

# Holonomic spaces

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**Abstract.** The purpose of this note is to give a synthetic description of the intrinsic metric of a fiber  $F$  of the connection metric on a vector bundle  $E \rightarrow M$  over a riemannian manifold  $M$  endowed with a euclidean metric and a compatible connection. These metrics are described in terms of the action of the holonomy group and several properties are derived thereafter. In the process, a strong metric geometric meaning is given to the holonomy group.

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In modern geometry, the tangent bundle has been central and has been extensively studied by Sasaki [16], Musso and Tricerri [11], Abbassi and Sarih [1], Beyounes et al. [3], among many others. In order to understand the Gromov-Hausdorff limits of tangent bundles, one must make a study of the properties of such bundles from the viewpoint of Metric Geometry, that is the synthetic properties of their induced metric-space structure. It is a well-known fact that the differential geometry of the tangent bundle with its Sasaki metric is fairly rigid. In particular, its fibers are totally geodesic and flat (see [13]). However, here we see that the metric properties of the embedding of these fibres depend very strongly upon the holonomy group of the manifold.

At any given point on a riemannian manifold there are three pieces of information that interplay: the tangent space, as a normed vector space  $V$ ; the holonomy group, as a subgroup  $H$  of the isometry group of the fiber; and a group-norm  $L$  on the holonomy group, given by considering the infimum

$$L(A) = \inf_{\gamma} \ell(\gamma) \tag{1}$$

of the lengths of the loops  $\gamma$  that yield a given holonomy element  $A$ .

A *holonomic space* is a triplet  $(V, H, L)$  consisting of a normed vector space  $V$ ; a group  $H$  of linear isometries of  $V$ ; and a group-norm  $L$  on such group; satisfying a local convexity property that relates them: For any element  $u \in V$  there is a ball around it such that for any two elements  $v, w$  in that ball the

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following inequality holds:

$$\|v - w\|^2 - \|av - w\|^2 \leq L^2(a) \quad (2)$$

for any element  $a \in H$ . See Definition 2.

By considering the following distance function,  $d_L : V \times V \rightarrow \mathbb{R}$ ,

$$d_L(u, v) = \inf_A \sqrt{L^2(A) + \|Au - v\|^2}, \quad (3)$$

one gets a modified metric-space structure on  $V$  that sheds light on the definition of a holonomic space: A triplet  $(V, H, L)$  is a holonomic space if and only if  $d_L$  is locally isometric to usual distance on  $V$ . This is Theorem 2, proved in section 1.

The measure of nontriviality of a holonomic space is controlled by the *holonomy radius*, a continuous function on  $V$ , given by the supremum of the radii of balls for which the local convexity property is satisfied. This function is finite if and only if  $H$  is nontrivial [Proposition 2].

Considering the holonomy radius at the origin already yields some information on the group-norm in the case when the normed vector space is actually an inner product space. Namely the following result.

**Theorem A.** Given a holonomic space  $(V, H, L)$ , the identity map on  $H$  is Lipschitz between the left invariant metrics on  $H$  induced by  $L(A^{-1}B)$  and  $\sqrt{2\|A - B\|}$  respectively, where  $\|\cdot\|$  stands for the operator norm. Moreover, the dilation is precisely the reciprocal of the holonomy radius  $\rho_0$  at the origin of  $V$ .

$$\sqrt{2\|A - B\|} \leq \frac{1}{\rho_0} L(A^{-1}B). \quad (4)$$

This is a consequence of Theorem 4 and Corollary 4 in section 1.

The relevance of this result is clear in the context of euclidean vector bundles. Recall that a Sasaki-type metric  $G$  on a euclidean vector bundle with compatible connections is given in terms of the connection map  $\kappa : TE \rightarrow E$ , uniquely determined by requiring that  $\kappa(\sigma_*x) = \nabla_x^E \sigma$ , as

$$G(\xi, \eta) = g(\pi_*\xi, \pi_*\eta) + h(\kappa\xi, \kappa\eta), \quad (5)$$

for vectors  $\xi, \eta \in TE$ .

**Theorem B.** Given a euclidean vector bundle with a compatible connection over a riemannian manifold, each point in the base space has a naturally associated holonomic space, with the fiber over that point being the underlying normed vector space.

Furthermore, if the total space is endowed with the corresponding Sasaki-type metric then the aforementioned modified metric-space structure coincides with the restricted metric on the fibers from the metric on the total space.

This result is stated more precisely in Proposition 5, Theorem 5, and Theorem 6.

The group-norm in Theorem B is given by (1). The study of this group-norm was already hidden in the work of Tapp [20] and Wilkins [22].

This group-norm induces a new topological group structure on the holonomy group that makes the group-norm continuous while retaining the continuity of the holonomy action [Lemma 2]. It should be noted that with the standard topology (i.e. that of a Lie group) of the holonomy group, this group-norm is not even upper semicontinuous. Wilkins [22] had already noted this (an immediate example is to consider a metric that is flat in a neighborhood of a point and consider the group-norm associated at that point). He proved that if the Lie group topology is compact then the group-norm topology is bounded, which is a surprising result given that the group-norm topology is finer.

Tapp [20] defines a ‘size’ for a given holonomy element as an infimum over *acceptable* smooth metrics on the holonomy group (quoted here as 7). As such, he proved that holonomy ‘size’ and the length group-norm (1) are comparable up to a constant that depends only on the base space and the norm of the curvature (see Theorem 9). One observation is the following.

**Theorem C.** The holonomy group of a connection on a vector bundle is compact if and only if the restricted holonomy group is compact and its associated length norm is bounded.

These results are discussed in more detail in the last section.

As an application of this constructions, the *holonomy radius* of a riemannian manifold is defined by assigning to each point the holonomy radius at the origin of the holonomic space associated to it by Theorem B in the case of its tangent bundle with the Levi-Civita connection [Definition 8]. In view of Theorem A, this function is positive [Theorem 11] and has a precise formulation in terms of the group-norms and of the usual linear action of the holonomy groups. In [18], this function is shown to be upper semicontinuous and to control the non collapsing of the fibers of a convergent sequence of vector bundles.

More specifically, in [18], the study of the Gromov-Hausdorff convergence of Sasaki-type metrics over a sequence of converging base spaces uses the notion of holonomic space as part of the description of the limits. In essence, it is proved that the fibers of a converging sequence of Sasaki-type metrics converge with their restricted metrics to a fiber in the limit. The degeneracy of these limits is determined by the action of a compact Lie group. This group is obtained by looking at the set

$$G_0 = \{ g \in O(V) \mid g = \lim_{i_n \rightarrow \infty} a_{i_n}, \lim_{i_n \rightarrow \infty} L_{i_n}(a_{i_n}) = 0 \}, \quad (6)$$

and considering the largest subgroup  $G$  of  $O(V)$  sharing the same orbits as  $G_0$ .

Of course, there are other groups that satisfy this. If the holonomy radius is uniformly bounded away from zero along the sequence, then  $G_0$  is trivial and the fibers of the limit remain vector spaces.

Similar ways to consider convergence of vector bundles—some imposing different conditions—have been proposed by Kasue [6], Rieffel [14], Tapp [21], or Rong and Xu [15]. Kotschwar [8] notices that the holonomy group is preserved under Ricci flow, hinting at the importance of understanding the metric geometric interactions of holonomy. In particular when additional information is known such as the presence of special holonomy, as in [2].

Another immediate application is the following result, the proof of which is straightforward.

**Theorem D.** The completion of the Sasaki metric on the tangent bundle  $TM$  is the total space of a submetry over the completion of  $M$ . All the new fibers are exactly obtained as limits of holonomic space metrics.

## 1 Initial consequences

### 1.1 Definitions

The notion of a *holonomic space* is introduced in this section. It will be seen in the sequel how these spaces occur as fibers of euclidean vector bundles with suitable conditions imposed; several properties of holonomic spaces are also analyzed here. Firstly, one important ingredient is that of group-norm.

**Definition 1.** Let  $G$  be a group. A function  $|\cdot| : G \rightarrow \mathbb{R}$  is a *group-norm* if it satisfies the following properties.

- (1) For any  $g \in G$ ,  $|g| \geq 0$ . (Positivity)
- (2) For any  $g \in G$ ,  $|g| = 0$  only when  $g$  is the identity. (Non-degeneracy)
- (3) For any  $g \in G$ ,  $|g^{-1}| = |g|$ . (Symmetry)
- (4) For any  $g, h \in G$ ,  $|gh| \leq |g| + |h|$ . (Subadditivity)

By setting  $d(g, h) = |g^{-1}h|$  one sees that a normed group is essentially the same as a group with a left-invariant metric. There are examples (e.g. word metrics for fundamental groups) where it is the norm which occurs first, while the metric comes as an afterthought. A good reference is the recent work of Bingham and Ostaszewski [5]. In this communication, it is also the case that the natural concept is the group-norm. For reasons that will become apparent by Theorem 5, the notation for a group-norm in the sequel is given by  $L(\cdot)$  rather than by  $|\cdot|$ .

**Definition 2.** Let  $(V, \|\cdot\|)$  be a normed vector space,  $H \leq \text{Aut}(V)$  a subgroup of norm preserving linear isomorphisms, and  $L : H \rightarrow \mathbb{R}$  a group-norm on  $H$ . The triplet  $(V, H, L)$  will be called a *holonomic space* if it further satisfies the following convexity property:

(P) For all  $u \in V$  there exists  $r = r_u > 0$  such that for all  $v, w \in V$  with  $\|v - u\| < r, \|w - u\| < r$ , and for all  $A \in H$ ,

$$\|v - w\|^2 - \|v - Aw\|^2 \leq L^2(A). \tag{7}$$

**Definition 3.** Let  $(V, H, L)$  be a holonomic space. The *holonomy radius* of a point  $u \in V$  is the supremum of the radii  $r > 0$  satisfying the convexity property (P) given by (7). It will be denoted by  $\text{HolRad}(u)$ . It may be infinite.

**Lemma 1.** *Given a holonomic space  $(V, H, L)$  as above, there exists  $r > 0$  such that for  $u \in V, \|u\| < r$ , and for any  $B \in H$ ,*

$$\|u - Bu\| \leq L(B). \tag{8}$$

*Proof.* Simply choose  $r = r_0$  as in 2,  $v = u, w = Bu$  and  $A = B^{-1}$ .  $\square$

**Definition 4.** Given a holonomic space  $(V, H, L)$ , the largest radius of a ball satisfying Lemma 1 is the *convexity radius of a holonomic space*. It may be infinite.

**Remark 1.** The convexity radius is in general larger than the holonomy radius at the origin, as can be seen in Example 1.

The group-norm  $L$  induces a metric topology on  $H$ , called here  $L$ -topology

**Lemma 2.** *Given a holonomic space  $(V, H, L)$ , the action  $H \times V \rightarrow V$  is continuous with respect to the  $L$ -topology on  $H$ . Furthermore, the bound depends only on the maximum norm when restricted to bounded domains.*

*Proof.* Let  $r_0$  be the convexity radius. Let  $(a, u) \in H \times V$ , with  $\|u\| \geq r_0$ . Fix  $\lambda > 0$  such that  $\lambda\|u\| < r$ , and let  $\varepsilon > 0$ . Let  $K = \sqrt{1 + \frac{1}{\lambda^2}}$ . Note that for any positive real numbers  $x, y \in \mathbb{R}$ ,

$$x + \frac{1}{\lambda}y \leq K\sqrt{x^2 + y^2}.$$

Now, if  $\delta = \min\{\frac{\varepsilon}{K}, \frac{r - \lambda\|u\|}{\lambda}\}$  and  $\sqrt{L^2(a^{-1}b) + \|u - v\|^2} < \delta$ , notice that

$$\|\lambda v\| \leq \lambda\|u - v\| + \|\lambda u\| < \lambda\delta + \lambda\|u\| \leq r_0.$$

Thus,

$$\begin{aligned}
\|au - bv\| &= \|u - a^{-1}bv\| = \frac{1}{\lambda} \|\lambda u - a^{-1}b\lambda v\| \\
&\leq \frac{1}{\lambda} (\|\lambda u - \lambda v\| + \|\lambda v - a^{-1}b\lambda v\|) \\
&\leq \|u - v\| + \frac{1}{\lambda} L(a^{-1}b) \\
&\leq K \sqrt{L^2(a^{-1}b) + \|u - v\|^2} \\
&= K \sqrt{L^2(a^{-1}b) + \|u - v\|^2} < K\delta \leq \varepsilon.
\end{aligned}$$

□

**Remark 2.** This implies that the  $L$ -topology is necessarily finer than the subgroup topology induced from  $O(V)$ . The fact that they be comparable is already restrictive on what  $L$  can be.

**Theorem 1.** *Let  $(V, H, L)$  be a holonomic space.*

$$d_L(u, v) = \inf_{A \in H} \left\{ \sqrt{L^2(A) + \|u - Av\|^2} \right\}, \quad (9)$$

is a metric on  $V$ .

*Proof.* By Lemma 2 one sees that the action  $H \times V \rightarrow V$  is continuous with respect to the  $L$ -topology on  $H$ . Observe that the  $V$  is homeomorphic to  $H \setminus (V \times H)$  and that the induced metric on  $V$  is given by (9). □

**Definition 5.** Given a holonomic space  $(V, H, L)$ . The metric given by (9) will be called *holonomic space metric*.

**Theorem 2.** *A triplet  $(V, H, L)$  is a holonomic space if and only if the identity map  $id_V : (V, \|\cdot\|) \rightarrow (V, d_L)$  is a locally isometry.*

*Proof.* By property (P), given any point  $u \in V$  there exists a radius  $r > 0$  such that for all  $v, w \in V$ , with  $\|v - u\| < r$  and  $\|w - u\| < r$ , and for all  $A \in H$ ,

$$\|v - w\| \leq \sqrt{L^2(A) + \|v - Aw\|^2}.$$

Hence, considering the infimum of the right-hand side, it follows that

$$\|v - w\| \leq d_L(v, w) \leq \sqrt{L^2(id_V) + \|v - w\|^2} = \|v - w\|.$$

Conversely, if the identity is a local isometry, property (P) in Definition 2 is also satisfied: On any ball around  $u \in V$  on which the identity map  $id_V$  restricts to an isometry and for any  $B \in H$  and any pair of points  $v, w$  in said ball,

$$\|v - w\| = \inf_{A \in H} \left\{ \sqrt{L^2(A) + \|v - Aw\|^2} \right\} \leq \sqrt{L^2(B) + \|v - Bw\|^2}.$$

$\square$

**Remark 3.** The holonomy radius is also the radius of the largest ball so that the restricted  $d_L$ -metric is euclidean.

**Proposition 1.** *Let  $(V, H, L)$  be a holonomic space. The original norm on  $V$  is recovered by the equation*

$$\|v\| = d_L(v, 0) \tag{10}$$

*Proof.* Because  $H$  acts by isometries on  $V$ ,

$$d_L(v, 0) = \inf_{A \in H} \left\{ \sqrt{L^2(A) + \|v\|^2} \right\}.$$

The conclusion now follows by letting  $A = id_V$ .

$\square$

**Corollary 1.** *Given a holonomic space  $(V, H, L)$  the rays emanating from the origin are geodesic rays with respect to  $d_L$ .*

## 1.2 Properties of the associated radii

**Proposition 2.** *Let  $(V, H, L)$  be a holonomic space. Then  $H = \{id_V\}$  if and only if there exists  $u \in V$  for which the holonomy radius is not finite.*

*Proof.* If  $H$  is trivial, then  $L \equiv 0$ , and so  $V$  is globally isometric to  $V$ , hence for any  $u \in V$  the holonomy radius is infinite. Conversely, if there exists  $u \in V$  with  $\text{HolRad}(u) = \infty$ , and there is  $a \in H$  with  $L(a) > 0$  (i.e.  $a \neq id_V$ ), then for any  $v \in V$ ,  $\|v - av\| \leq L(a)$  should hold. This is a contradiction since  $v \mapsto \|v - av\|$  is clearly not bounded unless  $a = id_V$ .

$\square$

**Corollary 2.** *Let  $(V, H, L)$  be a holonomic space. Then the function  $u \mapsto \text{HolRad}(u)$  is positive. Furthermore, it is finite provided  $H$  is nontrivial.*

**Proposition 3.** *Let  $(V, H, L)$  be a holonomic space. The function  $u \mapsto \text{HolRad}(u)$  is continuous.*

*Proof.* By Proposition 2 one can assume, with no loss of generality, that  $H \neq \{id_V\}$ . Let  $u \in V$  and let  $\varrho(u)$  be the holonomy radius at  $u$ . Let  $v \in V$  with  $\|v - u\| < \varrho(u)$ , i.e.  $v \in B_{\varrho(u)}(u)$ , then by maximality of  $\varrho(v)$ , it has to be at least as large as the radius of the largest ball around  $v$  completely contained in  $B_{\varrho(u)}(u)$ ,

$$\varrho(v) \geq \varrho(u) - \|u - v\|.$$

Also, by maximality of  $\varrho(u)$ , it follows that  $\varrho(v)$  cannot be strictly larger than the smallest ball around  $v$  that contains  $B_{\varrho(u)}(u)$ ,

$$\varrho(v) \leq \varrho(u) + \|u - v\|.$$

Therefore, at any given point  $u \in V$  and any  $\varepsilon > 0$ , there exists  $\delta = \min\{\varrho(u), \varepsilon\}$  such that for any  $v \in V$ ,  $\|u - v\| \leq \delta$  it follows that

$$|\varrho(u) - \varrho(v)| \leq \|u - v\| \leq \varepsilon.$$

□

**Theorem 3.** *Let  $(V, H, L)$  be a holonomic space. Then the convexity radius is given by*

$$\text{CvxRad} = \inf_{\substack{A \in H \\ A \neq id_V}} \frac{L(A)}{\|id_V - A\|}, \quad (11)$$

where for any  $T : V \rightarrow V$ ,  $\|T\|$  denotes its operator norm.

*Proof.* Let  $u \in V$  with  $\|u\| \leq \frac{L(A)}{\|id_V - A\|}$ , then

$$\|Au - u\| \leq \|A - id_V\| \|u\| \leq L(A)$$

which proves that

$$\text{CvxRad} \geq \inf_{\substack{A \in H \\ A \neq id_V}} \frac{L(A)}{\|id_V - A\|}.$$

Now, let  $\varrho > \frac{L(A)}{\|id_V - A\|}$  and let  $\varepsilon > 0$  be such that

$$\varepsilon < \|A - id_V\| \varrho - L(A) = \|A - id_V\| \left( \varrho - \frac{L(A)}{\|A - id_V\|} \right) > 0. \quad (12)$$

Then, by the definition of operator norm, there exists  $u \in V$  with  $\|u\| = \varrho$  such that

$$\|A - id_V\| \varrho \geq \|Au - u\| > \|A - id_V\| \varrho - \varepsilon.$$

The second inequality, together with (12), yields that

$$\|Au - u\| > \|A - id_V\| \varrho - \varepsilon > L(A).$$

This proves that  $\text{CvxRad}$  cannot be strictly larger than  $\frac{L(A)}{\|id_V - A\|}$  for any  $A$ , and thus for all. □



Notice that the operator norm and any composition of it with a non decreasing subadditive function is a group-norm; and that given a group-norm  $N$ , a left-invariant metric is obtained by  $d_N(g, h) = N(g^{-1}h)$ . With this, the group norm in the definition of a holonomic space, the usual operator norm and the convexity radius are related in the following Lipschitz condition.

**Corollary 3.** *Given a holonomic space  $(V, H, L)$ , with  $V$  an inner product space, then*

$$\|A - B\| \leq \frac{1}{\text{CvxRad}} L(A^{-1}B),$$

for all  $A, B \in H$ .

**Theorem 4.** *Let  $(V, H, L)$  be a holonomic space and suppose further that  $V$  is an inner product space and that the norm is given by  $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ . Then the holonomy radius at the origin is given by*

$$\text{HolRad}(0) = \inf_{\substack{A \in H \\ A \neq id_V}} \frac{L(A)}{\sqrt{2\|id_V - A\|}}, \quad (13)$$

where for any  $T : V \rightarrow V$ ,  $\|T\|$  denotes its operator norm.

*Proof.* Using the inner product, the symmetry of the group-norm  $L$ ,  $L(A^{-1}) = L(A)$ , and that  $H$  acts by isometries, (7) is equivalent to

$$\|v - w\|^2 - \|Av - w\|^2 \leq L^2(A),$$

which when expanded out yields,

$$\|v\|^2 + \|w\|^2 - 2\langle v, w \rangle - \|v\|^2 - \|w\|^2 + 2\langle Av, w \rangle \leq L^2(A),$$

whence

$$2\langle Av - v, w \rangle \leq L^2(A).$$

Thus, if  $\|v\|, \|w\| \leq \frac{L(A)}{\sqrt{2\|id_V - A\|}}$  then

$$2\langle Av - v, w \rangle \leq 2\|A - id_V\| \|v\| \|w\| \leq L^2(A).$$

Since the inequality has to hold for any  $A$ , it follows that

$$\text{HolRad}(0) \geq \inf_{\substack{A \in H \\ A \neq id_V}} \frac{L(A)}{\sqrt{2\|id_V - A\|}}.$$

Furthermore, for  $\rho > \frac{L(A)}{\sqrt{2\|id_V - A\|}}$ , let  $\varepsilon > 0$  such that

$$\varepsilon < \|id_V - A\| \rho - \frac{L^2(A)}{2\rho} = \frac{\|id_V - A\|}{\rho} \left( \rho^2 - \frac{L^2(A)}{2\|id_V - A\|} \right) > 0.$$

By the definition of operator norm, there exists  $v \in V$ , with  $\|v\| = \rho$  and

$$\|id_V - A\|\rho \geq \|Av - v\| > \|id_V - A\|\rho - \varepsilon.$$

Set  $w = \frac{\rho}{\|Av - v\|}(Av - v)$ . It now follows that

$$2\langle Av - v, w \rangle = 2\rho\|Av - v\| > 2\rho(\|id_V - A\|\rho - \varepsilon) > L^2(A),$$

by the previous second inequality.

Thus, (7) cannot hold for  $\rho > \frac{L(A)}{\sqrt{2\|id_V - A\|}}$  and the claim follows.  $\square$  *QED*

**Corollary 4** (cf. Corollary 3). *Given a holonomic space  $(V, H, L)$ , with  $V$  an inner product space, then*

$$\sqrt{2\|A - B\|} \leq \frac{1}{\text{HolRad}(0)} L(A^{-1}B),$$

for all  $A, B \in H$ .

### 1.3 Examples

**Example 1.** The existence of an  $r > 0$  satisfying (8) (guaranteed for holonomic spaces by 1) is not equivalent to the existence of an  $r' > 0$  satisfying (7). This follows from (13) by considering the following action: Let  $V = \mathbb{C}^2$ ,  $H = \mathbb{R}$ ,

$$t \cdot (z, w) = (e^{it}z, e^{\sqrt{2}it}w),$$

and  $L(t) = |t|$ . Indeed, by Theorem 4,

$$\text{HolRad}(0) = \inf_{\substack{A \in H \\ A \neq id_V}} \frac{L(A)}{\sqrt{2\|id_V - A\|}} = \lim_{t \rightarrow 0^+} \frac{|t|}{\sqrt{2\sqrt{2 - 2\cos(\sqrt{2}t)}}} = 0,$$

whereas, any positive  $r \leq \frac{1}{\sqrt{2}}$  will make (8) hold. Hence, by Theorem 3,

$$\text{CvxRad} \geq \frac{\sqrt{2}}{2}.$$

Finally, consider the following example.

**Example 2.** Let  $r > 0$  and let  $H$  be the group generated by a rotation by  $0 < \alpha < \pi$ . Let  $L_r$  be the group-norm given by

$$L(A) = \begin{cases} 2r & a \neq e, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Consider  $V$  to be  $\mathbb{R}^2$  with the standard inner product. Then  $(V, H, L_r)$  is a holonomic space. This can be verified directly. It will also follow from Theorem 6 by considering the flat metric

$$ds^2 = dr^2 + \left(\frac{\alpha r}{2\pi}\right)^2 d\theta^2$$

on  $\mathbb{R}^2 \setminus \{0\}$  (see [18]).

### 1.4 Fibers as holonomic spaces

A euclidean vector bundle  $E$  is a vector bundle with a fiberwise positive-definite inner product  $h$ . A compatible connection  $\nabla$  is then a derivation with respect to which  $h$  is parallel. Let  $\alpha : [0, 1] \rightarrow M$  be a piecewise smooth curve. Denote by  $P_t^\alpha$  the parallel translation transformation from the fiber at  $\alpha(0)$  to the fiber at  $\alpha(t)$ . Let  $e, u$  be vectors on the same fiber in  $E$ . Denote by  $e^v(u)$  the tangent vector at  $t = 0$  to the curve  $t \mapsto u + te$ . Given a vector  $u$  in  $E$  and a tangent vector  $x$  at  $\pi(x)$ , denote by  $x^h(u)$  the tangent vector at  $t = 0$  to the curve  $t \mapsto P_t^\alpha(u)$  along any  $\alpha$  such that  $\pi(u) = \alpha(0)$  and  $x = \dot{\alpha}(0)$ .

**Definition 6** ([16]). Given a vector bundle with metric and compatible connection  $(E, \pi, h, \nabla^E)$  over a riemannian manifold  $(M, g)$ , the *Sasaki-type metric*  $G = G(g, h, \nabla^E)$  is defined as follows

$$G(e^v, f^v) = h(e, f) \tag{15}$$

$$G(e^v, x^h) = 0 \tag{16}$$

$$G(x^h, y^h) = g(x, y), \tag{17}$$

**Remark 4.** An equivalent phrasing of  $G$  can be given in terms of the connection map [7],  $\kappa : TE \rightarrow E$ , uniquely determined by requiring that

$$\kappa(\sigma_*x) = \nabla_x^E \sigma; \tag{18}$$

so that  $G$  becomes

$$G(\xi, \eta) = g(\pi_*\xi, \pi_*\eta) + h(\kappa\xi, \kappa\eta), \tag{19}$$

for vectors  $\xi, \eta \in T_eTE$ .

**Proposition 4.** *Given a curve  $\alpha : I \rightarrow M$  (parametrized by arc length), the (trivial) pullback bundle  $\alpha^*E$  is further isometric to  $I \times \mathbb{R}^k$  where  $k$  is the rank of  $E$ .*

*Proof.* By parallel translation one gets that

$$\alpha^*G = \ell(\alpha)^2 dt^2 + \alpha^*h_p,$$

where  $p = \alpha(0)$ , and  $\ell$  denotes the length of  $\alpha$ .  $\square$

**Proposition 5.** *The length distance on  $(E, \mathbb{G})$  is expressed as follows. Let  $u, v \in E$ , then*

$$d_E(u, v) = \inf \left\{ \sqrt{\ell(\alpha)^2 + \|P_1^\alpha u - v\|^2} \mid \begin{array}{l} \alpha : [0, 1] \rightarrow M \\ \alpha(0) = \pi u \\ \alpha(1) = \pi v \end{array} \right\}. \quad (20)$$

Furthermore, if  $\pi u = \pi v$  then

$$d_E(u, v) = \inf \{ \sqrt{L(a)^2 + \|au - v\|^2} : a \in Hol_p \}, \quad (21)$$

with  $L$  being the infimum of lengths of loops yielding a given holonomy element.

*Proof.* The first expression is essentially the definition of distances in view of 4. In the case  $\pi u = \pi v$ , (21) follows by partitioning the set of all curves  $\alpha$  according to the holonomy element they generate.  $\square$

**Theorem 5.** *Let  $Hol_p$  be the holonomy group over a point  $p \in M$  of a bundle with metric and connection and suppose that  $M$  is riemannian. Then the function  $L_p : Hol_p \rightarrow \mathbb{R}$ ,*

$$L_p(A) = \inf \{ \ell(\alpha) \mid \alpha \in \Omega_p, P_1^\alpha = A \}, \quad (22)$$

*is a group-norm for  $Hol_p$ , where  $\Omega_p$  denotes the space of piecewise smooth loops based at  $p$ .*

*Proof.* Positivity is immediate from the fact that it is defined as an infimum of positive numbers. To prove non-degeneracy suppose that an element  $A \neq I$  has zero length. There exists  $u \in E_p$  such that  $Au \neq u$ ; thus, by (21), choosing  $a = A$  yields  $d(u, Au) = 0$ . A contradiction.

The length of the inverse of any holonomy element is the same because the infimum is taken essentially over the same set. Finally, to establish the triangle inequality, note that the set of loops that generate  $AB$  contains the concatenation of loops generating  $A \in Hol_p$  with loops generating  $B \in Hol_p$ .  $\square$

**Definition 7.** The function  $L_p$ , defined by (22) will be called *length-norm* of the holonomy group induced by the riemannian metric at  $p$ .

**Remark 5.** The fact that the connection be metric is used twice in proof that  $L_p$  is indeed non-degenerate. This is because once can then produce a riemannian metric on the total space of the bundle, namely that of Sasaki type.

**Theorem 6.** *Let  $E_p$  be the fiber of a vector bundle  $E$  with metric and connection over a riemannian manifold  $M$  at a point  $p$ . Let  $Hol_p$  denote the associated holonomy group at  $p$  and let  $L_p$  be the group-norm given by (22). Then  $(E_p, Hol_p, L_p)$  is a holonomic space. Moreover, if  $E$  is endowed with the corresponding Sasaki-type metric, the associated holonomic space metric coincides with the restricted metric on  $E_p$  from  $E$ .*

*Proof.* According to the definition given in 2, the only remaining condition is given by (7). To see this, one needs only to note that the fiber  $E_p$  is a totally geodesic submanifold of  $E$ . With this, given any point  $u \in E_p$ , let  $r = CvxRad_p(E) > 0$ , the convexity radius; thus, for any pair of points  $v, w \in B_r^E(p) \cap E_p$  there exists a unique geodesic from  $v$  to  $w$ . This geodesic is necessarily  $t \mapsto u - t(v - u) \in E_p$ , and thus the distance

$$d(u, v) = \|u - v\|.$$

□

## 2 Further constructions

### 2.1 Length of loops—New topologies for the holonomy group

Controlling the length of loops that generate a given holonomy element has many applications, as pointed out by Montgomery [9], in Control Theory, Quantum Mechanics, or sub-riemannian geometry (see [10]).

Considering the infimum  $L(A)$  of lengths of loops that generate a given holonomy element  $a$  is a natural pick, and exhibits the fibers the vector bundle as a holonomic space as seen in 6.

Although the function  $A \mapsto L(A)$  is in general not even upper-semicontinuous when regarded as a function on the holonomy group with the subspace topology (or even its Lie group topology), as pointed out by Wilkins [22], the following results gives a more positive outcome.

**Theorem 7.** *Let  $H$  be the holonomy group of a metric connection on a vector bundle  $E$  over a riemannian manifold. There exists a finer metrizable topology on  $H$ , given by  $d(A, B) = L(A^{-1}B)$ , so that the function  $A \mapsto L(A)$  is continuous with respect to this topology and furthermore, the group action  $H \times E_p \rightarrow E_p$  remains continuous.*

*Proof.* By 2 the action map  $H \times E_p \rightarrow E_p$ , is continuous. Thus, the identity map is continuous from the  $L$ -topology to the Lie topology. Furthermore, clearly  $L$  is continuous with respect to the  $L$ -topology. □

Now, the following fact hints a type of ‘wrong way’ inheritance.

**Proposition 6** (Schroeder and Strake [22], Wilkins [17]). *Let  $\pi : P \rightarrow M$  be a smooth principal bundle over a smooth manifold  $M$ , let a smooth connection on  $\pi : P \rightarrow M$  be given, and let  $H_p$  denote the holonomy group of this connection attached to some element  $p$  of  $P$ . Suppose that  $H_p$  is compact. Then there exists a constant  $K$  such that every element of  $H_p$  can be generated by a loop of length not exceeding  $K$ .*

So, in the language of the induced length structure the following is true.

**Theorem 8.** *Let  $E \rightarrow M$  be a vector bundle with bundle metric and compatible connection. Let  $H$  be the holonomy group of this connection. If  $H$  is compact with the standard Lie group topology (in particular bounded with respect to any —invariant— metric), then  $H$  with the induced length metric given by (22) is bounded.*

Tapp [20] introduces a way to measure the size of a holonomy transformation as a supremum over *acceptable* left invariant metrics. A smooth invariant metric  $m$  is acceptable if for any  $X \in \mathfrak{k} = \mathfrak{g}(\Phi)$ , the Lie algebra of  $\Phi$ ,

$$\|X\|_m \leq \sup_{v, \|v\|=1} \|X(v)\|, \quad (23)$$

where  $X(v)$  means the evaluation of the fundamental vector on  $F$  associated with  $X$ . The size of a holonomy transformation  $A$  is then defined as the supremum of its distances to the identity  $dist_m(A, Id)$  over acceptable metrics  $m$ . And the following fact relates this ‘size’ to the norm defined by (22), whenever there are curvature bounds.

**Proposition 7** (Tapp [20]. Proposition 7.1). *Let  $E \rightarrow B$  be a riemannian vector bundle over a compact simply connected manifold  $B$ . Let  $\nabla$  be a compatible metric connection and let its curvature  $R$  be bounded in norm,  $|R| \leq C_R$ . Fix a point  $x \in B$  and let  $Hol(\nabla)$  be the corresponding holonomy group at  $x$ . Then there exists a constant  $C(B)$  such that for any loop  $\alpha$  in  $B$ ,  $|P_\alpha| \leq C \cdot C_R \cdot \ell(\alpha)$ , where  $P_\alpha \in Hol(\nabla)$  stands for the holonomy transformation induced by  $\alpha$ .*

**Theorem 9.** *With the assumptions as in the previous statement, the norm given by (22) and Tapp’s holonomy size are related by  $|g| \leq C \cdot C_R \cdot L(g)$ , so that the induced length topology is finer than that of Tapp’s holonomy size.*

*Proof.* This is immediate from the inequality, since the infimum is taken over loops with the same holonomy transformation associated.  $\square$

Finally, with the additional assumption of completeness, the following result gives a converse to Proposition 6.

**Theorem 10.** *Let  $E \rightarrow B$  be a riemannian vector bundle with compatible connection over a complete riemannian manifold  $B$ . The holonomy group is compact if and only if the restricted holonomy group is compact and its associated length norm is bounded.*

*Proof.* The necessity is the content of Proposition 6. For the sufficiency, suppose that the holonomy group has infinitely many connected components. Consider a sequence  $\{A_i\}$  of inequivalent classes. Let  $\gamma_i$  be a loop generating  $A_i$  such that  $L(A_i) = \ell(\gamma_i)$ ; these exist by the completeness of the metric and an application of Arzelà-Ascoli Theorem as pointed out by Montgomery [9]. Since the lengths of the  $\gamma_i$ 's are bounded by assumption, another application of Arzelà-Ascoli Theorem, now to the sequence  $\gamma_i$ , yields a uniformly convergent subsequence, also denoted by  $\{\gamma_i\}$ . Therefore, for  $i \gg 0$ , all loops are homotopic. This is a contradiction to the following fact: different connected components of the holonomy group represent different homotopy classes.  $\square$

**Corollary 5.** *In the case of complete riemannian manifolds, the holonomy group is compact if and only if the length norm is bounded.*

*Proof.* By virtue of the classification theorem of Berger [4], the restricted holonomy group is always compact.  $\square$

**Remark 6.** Completeness is really essential as looking again at a cone metric on the punctured plane shows (Solórzano [18]).

## 2.2 Holonomy radius of a riemannian manifold

Given a riemannian manifold  $(M, g)$ , in view of the fundamental theorem of riemannian Geometry, one immediately obtains a vector bundle, a connection and a bundle metric compatible with the connection; i.e. the tangent bundle, the Levi-Civita connection and the metric itself. This is the metric introduced by Sasaki [16].

**Definition 8.** Let  $(M, g)$  be a riemannian manifold and let  $p \in M$ . The *holonomy radius* of  $M$  at  $p$  and denoted by  $\text{HolRad}_M(p)$  is defined to be the supremum of  $r > 0$  such that for all  $u, v \in T_pM$  with  $\|u\|, \|v\| \leq r$  and for all  $A \in \text{Hol}_p$

$$\|u - v\|^2 - \|Au - v\|^2 \leq L_p^2(A), \tag{24}$$

where  $L_p$  is the associated length norm on  $\text{Hol}_p$ .

**Remark 7.** This is simply the holonomy radius at the origin of the holonomic space  $(T_pM, \text{Hol}_p, L_p)$  (see Definition 3).

**Theorem 11.** *Given a riemannian manifold  $M$ . The function that assigns to each point its holonomy radius is strictly positive.*

*Proof.* This is a direct consequence of 2 and the fact that the tangent spaces are holonomic by 6.  $\square$

**Remark 8.** This fact also follows directly from geometric considerations given that  $0 < \text{CvxRad}_{TM}(0_p) \leq \text{HolRad}_M(p)$ , where  $\text{CvxRad}_{TM}$  is the convexity radius of  $TM$  with its Sasaki metric.

**Remark 9.** The continuity of  $\text{HolRad} : M \rightarrow \mathbb{R}$  is addressed in [18]. In particular, its upper semicontinuity is established.

**Proposition 8.** *If there exists a point  $p$  in a Riemannian manifold  $M$  for which the holonomy radius is not finite, then  $M$  is flat.*

*Proof.* by 2, the existence of such point is equivalent to the group being trivial. In particular, the restricted holonomy group is trivial, which in turn is equivalent to flatness.  $\square$

**Remark 10.** The converse is certainly not true. Consider for example a cone metric on  $\mathbb{R}^2 \setminus \{0\}$  [18].

**Corollary 6.** *Let  $M$  be a simply connected riemannian manifold. If there is a point on  $M$  with infinite holonomy radius, then  $M$  is isometric to a euclidean space.*

### 2.3 Two-dimensional examples

In the two-dimensional case more can be obtained from the Gauß-Bonnet Theorem. Furthermore, in the particular case of the  $\mathbb{S}^2$  or  $\mathbb{H}^2$ ,  $L$  can be computed explicitly by virtue of the isoperimetric inequality.

Recall the following classical result.

**Lemma 3.** *Let  $(M^2, g)$  be a 2-dimensional riemannian manifold and let  $\gamma : [0, \ell] \subseteq \mathbb{R} \rightarrow M$  be any curve parametrized by arc length. Let  $k$  be a signed geodesic curvature of  $\gamma$  with respect to an orientation of  $\gamma^*TM$ . Let  $\theta(t)$  be the angle between  $\dot{\gamma}$  and its parallel translate at time  $t$ . Then*

$$2\pi - \theta(t) = \int_0^t k \quad (25)$$

*Assume further that  $\gamma$  is a loop. Then, possibly up to a reversal in orientation, the holonomy action of  $\gamma$  at  $p = \gamma(0)$  is the rotation by  $2\pi - \int_0^\ell k - \alpha$ , where  $\alpha$  is the angle between  $\dot{\gamma}(0)$  and  $\dot{\gamma}(1)$ .*



*Proof.* Consider a compatible parallel almost complex structure on  $\gamma^*TM$ ,  $J$ . With respect to the orthonormal frame given by  $\{\dot{\gamma}, J(\dot{\gamma})\}$ ,  $\nabla_{\dot{\gamma}}\dot{\gamma} = kJ(\dot{\gamma})$ , and thus the equation for any parallel vector field  $P = a\dot{\gamma} + bJ(\dot{\gamma})$  along  $\gamma$  is given by

$$\begin{aligned} \dot{a} &= kb \\ \dot{b} &= -ka \end{aligned}$$

which integrates to a rotation by  $-\int k$  as claimed.  $\square$

**Theorem 12.** *Let  $M^2$  be a complete simply connected two-dimensional non-flat space-form with curvature  $K$ . Let  $L : \mathbb{S}^1 \rightarrow \mathbb{R}$  be the associated length-norm on the holonomy group. Then*

$$L(\theta) = \frac{\sqrt{4\pi|\theta| \pm \theta^2}}{\sqrt{|K|}}, \tag{26}$$

for  $-\pi \leq \theta \leq \pi$ , where the sign is opposite to the sign of the curvature.

*Proof.* By the Gauß-Bonnet Theorem,  $\theta = 2\pi - \int k - \alpha = K\mathcal{A}$ , where  $\mathcal{A}$  is the area of the region enclosed by any loop  $\gamma$ , so that

$$\mathcal{A} = \left| \frac{\theta}{K} \right|. \tag{27}$$

Now, the isoperimetric inequality in this case (see [12]) is given by

$$\ell^2 \geq 4\pi\mathcal{A} - K\mathcal{A}^2, \tag{28}$$

where the equality is achieved when  $\gamma$  is a metric circle. So, by direct substitution of (27) into (28) the claim follows.  $\square$

**Corollary 7.** *Let  $M^2$  be a simply connected two-dimensional non-flat space-form with curvature  $K$ . The holonomy radius at any point  $p \in M$  is given by*

$$\text{HolRad}_M \equiv \inf_{0 < \theta \leq \pi} \sqrt{\frac{4\pi\theta \pm \theta^2}{2|K|\sqrt{2 - 2\cos(\theta)}}}. \tag{29}$$

*Proof.* In view of (13), the only remain part is to compute  $\|a - id\|$  for any holonomy element  $a$ . Since all of them are rotations by some angle  $\theta$ , it follows that  $\|au - u\| = \|a - id\|\|u\|$  for any given  $u \in T_pM$ . Hence a direct application of the law of cosines yields that

$$\|a - id_V\| = \sqrt{2 - 2\cos\theta} \tag{30}$$

and hence the result.  $\square$

**Remark 11.** In particular, for the hyperbolic plane  $\mathbb{H}^2(-K)$ ,

$$\text{HolRad}_{\mathbb{H}^2(-K)} \equiv \sqrt{\frac{2\pi}{K}}, \quad (31)$$

whereas for the spheres  $\mathbb{S}^2(K)$  it is a more elusive minimum of (29).

(32)

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