $U' \in \tau(Y)$ drawn in the figure 2 does not belong to $\tau(X)$.

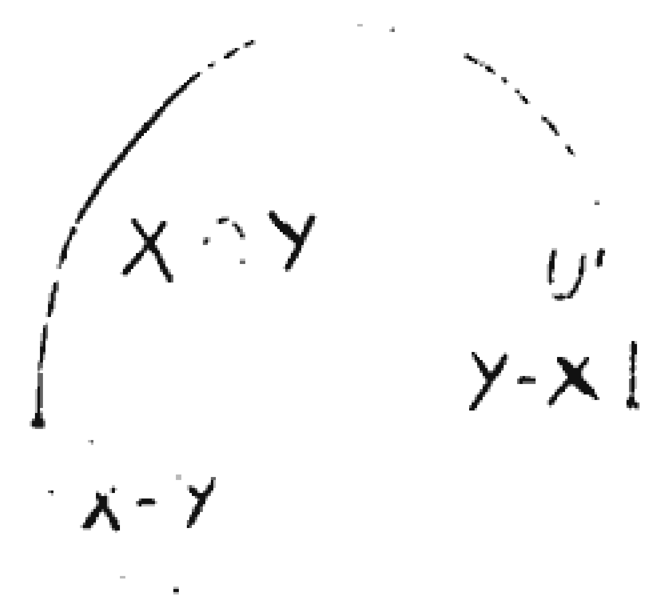
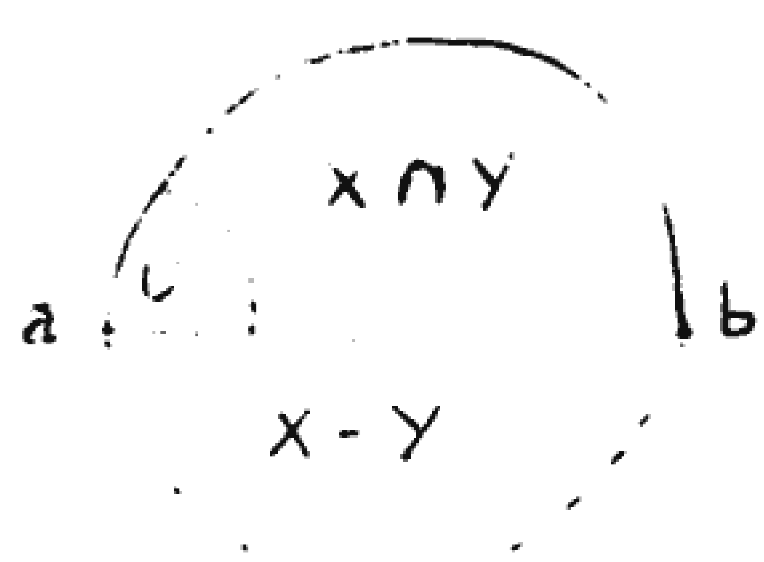


Fig. 2

Example 3. The condition X-intX = Y-intY = X' is verified but $cl(X-Y) \cap X = \{a\} \neq \emptyset$. Also in figure 3 an open set is drawn $U' \in \tau(Y)$ that does not belong to $\tau(X)$; of course, $\tau(X) \neq \tau(Y)$.



$a \in X'$, $b \notin X'$

$a \in U'$

Fig. 3

3. THE MAIN RESULTS.

PROPOSITION 2. *Given two topologies* τ, τ' *on* S *and assuming that* $\tau' = \tau(X)$ *is a simple extension of* τ, *it follows that* X-cl(intX) *is the union of those open sets of* τ' *that do not contain non-empty open sets of* τ .

Proof. Let $p \in X-cl(intX)$ and let $N \in \tau(p)$ verify the condition $N \cap cl(intX) = \emptyset$. The only open set in τ included in $N' = N \cap X \in \tau'$ is the empty set; so $X-cl(intX)$ is included in the union of those open sets of τ' that do not contain non-empty open sets of τ

Conversely let $A,B \in \tau$, $U' = A \cup (B \cap X) \in \tau'$ and $U' \neq \emptyset$

If U' does not contain non-empty open sets of τ, then $A = \emptyset$; thus $U' = B \cap X$. If on the other hand we had $U' \cap C(X-cl(intX)) \neq \emptyset$, it would follow $B \cap X \cap cl(intX) \neq \emptyset$; $B \cap intX$ would therefore be a non-empty open set of τ included in U', a contradiction. So $U' \subseteq X-cl(intX)$ and the assertion is true.

The following corollary is now an easy consequence.

COROLLARY 4. *If* (S, τ') *is a simple extension of* (S, τ), $\tau' = \tau(X)$, *and if* $intX = \emptyset$, *then* X *is the union of those open sets of* τ' *containing no non-empty open sets of* τ .

PROPOSITION 3. *Let* (S, τ') *be a simple extension of* (S, τ), $\tau' = \tau(X)$, *then*

$$X' = X-intX = \bigcup_{U' \in \tau'} (U'-intU').$$

Proof. Trivially $X' \subseteq \bigcup_{U' \in \tau'} (U' - intU')$ since $X \in \tau'$.

For the inverse inclusion let us take $U' = A \cup (B \cap X) \in \tau'$ with $A,B \in \tau$. We have $intU' \supseteq A \cup (B \cap intX)$, hence $U'-intU' \subseteq (A \cup (B \cap X)) - (A \cup (B \cap intX)) \subseteq$ $\subseteq (B \cap X) - (B \cap intX) = B \cap (X-intX) \subseteq X'$.

PROPOSITION 4. *Let* (S,τ) *be a topological space*, $X \subseteq S$ *and* $X' =$ $= X-intX$. *If* F *is a closed set in* τ, $F \cap X = X'$, $cl(F-X') = F$ *and* $X' \subseteq int(F \cup X)$ *then setting* $Y = CF \cup X'$ *we have*

$$\tau(X) = \tau(Y).$$

Proof. Since $Y = CF \cup X' = (CF\cup X)\cap(CF\cup \mathrm{Cint}X) = CF\cup X$, we have $Y\text{-int}Y = (CF\cup X)\cap \mathrm{cl}(F\cap CX') = (CF\cup X)\cap F = X\cap F = X'$.

Furthermore $\mathrm{cl}(Y-X)\cap X' = \mathrm{cl}(CF\cap CX)\cap X' = \mathrm{Cint}(F\cup X)\cap X' = \emptyset$ and $X \subseteq Y$, hence $\mathrm{cl}(X-Y) = \emptyset$.

The assertion follows now from proposition 1.

PROPOSITION 5. *Let* $\tau \subseteq \tau'$ *be two topologies on* S *and let* $X' = \bigcup_{U'\in\tau'}(U'-\mathrm{int}U')$.

A necessary and sufficient condition for τ' *to be a simple extension of* τ *is that there exists a closed set* F *in* τ *containing* X' *and verifying the conditions*

1) $\mathrm{cl}(F-X') = F$

2) $\forall p\in X'$ *and* $\forall N' \in \tau'(p)$ $\exists N \in \tau(p)$ *such that* $N\cap(CF\cup X')\in \tau'(p)$ *and* $N\cap(CF\cup X') \subseteq N'$.

In that case we have $\tau' = \tau(X)$, *where* $X = CF \cup X'$.

Proof. (Sufficiency). Let us consider a closed set F verifying the conditions 1) and 2) and let $X = CF\cup X'$.

Then it follows that $\tau' \subseteq \tau(X)$; in fact if $N'\in\tau'$ and $N'\cap X' = \emptyset$ then $N'-\mathrm{int}N' = \emptyset$ and $N'\in \tau \subseteq \tau(X)$; if, on the contrary. $N'\cap X' = L \neq \emptyset$ and $p\in L$, then there exists $N_p \in \tau(p)$ with $N_p\cap X \subseteq N'$ and $N_p \cap X \in \tau'(p)$.

So the open set $N = \bigcup_{p\in L} N_p \in\tau$ contains L and intersect X in a subset of N'; consequently $N' = L\cup \mathrm{int}N' = \mathrm{int}N'\cup(N\cap X) \in \tau(X)$.

On the other hand we shall prove $\tau\cap X\subset\tau'$; in fact by $X\text{-int}X = (CF\cup X')\cap F = X'$ we have

if $N\in\tau$ and $N \subseteq X$, then $N\cap X = N \in \tau'$;

if $N \in \tau$, $N\cap CX \neq \emptyset$ and $N\cap X' = \emptyset$, then $N\cap X \subseteq \mathrm{int}X$ and $N\cap X\in\tau\subseteq \tau'$;

if $N \in \tau$ and $N \cap X' = L \neq \emptyset$ we have $N \cap intX \in \tau \subseteq \tau'$ and each p in L must belong to an open neighbourhood $N_p \subseteq N$ such that $N_p \cap X \in \tau'(p)$; if we put

$$N' = \bigcup_{p \in L} N_p \cap X \in \tau', \text{ we have } N \cap X = (N \cap intX) \cup N' \in \tau'.$$

(Necessity). If $\tau' = \tau(X)$, then we have (see proposition 3) $X-intX=X'$. Trivially the closed set $F = CintX$ in τ contains X' and verifies the condition $cl(F-X') = F$.

Now, if $p \in X'$ and $N' \in \tau'(p)$, namely $N' = A \cup (B \cap X)$ where $A,B \in \tau$, then the open neighbourhood $N = A \cup B \in \tau(p)$ is such that $N \cdot \cap (CF \cup X') = (A \cap X) \cup (B \cap X) \in \tau'(p)$ and $N \cap (CF \cup X') \subseteq N$.

REMARK 2. If τ' is a simple extension of τ, the subset $X' = \bigcup_{U' \in \tau'} (U'-intU')$ must contain no interior point. (This follows trivially from proposition 3).

REMARK 3. If the assumptions of proposition 5 hold and furthermore the subset $X' = \bigcup_{U' \in \tau'} (U'-intU')$ belong to τ' and contains no interior point in τ, then the closed set S in τ verifies the conditions 1) and 2) and consequently $\tau' = \tau(X')$.

Now we give two examples in which we recognize that a given refinemennt τ' of τ is a simple extension of τ.

Example 4. Let $S = \{a,b,c,d,f\}$, and let $\tau = \{\emptyset,\{a\},\{b\},\{d,f\},\{a,b\}, \{a,d,f\},\{b,d,f\}, \{a,b,d,f,\},S\}$ be a topology on S. $\tau' = \tau \cup (\{d\},\{a,d\},\{b,d\}, \{c,d),\{a,c,d\},\{b,c,d\},\{a,b,d\},\{c,d,f\},\{a,b,c,d\},\{a,c,d,f\},\{b,c,d,f)\}$ is an expansion of τ' and we have $X' = \bigcup_{N' \in \tau'} (N'-intN') = (c,d)$.

Since $X' \in \tau'$ and $intX' = \emptyset$ we have, by remark 3, $\tau' = \tau(X')$.

Example 5. $S = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},\{c,d\},\{a,c,d\}, \{b,c,d\},S\}$; $\tau' = \tau \cup \{\{c\},\{a,c\},\{b,c\},\{a,b,c\}\}$ is an expansion of τ and we have $X' = \bigcup_{N' \in \tau'} (N'-intN') = \{c\}$.

It is easily seen that the closed subset $F = \{c,d\}$ in τ contains X' and verifies the condition $cl(F-X') = F$; if furthermore $N' \in \tau'(c)$ and $N = \{c,d\} \in \tau(c)$, we have $N \cap (CF \cup X') = \{c\} \in \tau'(c)$ and by proposition 5 we have $\tau' = \tau(\{a,b,c\})$.

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