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Inequalities concerning the (p, k)-gamma and (p, k)-polygamma functions

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Abstract. In this paper, we establish several inequalities involving the (p, k)-gamma and (p, k)-polygamma functions. Among other tools, we employ the mean value therorem, the Hermite-Hadamard's inequality, Petrovic's inequality and the Holder's inequality. Upon some parameter variations, we recover some known results as special cases of the established results.

Keywords: (p, k)-gamma function, (p, k)-digamma function, (p, k)-polygamma functions, inequality

MSC 2000 classification: 33B15, 26D07, 26D15

1 Introduction

In [15], the authors introduced a two-parameter deformation of the classical gamma function which is called the (p, k)-gamma function. It is defined for $p \in \mathbb{N}$, $k > 0$ and $x > 0$ as

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p dt,$$

or

$$\Gamma_{p,k}(x) = \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)},$$

and satisfies the properties

$$\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \tag{1}$$
$$\Gamma_{p,k}(k) = 1.$$

Closely related to the (p,k)-gamma function is the (p,k)-digamma function which is defined as follows.

$$\psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k}(x) = \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x} \tag{2}$$

$$= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt. \tag{3}$$

By this definition, the functional equation (1) gives

$$\psi_{p,k}(x+k) - \psi_{p,k}(x) = \frac{1}{x} - \frac{1}{x+pk+k}. \tag{4}$$

Also, the (p,k)-polygamma function is defined for $v \in \mathbb{N}$ as

$$\psi_{p,k}^{(v)}(x) = \frac{d^v}{dx^v}\psi_{p,k}(x) = (-1)^{v+1}v!\sum_{n=0}^{p}\frac{1}{(nk+x)^{v+1}} \tag{5}$$

$$= (-1)^{v+1}\int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}t^v e^{-xt}\,dt, \tag{6}$$

where, $\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)$. By successive differentiations of (4), one obtains

$$\psi_{p,k}^{(v)}(x+k) - \psi_{p,k}^{(v)}(x) = \frac{(-1)^v v!}{x^{v+1}} - \frac{(-1)^v v!}{(x+pk+k)^{v+1}} \tag{7}$$

for $v \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1,2,3,\dots\}$. Also, it can be deduced from (5) and (6) that:

(a) $\psi_{p,k}(x)$ is increasing,

(b) $\psi_{p,k}^{(v)}(x)$ is positive and decreasing if $v \in \{2n+1 : n \in \mathbb{N}_0\}$,

(c) $\psi_{p,k}^{(v)}(x)$ is negative and increasing if $v \in \{2n : n \in \mathbb{N}\}$.

Furthermore, the (p,k)-gamma and (p,k)-polygamma functions satisfy the limit relations

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p\to\infty} & \Gamma_k(x) \\ {\scriptstyle k\to1}\big\downarrow & & \big\downarrow{\scriptstyle k\to1} \\ \Gamma_p(x) & \xrightarrow[p\to\infty]{} & \Gamma(x) \end{array} \qquad \begin{array}{ccc} \psi_{p,k}^{(v)}(x) & \xrightarrow{p\to\infty} & \psi_k^{(v)}(x) \\ {\scriptstyle k\to1}\big\downarrow & & \big\downarrow{\scriptstyle k\to1} \\ \psi_p^{(v)}(x) & \xrightarrow[p\to\infty]{} & \psi^{(v)}(x) \end{array}$$

where, $\Gamma(x)$ and $\psi^{(v)}(x)$ are the ordinary gamma and polygamma functions; $\Gamma_p(x)$ and $\psi_p^{(v)}(x)$ are the p-gamma and p-polygamma functions (see [8],[9]); and $\Gamma_k(x)$ and $\psi_k^{(v)}(x)$ are the k-gamma and k-polygamma functions [5].

The (p,k)-gamma and (p,k)-polygamma functions have been studied in various ways. See for instance the recent works [6], [11], [12], [13], [14], [16], [17], [19] and [20].

In the present work, our goal is to establish some inequalities involving the (p,k)-gamma and (p,k)-polygamma functions. Among other tools, we shall make use of the mean value therorem, the Hermite-Hadamard's inequality, Petrovic's inequality and the Holder's inequality. Upon some parameter variations, we recover some known results as special cases of the established results. We present our results in the following section.

2 Main Results

Theorem 1. *Let $p \in \mathbb{N}$ and $k > 0$. Then for $0 \leq x \leq k$, the inequality*

$$\left(\frac{pk}{p+2}\right) e^{(x-k)\psi_{p,k}(2k)} \leq \Gamma_{p,k}(x+k) \leq \left(\frac{pk}{p+2}\right) e^{(x-k)\psi_{p,k}(x+k)}, \tag{8}$$

is satisfied. Equality holds if and only if $x = k$.

Proof. The situations where $x = 0$ and $x = k$ are obvious. Consider the function $\ln \Gamma_{p,k}(x+k)$ on the interval (x,k). Then by the classical mean value theorem, there exists a $\lambda \in (x,k)$ such that

$$\frac{\ln \Gamma_{p,k}(2k) - \ln \Gamma_{p,k}(x+k)}{k-x} = \psi_{p,k}(\lambda + k).$$

Since $\psi_{p,k}(z)$ is increasing for all $z > 0$, we have

$$\psi_{p,k}(x+k) < \frac{\ln \Gamma_{p,k}(2k) - \ln \Gamma_{p,k}(x+k)}{k-x} < \psi_{p,k}(2k),$$

which implies that

$$(k-x)\psi_{p,k}(x+k) < \ln\left(\frac{pk}{p+2}\right) - \ln \Gamma_{p,k}(x+k) < (k-x)\psi_{p,k}(2k).$$

This further implies that

$$\ln\left(\frac{pk}{p+2}\right) + (k-x)\psi_{p,k}(2k) < \ln \Gamma_{p,k}(x+k) < \ln\left(\frac{pk}{p+2}\right) + (k-x)\psi_{p,k}(x+k),$$

and by taking exponents, we obtain inequality (8). QED

Remark 1. By letting $k = 1$ and $p \to \infty$ in Theorem 1, we obtain

$$e^{(1-x)(1-\gamma)} \leq \Gamma(x+1) \leq e^{(1-x)\psi(x+1)},$$

as presented in [10] for $0 \leq x \leq 1$ where γ is the Euler-Mascheroni constant.

Theorem 2. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$e^{\psi_{p,k}(k)} < [\Gamma_{p,k}(x+k)]^{\frac{1}{x}} < e^{\psi_{p,k}(x+k)}, \tag{9}$$

holds for all $x > 0$.

Proof. Consider the function $\ln \Gamma_{p,k}(x)$ on the interval $(k, x+k)$. Then by the mean value theorem, there exists a $\lambda \in (k, x+k)$ such that

$$\frac{\ln \Gamma_{p,k}(k+x) - \ln \Gamma_{p,k}(k)}{x} = \psi_{p,k}(\lambda).$$

Similarly, since $\psi_{p,k}(z)$ is increasing for all $z > 0$, we have

$$\psi_{p,k}(k) < \frac{\ln \Gamma_{p,k}(x+k)}{x} < \psi_{p,k}(x+k),$$

which upon taking exponents, gives inequality (9). QED

By applying the Hermite-Hadamard inequality, we obtain a similar inequality as shown in the following theorem.

Theorem 3. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$e^{\frac{1}{2}(\psi_{p,k}(k)+\psi_{p,k}(x+k))} < [\Gamma_{p,k}(x+k)]^{\frac{1}{x}} < e^{\psi_{p,k}(\frac{x}{2}+k)}, \tag{10}$$

holds for all $x > 0$.

Proof. It is seen from (5) and (6) that, the function $\psi_{p,k}(z)$ is concave for all $z > 0$. Then by applying the Hermite-Hadamard inequality on the interval $(k, x+k)$, we obtain

$$\frac{1}{2}(\psi_{p,k}(k) + \psi_{p,k}(x+k)) < \frac{1}{x}\int_k^{x+k} \psi_{p,k}(z)\, dz < \psi_{p,k}\left(\frac{x}{2}+k\right)$$

which gives

$$\frac{1}{2}(\psi_{p,k}(k) + \psi_{p,k}(x+k)) < \ln [\ln \Gamma_{p,k}(x+k)]^{\frac{1}{x}} < \psi_{p,k}\left(\frac{x}{2}+k\right).$$

Then, by taking exponents, we obtain inequality (10). QED

Remark 2. Clearly, $\psi_{p,k}(\frac{x}{2}+k) < \psi_{p,k}(x+k)$ and also, by the arithmetic-geometric mean inequality, we have

$$\frac{1}{2}(\psi_{p,k}(k)+\psi_{p,k}(x+k)) > \sqrt{\psi_{p,k}(k)\psi_{p,k}(x+k)} > \sqrt{(\psi_{p,k}(k))^2} = \psi_{p,k}(k).$$

Thus, (10) is sharper than (9).

Remark 3. By letting $k = 1$ and $p \to \infty$ in (9), we obtain

$$e^{-\gamma} < [\Gamma(x+1)]^{\frac{1}{x}} < e^{\psi(x+1)},$$

which is the same as what was established in Theorem 4.2 of [10].

Remark 4. Also, by letting $k = 1$ and $p \to \infty$ in (10), we obtain

$$e^{\frac{1}{2}(-\gamma+\psi(x+1))} < [\Gamma(x+1)]^{\frac{1}{x}} < e^{\psi(\frac{x}{2}+1)}.$$

The following lemma is known in the literature as Petrovic's inequality for convex functions (see [1]).

Lemma 1. *Suppose that $f : I \subseteq [0,\infty) \to \mathbb{R}$ is a convex function. If $x_i \in I$ for $i = 1, 2, \ldots, n$ and $x_1 + x_2 + \cdots + x_n \in I$, then*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \le f(x_1 + x_2 + \cdots + x_n) + (n-1)f(0),$$

with equality if and only if $n = 1$ or $x_1 = x_2 = \cdots = x_n = 0$.

Theorem 4. *Let $p \in \mathbb{N}$, $k > 0$ and $x_i > 0$ for $i = 1, 2, \ldots, n$. Then the inequality*

$$\frac{\Gamma_{p,k}(x_1)\Gamma_{p,k}(x_2)\ldots\Gamma_{p,k}(x_n)}{\Gamma_{p,k}(x_1+x_2+\cdots+x_n)} \le \frac{\left(\frac{pk(x_1+x_2+\cdots+x_n)}{(x_1+x_2+\cdots+x_n)+pk+k}\right)}{\left(\frac{(pk)^n(x_1.x_2\ldots x_n)}{(x_1+pk+k)(x_2+pk+k)\ldots(x_n+pk+k)}\right)}, \tag{11}$$

holds.

Proof. It has been shown in [15, Theorem 2.1] that the function $f(x) = \ln\Gamma_{p,k}(x+k)$ is convex on $I = (0,\infty)$. Now let $x_i \in (0,\infty)$ for $i = 1, 2, \ldots, n$. Then by Lemma 1, we obtain

$$\ln\Gamma_{p,k}(x_1+k)+\ln\Gamma_{p,k}(x_2+k)+\cdots+\ln\Gamma_{p,k}(x_n+k) \le \ln\Gamma_{p,k}(x_1+x_2+\cdots+x_n+k),$$

since $f(0) = 0$. That is,

$$\Gamma_{p,k}(x_1+k)\Gamma_{p,k}(x_2+k)\ldots\Gamma_{p,k}(x_n+k) \le \Gamma_{p,k}(x_1+x_2+\cdots+x_n+k).$$

Then by (1) we obtain

$$\frac{(pk)^n(x_1.x_2\dots x_n)}{(x_1+pk+k)(x_2+pk+k)\dots(x_n+pk+k)}\Gamma_{p,k}(x_1)\Gamma_{p,k}(x_2)\dots\Gamma_{p,k}(x_n)$$
$$\leq \frac{pk(x_1+x_2+\cdots+x_n)}{(x_1+x_2+\cdots+x_n)+pk+k}\Gamma_{p,k}(x_1+x_2+\cdots+x_n),$$

which when rearranged, gives inequality (11). QED

Remark 5. By letting $k=1$ and $p\to\infty$ in (11), we obtain

$$\frac{\Gamma(x_1)\Gamma(x_2)\dots\Gamma(x_n)}{\Gamma(x_1+x_2+\cdots+x_n)}\leq\frac{x_1+x_2+\cdots+x_n}{x_1.x_2\dots x_n}$$

which gives an upper bound for the beta function of n variables [2].

Remark 6. In particular, let $n=2$, $x_1=x$ and $x_2=y$ in (11). Then we obtain

$$\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}\leq\frac{x+y}{xy}\cdot\frac{(x+pk+k)(y+pk+k)}{pk(x+y+pk+k)} \tag{12}$$

which gives an upper bound for the (p,k)-beta function. Moreover, if $k=1$ and $p\to\infty$ in (12), the we obtain

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\leq\frac{x+y}{xy}, \tag{13}$$

for $x>0$ and $y>0$.

Theorem 5. *Let $p\in\mathbb{N}$ and $k>0$. Then the inequality*

$$\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}\leq\left(\frac{1}{xy}\right)^{\frac{1}{2}}\left(\frac{x+pk+k}{pk}\cdot\frac{y+pk+k}{pk}\right)^{\frac{1}{2}} \tag{14}$$

holds for $x\geq k$ and $y\geq k$, with equality if and only if $x=y=k$.

Proof. Consider the (p,k)-beta function, $B_{p,k}(x,y)=\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}$ for $x\geq k$ and $y\geq k$. With no loss of generality, let y be fixed. Then logarithmic differntiation yields

$$\frac{\partial}{\partial x}B_{p,k}(x,y)=B_{p,k}(x,y)\left(\psi_{p,k}(x)-\psi_{p,k}(x+y)\right)<0.$$

Thus, $B_{p,k}(x,y)$ is decreasing in x. Then for $x\geq k$, we have

$$\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}\leq\frac{\Gamma_{p,k}(k)\Gamma_{p,k}(y)}{\Gamma_{p,k}(k+y)}=\frac{\Gamma_{p,k}(y)}{\frac{pky}{y+pk+k}\Gamma_{p,k}(y)}=\frac{y+pk+k}{pky}. \tag{15}$$

Likewise, fixing x yields

$$\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)} \leq \frac{x+pk+k}{pkx}. \tag{16}$$

Now, (15) and (16) imply

$$\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)} \leq \left(\frac{x+pk+k}{pkx} \cdot \frac{y+pk+k}{pky}\right)^{\frac{1}{2}},$$

which completes the proof. QED

Remark 7. By letting $k=1$ and $p \to \infty$ in (14), we obtain

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \leq \left(\frac{1}{xy}\right)^{\frac{1}{2}}, \tag{17}$$

for $x \geq 1$ and $y \geq 1$. However, this is weaker than the main result of [7]. Additional information on inequalities of type (13) and (17) can be found in [3], [4], and the related references therein.

Theorem 6. *Let $p \in \mathbb{N}$, $k > 0$, $u \geq 0$, $s \in \{2n : n \in \mathbb{N}_0\}$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then the inequalities*

$$(k-u)\psi_{p,k}^{(s+1)}(x+k) \leq \psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x+u) \leq (k-u)\psi_{p,k}^{(s+1)}(x+u), \tag{18}$$

$$(k-u)\psi_{p,k}^{(r+1)}(x+u) \leq \psi_{p,k}^{(r)}(x+k) - \psi_{p,k}^{(r)}(x+u) \leq (k-u)\psi_{p,k}^{(r+1)}(x+k), \tag{19}$$

are valid for all $x > 0$. Equality holds if and only if $u = k$.

Proof. The case where $u = k$ is obvious. Now, let $0 \leq u < k$ and for $s \in \{2n : n \in \mathbb{N}_0\}$, consider the function $\psi_{p,k}^{(s)}(x)$ on the interval $(x+u, x+k)$. Then by the mean value theorem, there exist a $\lambda \in (x+u, x+k)$ such that

$$\frac{\psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x+u)}{k-u} = \psi_{p,k}^{(s+1)}(\lambda).$$

Since $\psi_{p,k}^{(s+1)}(z)$ is decreasing for all $z > 0$, we obtain

$$\psi_{p,k}^{(s+1)}(x+k) < \frac{\psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x+u)}{k-u} < \psi_{p,k}^{(s+1)}(x+u),$$

which gives inequality (18). By the same procedure, the case for $u > k$ yields the same result. Hence we omit the details.

Similarly, let $0 \leq u < k$ and for $r \in \{2n+1 : n \in \mathbb{N}_0\}$, consider the function $\psi_{p,k}^{(r)}(x)$ on the interval $(x+u, x+k)$. The mean value theorem gives

$$\frac{\psi_{p,k}^{(r)}(x+k) - \psi_{p,k}^{(r)}(x+u)}{k-u} = \psi_{p,k}^{(r+1)}(\delta),$$

for $\delta \in (x+u, x+k)$. Since $\psi_{p,k}^{(r+1)}(z)$ is increasing for all $z > 0$, we have

$$\psi_{p,k}^{(r+1)}(x+u) < \frac{\psi_{p,k}^{(r)}(x+k) - \psi_{p,k}^{(r)}(x+u)}{k-u} < \psi_{p,k}^{(r+1)}(x+k),$$

which gives inequality (19). The case for $u > k$ gives the same result and so, we omit the details. QED

Corollary 1. *Let $p \in \mathbb{N}$, $k > 0$, $s \in \{2n : n \in \mathbb{N}_0\}$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then the inequalities*

$$k\psi_{p,k}^{(s+1)}(x+k) < \frac{s!}{x^{s+1}} - \frac{s!}{(x+pk+k)^{s+1}} < k\psi_{p,k}^{(s+1)}(x), \tag{20}$$

$$k\psi_{p,k}^{(r+1)}(x) < \frac{r!}{(x+pk+k)^{r+1}} - \frac{r!}{x^{r+1}} < k\psi_{p,k}^{(r+1)}(x+k), \tag{21}$$

are valid for all $x > 0$.

Proof. By letting $u = 0$ in Theorem 6, we obtain

$$k\psi_{p,k}^{(s+1)}(x+k) < \psi_{p,k}^{(s)}(x+k) - \psi_{p,k}^{(s)}(x) < k\psi_{p,k}^{(s+1)}(x), \tag{22}$$

$$k\psi_{p,k}^{(r+1)}(x) < \psi_{p,k}^{(r)}(x+k) - \psi_{p,k}^{(r)}(x) < k\psi_{p,k}^{(r+1)}(x+k). \tag{23}$$

Then by applying (7) to (22) and (23), we obtain respectively (20) and (21). QED

Remark 8. By letting $k = 1$ and $p \to \infty$ in Theorem 6 and Corollary 1, we obtain the results of Theorem 5.4 and Corollary 5.5 of [10].

Remark 9. If $k = 1$, $s = 0$ and $r = 1$, then (20) and (21) reduces to

$$\psi_p'(x+1) < \frac{1}{x} - \frac{1}{x+p+1} < \psi_p'(x),$$

$$\psi_p''(x) < \frac{1}{(x+p+1)^2} - \frac{1}{x^2} < \psi_p''(x+1),$$

where, $\psi_p(x)$ is the p-digamma function.

Lemma 2. *Let $p \in \mathbb{N}$, $k > 0$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then, the inequality*

$$\psi_{p,k}^{(r)}(x)\psi_{p,k}^{(r+2)}(x) - \left[\psi_{p,k}^{(r+1)}(x)\right]^2 \geq 0,$$

holds for $x > 0$.

Proof. See Corollary 2.3 of [15]. QED

Lemma 3. *Let $p \in \mathbb{N}$, $k > 0$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then, the function $\frac{\psi_{p,k}^{(m+1)}(x)}{\psi_{p,k}^{(m)}(x)}$ is increasing on $(0,\infty)$.*

Proof. Direct differentiation yields

$$\left(\frac{\psi_{p,k}^{(r+1)}(x)}{\psi_{p,k}^{(r)}(x)}\right)' = \frac{\psi_{p,k}^{(r)}(x)\psi_{p,k}^{(r+2)}(x) - [\psi_{p,k}^{(r+1)}(x)]^2}{[\psi_{p,k}^{(r)}(x)]^2} \geq 0,$$

which follows directly from Lemma 2. QED

Theorem 7. *Let $p \in \mathbb{N}$, $k > 0$, $u \geq 0$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then the inequality*

$$\exp\left\{(k-u)\frac{\psi_{p,k}^{(r+1)}(x+u)}{\psi_{p,k}^{(r)}(x+u)}\right\} \leq \frac{\psi_{p,k}^{(r)}(x+k)}{\psi_{p,k}^{(r)}(x+u)} \leq \exp\left\{(k-u)\frac{\psi_{p,k}^{(r+1)}(x+k)}{\psi_{p,k}^{(r)}(x+k)}\right\}, \tag{24}$$

valid for all $x > 0$. Equality holds if and only if $u = k$.

Proof. Suppose that $0 \leq u < k$ and consider the function $\ln \psi_{p,k}^{(r)}(x)$ on the interval $(x+u, x+k)$. Then the mean value theorem yields

$$\frac{\ln \psi_{p,k}^{(r)}(k+x) - \ln \psi_{p,k}^{(r)}(x+u)}{k-u} = \frac{\psi_{p,k}^{(r+1)}(c)}{\psi_{p,k}^{(r)}(c)},$$

where, $c \in (x+u, x+k)$. Then, since $\frac{\psi_{p,k}^{(r+1)}(z)}{\psi_{p,k}^{(r)}(z)}$ is increasing for all $z > 0$, we have

$$\frac{\psi_{p,k}^{(r+1)}(x+u)}{\psi_{p,k}^{(r)}(x+u)} < \frac{1}{k-u} \ln \frac{\psi_{p,k}^{(r)}(x+k)}{\psi_{p,k}^{(r)}(x+u)} < \frac{\psi_{p,k}^{(r+1)}(x+k)}{\psi_{p,k}^{(r)}(x+k)},$$

which upon taking exponents, gives inequality (24). The case where $u > k$ gives the same result by following a similar procedure. QED

Remark 10. By letting $k = 1$ and $p \to \infty$ in (24), we recover inequality (8.1) of [10]. However, we noticed that, the proof of inequality (8.2) of [10] has a drawback. The expression $\ln \psi^{(2n)}(x+1) - \ln \psi^{(2n)}(x+\lambda)$ is not defined since $\psi^{(i)}(x) < 0$ for even values of i.

Theorem 8. *Let $p \in \mathbb{N}$, $k > 0$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$. Then the inequality*

$$\psi_{p,k}^{(r)}(x)\psi_{p,k}^{(r)}(x+y+z) - \psi_{p,k}^{(r)}(x+y)\psi_{p,k}^{(r)}(x+z) > 0, \tag{25}$$

holds for positive real numbers x, y and z.

Proof. For positive real numbers x, y and z, let Q be defined as

$$Q(x) = \frac{\psi_{p,k}^{(r)}(x+z)}{\psi_{p,k}^{(r)}(x)}.$$

Then

$$Q'(x) = \frac{\psi_{p,k}^{(r)}(x+z)}{\psi_{p,k}^{(r)}(x)} \left[\frac{\psi_{p,k}^{(r+1)}(x+z)}{\psi_{p,k}^{(r)}(x+z)} - \frac{\psi_{p,k}^{(r+1)}(x)}{\psi_{p,k}^{(r)}(x)} \right] > 0,$$

since $\frac{\psi_{p,k}^{(r+1)}(x)}{\psi_{p,k}^{(r)}(x)}$ is increasing. Hence $Q(x)$ is increasing. Therefore, $Q(x+y) > Q(x)$ which gives inequality (25). QED

Theorem 9. *Let $p \in \mathbb{N}$, $k > 0$, $r \in \{2n+1 : n \in \mathbb{N}_0\}$ and $s \in \{2n : n \in \mathbb{N}_0\}$. Then the inequalities*

$$\psi_{p,k}^{(r)}(x+y) < \psi_{p,k}^{(r)}(x) + \psi_{p,k}^{(r)}(y), \tag{26}$$

$$\psi_{p,k}^{(s)}(x+y) > \psi_{p,k}^{(s)}(x) + \psi_{p,k}^{(s)}(y), \tag{27}$$

holds for $x > 0$ and $y > 0$.

Proof. Let G be defined for $x > 0$, $y > 0$ and $r \in \{2n+1 : n \in \mathbb{N}_0\}$ as

$$G(x,y) = \psi_{p,k}^{(r)}(x+y) - \psi_{p,k}^{(r)}(x) - \psi_{p,k}^{(r)}(y).$$

Without loss of generality, let y be fixed. Then

$$\frac{\partial}{\partial x} G(x,y) = \psi_{p,k}^{(r+1)}(x+y) - \psi_{p,k}^{(r+1)}(x) > 0,$$

since $\psi_{p,k}^{(v)}(x)$ is increasing for $v \in \{2n : n \in \mathbb{N}_0\}$. Thus, $G(x,y)$ is increasing in x. Furthermore, by using representation (6), we obtain

$$\lim_{x\to\infty} G(x,y) = -\int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} t^{r+1} e^{-yt} < 0.$$

Therfore, $G(x,y) < \lim_{x\to\infty} G(x,y) < 0$ which gives (26). The proof of (27) follows a similar procedure. Hence we omit the details. QED

Remark 11. Theorem 9 generalizes Theorem 2.2 and Theorem 2.4 of [18].

Theorem 10. *Let* $p \in \mathbb{N}$, $k > 0$, $r \in \{2n+1 : n \in \mathbb{N}_0\}$, $a > 1$ *and* $\frac{1}{a} + \frac{1}{b} = 1$. *Then the inequality*

$$\psi_{p,k}^{(r)}(x+y) \leq \left(\psi_{p,k}^{(r)}(x)\right)^{\frac{1}{a}} \left(\psi_{p,k}^{(r)}(y)\right)^{\frac{1}{b}} \tag{28}$$

is valid for $x > 0$ *and* $y > 0$.

Proof. By utilizing the Hölder's inequality for finite sums, we obtain

$$\begin{aligned}
\psi_{p,k}^{(r)}(x+y) &= \sum_{n=0}^{p} \frac{r!}{(nk+x+y)^{r+1}} \\
&= \sum_{n=0}^{p} \frac{(r!)^{\frac{1}{a}}(r!)^{\frac{1}{b}}}{(nk+x+y)^{\frac{r+1}{a}}(nk+x+y)^{\frac{r+1}{b}}} \\
&\leq \sum_{n=0}^{p} \frac{(r!)^{\frac{1}{a}}}{(nk+x)^{\frac{r+1}{a}}} \cdot \frac{(r!)^{\frac{1}{b}}}{(nk+y)^{\frac{r+1}{b}}} \\
&\leq \left(\sum_{n=0}^{p} \frac{r!}{(nk+x)^{r+1}}\right)^{\frac{1}{a}} \left(\sum_{n=0}^{p} \frac{r!}{(nk+y)^{r+1}}\right)^{\frac{1}{b}} \\
&= \left(\psi_{p,k}^{(r)}(x)\right)^{\frac{1}{a}} \left(\psi_{p,k}^{(r)}(y)\right)^{\frac{1}{b}},
\end{aligned}$$

which completes the proof. QED

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