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A note on the Formanek Weingarten function

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Abstract. The aim of this note is to compare work of Formanek [7] on a certain construction of central polynomials with that of Collins [3] on integration on unitary groups.

These two quite disjoint topics share the construction of the same function on the symmetric group, which the second author calls *Weingarten function*.

By joining these two approaches we succeed in giving a simplified and *very natural* presentation of both Formanek and Collins's Theory.

Keywords: symmetric group, unitary group, Schur-Weyl duality, invariants, representative functions

MSC 2020 classification: primary 20CXX, secondary 20C30

1 Schur Weyl duality

1.1 Basic results

We need to recall some basic facts on the representation Theory of the symmetric and the linear group.

Let V be a vector space of finite dimension d over a field F which in this note can be taken as $\mathbb{Q}$ or $\mathbb{C}$. On the tensor power $V^{\otimes k}$ act both the symmetric group S_k and the linear group $GL(V)$, Formula (1.1), furthermore if $F=\mathbb{C}$ and V is equipped with a Hilbert space structure one has an induced Hilbert space structure on $V^{\otimes k}$. The unitary group $U(d)\subset GL(V)$ acts on $V^{\otimes k}$ by unitary matrices.

$$\sigma\cdot u_1\otimes u_2\otimes\ldots\otimes u_k:=u_{\sigma^{-1}(1)}\otimes u_{\sigma^{-1}(2)}\otimes\ldots\otimes u_{\sigma^{-1}(k)},$$

$$g\cdot u_1\otimes u_2\otimes\ldots\otimes u_k:=gu_1\otimes gu_2\otimes\ldots\otimes gu_k,\ \sigma\in S_k,\ g\in GL(V). \tag{1.1}$$

The first step of Schur Weyl duality is the fact that the two operator algebras $\Sigma_k(V), B_{k,d}$ generated respectively by S_k and $GL(V)$ acting on $V^{\otimes k}$, are both semisimple and each the centralizer of the other.

In particular the algebra $\Sigma_k(V) \subset End(V^{\otimes k}) = End(V)^{\otimes k}$ equals the subalgebra $\Sigma_k(V) = \left(End(V)^{\otimes k}\right)^{GL(V)}$ of invariants under the conjugation action of the group $GL(V) \to End(V)^{\otimes k},\ g \mapsto g \otimes g \otimes \ldots \otimes g$.

From this, the double centralizer Theorem and work of Frobenius and Young one has that, under the action of these two commuting groups, the space $V^{\otimes k}$ decomposes into the direct sum

$$V^{\otimes k} = \oplus_{\lambda \vdash k,\ ht(\lambda) \leq d} M_\lambda \otimes S_\lambda(V) \tag{1.2}$$

over all partitions λ of k of height $\leq d$, (the height $ht(\lambda)$ denotes the number of elements or *rows*, nonzero, of λ).

M_λ is an irreducible representation of S_k while $S_\lambda(V)$, called a *Schur functor* is an irreducible polynomial representation of $GL(V)$, which remains irreducible also when restricted to $U(d)$. The partition with a single row k corresponds to the trivial representation of S_k and to the symmetric power $S^k(V)$ of V. The partition with a single column k corresponds to the sign representation of S_k and to the exterior power $\bigwedge^k(V)$ of V.

The character theory of the two groups can be deduced from these representations. We shall denote by $\chi_\lambda(\sigma)$ the character of the permutation σ on M_λ. As for $S_\lambda(V)$ its character is expressed by a symmetric function $S_\lambda(x_1, \ldots, x_d)$ restriction to the first d variables of a stable symmetric function called *Schur function.* Of this deep and beautiful Theory, see [15], [9], [10], [28], [21], we shall use only two remarkable formulas, the *hook formula* due to Frame, Robinson and Thrall [22], expressing the dimension $\chi_\lambda(1)$ of M_λ and the *hook-content formula* of Stanley, cf. [25, Corollary 7.21.4]) expressing the dimension $s_\lambda(d) := S_\lambda(1, \ldots, 1) = S_\lambda(1^d)$ of $S_\lambda(V)$.

We display partitions by Young diagrams, as in the figure below.

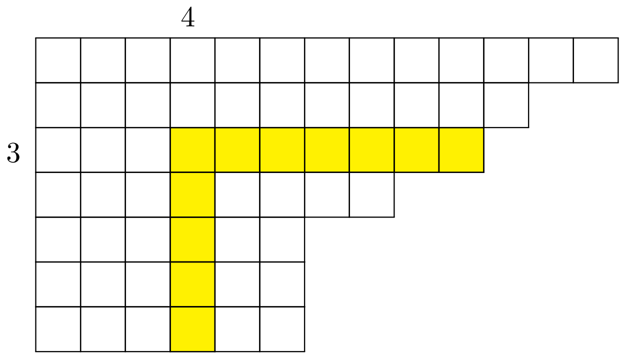
By $\tilde{\lambda}$ we denote the dual partition obtained by exchanging rows and columns. The *boxes*, cf. (2), of the diagram are indexed by pairs (i, j) of coordinates. [1] Given then one of the boxes u we define its *hook number* h_u and its *content* c_u as follows:

Definition 1. Let λ be a partition of n and let $u = (i, j) \in \lambda$ be a box in the corresponding Young diagram. The *hook number* $h_u = h(i, j)$ and the *content* c_u are defined as follows:

$$h_u = h(i, j) = \lambda_i + \check{\lambda}_j - i - j + 1, \quad c_u = c(i, j) := j - i. \tag{1.3}$$

Example 1. Note that the box $u = (3, 4)$ defines a hook in the diagram λ, and h_u equals the length (number of boxes) of this hook:

[1] We use the *english notation*



In this figure, we have $\lambda = (13, 11, 10, 8, 6^3)$, $ht(\lambda) = 7$ with $u = (3, 4)$.

Then $\check{\lambda} = (7^6, 4^2, 3^2, 2, 1^2)$ and $h_u = \lambda_3 + \check{\lambda}_4 - 3 - 4 + 1 = 10 + 7 - 6 = 11$.

Here is another example: In the following diagram of shape $\lambda = (8, 3, 2, 1)$, each hook number h_u, respectively content c_u is written inside its box in the diagram λ:

11	9	7	5	4	3	2	1
5	3	1					
3	1						
1							

,

0	1	2	3	4	5	6	7
-1	0	1					
-2	-1						
-3							

Theorem 1 (The hook and hook–content formulas)**.** *Let $\lambda \vdash k$ be a partition of k and $\chi_\lambda(1)$ and $s_\lambda(d)$ be the dimension of the corresponding irreducible representation M_λ of S_k and $S_\lambda(V)$ of $GL(V)$, $\dim(V) = d$. Then*

$$s_\lambda(d) = \prod_{u \in \lambda} \frac{d + c_u}{h_u}, \qquad \chi_\lambda(1) = \frac{k!}{\prod_{u \in \lambda} h_u}. \tag{1.4}$$

The remarkable Formula of Stanley, Theorem 15.3 of [24], exhibits $s_\lambda(d)$ as a polynomial of degree $k = |\lambda|$ in d with zeroes the integers $-c_u$ and leading coefficient $\prod_{u \in \lambda} h_u^{-1}$, see §3.3 for a proof.

1.1.1 Matrix invariants

The dual of the algebra $End(V)^{\otimes k}$ can be identified, in a $GL(V)$ equivariant way, to $End(V)^{\otimes k}$ by the pairing formula:

$$\langle A_1 \otimes A_2 \cdots \otimes A_k \mid B_1 \otimes B_2 \cdots \otimes B_k \rangle := \operatorname{tr}(A_1 \otimes A_2 \cdots \otimes A_k \circ B_1 \otimes B_2 \cdots \otimes B_k)$$

$$= \operatorname{tr}(A_1 B_1 \otimes A_2 B_2 \cdots \otimes A_k B_k) = \prod_{i=1}^{k} \operatorname{tr}(A_i B_i).$$

Under this isomorphism the multilinear invariants of matrices are identified with the $GL(V)$ invariants of $End(V)^{\otimes m}$ which in turn are spanned by the elements of the symmetric group, hence by the elements of Formula (1.5). These are explicited by Formula (1.6) as in Kostant [14].

Proposition 1. *The space $\mathcal{T}_d(k)$ of multilinear invariants of k, $d \times d$ matrices is identified with $End_{GL(V)}(V^{\otimes k})$ and it is linearly spanned by the functions:*

$$T_\sigma(X_1, X_2, \ldots, X_d) := \operatorname{tr}(\sigma^{-1} \circ X_1 \otimes X_2 \otimes \cdots \otimes X_d), \ \sigma \in S_k. \tag{1.5}$$

If $\sigma = (i_1 i_2 \ldots i_h) \ldots (j_1 j_2 \ldots j_\ell)(s_1 s_2 \ldots s_t)$ is the cycle decomposition of σ then we have that $T_\sigma(X_1, X_2, \ldots, X_d)$ equals

$$= \operatorname{tr}(X_{i_1} X_{i_2} \ldots X_{i_h}) \ldots \operatorname{tr}(X_{j_1} X_{j_2} \ldots X_{j_\ell}) \operatorname{tr}(X_{s_1} X_{s_2} \ldots X_{s_t}). \tag{1.6}$$

Proof. Since the identity of Formula (1.6) is multilinear it is enough to prove it on the decomposable tensors of $End(V) = V \otimes V^*$ which are the endomorphisms of rank 1, $u \otimes \varphi : v \mapsto \langle \varphi \,|\, v \rangle u$.

So given $X_i := u_i \otimes \varphi_i$ and an element $\sigma \in S_k$ in the symmetric group we have

$$\sigma^{-1} \circ u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \ldots \otimes u_k \otimes \varphi_k (v_1 \otimes v_2 \otimes \ldots \otimes v_k)$$
$$\overset{(1.1)}{=} \prod_{i=1}^k \langle \varphi_i \,|\, v_i \rangle u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(k)}$$
$$u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \ldots \otimes u_k \otimes \varphi_m \circ \sigma^{-1}(v_1 \otimes v_2 \otimes \ldots \otimes v_k)$$
$$= \prod_{i=1}^m \langle \varphi_i \,|\, v_{\sigma(i)} \rangle u_1 \otimes u_2 \otimes \ldots \otimes u_k = \prod_{i=1}^k \langle \varphi_{\sigma^{-1}(i)} \,|\, v_i \rangle u_1 \otimes u_2 \otimes \ldots \otimes u_k$$
$$\implies \sigma^{-1} \circ u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \ldots \otimes u_m \otimes \varphi_k = u_{\sigma(1)} \otimes \varphi_1 \otimes u_{\sigma(2)} \otimes \varphi_2 \otimes \ldots \otimes u_{\sigma(k)} \otimes \varphi_k$$
$$\implies u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \ldots \otimes u_k \otimes \varphi_k \circ \sigma = u_1 \otimes \varphi_{\sigma(1)} \otimes u_2 \otimes \varphi_{\sigma(2)} \otimes \ldots \otimes u_k \otimes \varphi_{\sigma(k)}. \tag{1.7}$$

So we need to understand in matrix formulas the invariants

$$\operatorname{tr}(\sigma^{-1} u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \ldots \otimes u_k \otimes \varphi_k) = \prod_{i=1}^k \langle \varphi_i \,|\, u_{\sigma(i)} \rangle. \tag{1.8}$$

We need to use the rules

$$u \otimes \varphi \circ v \otimes \psi = u \otimes \langle \varphi \,|\, v \rangle \psi, \quad \operatorname{tr}(u \otimes \varphi) = \langle \varphi \,|\, u \rangle$$

from which the formula easily follows by induction. $\boxed{QED}$

Remark 1. We can extend the Formula (1.5) to the group algebra

$$t(\sum_{\tau \in S_d} a_\tau \tau)(X_1, \ldots, X_d) := \sum_{\tau \in S_d} a_\tau T_\tau(X_1, X_2, \ldots, X_d). \tag{1.9}$$

1.2 The symmetric group

The algebra of the symmetric group S_k decomposes into the direct sum

$$F[S_k] = \oplus_{\lambda \vdash k} End(M_\lambda)$$

of the matrix algebras associated to the irreducible representations M_λ of partitions $\lambda \vdash k$. Denote by χ_λ the corresponding character of S_k and by $e_\lambda \in End(M_\lambda) \subset F[S_k]$ the corresponding central unit. These elements form a basis of orthogonal idempotents of the center of $F[S_k]$.

For a finite group G let e_i be the central idempotent of an irreducible representation with character χ_i. One has the Formula:

$$I) \quad e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \bar{\chi}_i(g) g, \quad II) \quad \chi_i(e_j) = \begin{cases} \chi_i(1) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} . \tag{1.10}$$

This is equivalent to the *orthogonality of characters*

$$\frac{1}{|G|} \sum_{g \in G} \bar{\chi}_i(g) \chi_j(g) = \delta^i_j. \tag{1.11}$$

As for the algebra $\Sigma_k(V)$, it is isomorphic to $F[S_k]$ if and only if $d \geq k$. Otherwise it is a homomorphic image of $F[S_k]$ with kernel the ideal generated by any antisymmetrizer in $d+1$ elements. This ideal is the direct sum of the $End(M_\lambda)$ with $ht(\lambda) > d$, where $ht(\lambda)$, the *height* of λ, cf. page 70 is also the length of its first column. So that

$$\Sigma_k(V) = \oplus_{\lambda \vdash k,\ ht(\lambda) \leq d} End(M_\lambda) \tag{1.12}$$

1.3 The function $Wg(d, \mu)$

We start with a computation of a character.

Definition 2. Given a permutation $\rho \in S_k$ we denote by $c(\rho)$ the number of cycles into which it decomposes, and $\pi(\rho) \vdash k$ the partition of k given by the lengths of these cycles. Notice that $c(\rho) = ht(\pi(\rho))$.

Given a partition $\mu \vdash k$ we denote by

$$(\mu) := \{\rho \mid \pi(\rho) = \mu\}, \quad C_\mu := \sum_{\rho \mid \pi(\rho) = \mu} \rho = \sum_{\rho \in (\mu)} \rho. \tag{1.13}$$

The sets $(\mu) := \{\rho \mid \pi(\rho) = \mu\}$ are the conjugacy classes of S_k and, thinking of $F[S_k]$ as functions from S_k to F we have that C_μ is the characteristic function of the corresponding conjugacy class. Of course the elements C_μ form a basis of the center of the group algebra $F[S_k]$.

Proposition 2. *(1) For every pair of positive integers k, d the function P on S_k given by $P : \rho \mapsto d^{c(\rho)}$ is the character of the permutation action of S_k on $V^{\otimes k}$, $\dim_F(V) = d$.*

(2) The symmetric bilinear form on $F[S_k]$ given by $\langle \sigma \mid \tau \rangle := d^{c(\sigma\tau)}$ has as kernel the ideal generated by the antisymmetrizer on $d+1$ elements. In particular if $k \leq d$ it is non degenerate.

Proof. (1) If $e_1, \ldots, e_d$ is a given basis of V we have the induced basis of $V^{\otimes k}$, $e_{i_1} \otimes \ldots \otimes e_{i_k}$ which is permuted by the symmetric group. For a permutation representation the trace of an element σ equals the number of the elements of the basis fixed by σ.

If $\sigma = (1, 2, \ldots, k)$ is one cycle then $e_{i_1} \otimes \ldots \otimes e_{i_k}$ is fixed by σ if and only if $i_1 = i_2 = \ldots = i_k$ are equal, so equal to some e_j so $\operatorname{tr}(\sigma) = d$.

For a product of a cycles of lengths $b_1, b_2, \ldots b_a$ which up to conjugacy we may consider as

$$(1, 2, \ldots, b_1)(b_1 + 1, b_1 + 2, \ldots, b_1 + b_2) \ldots (k - b_a, \ldots, k)$$

we see that $e_{i_1} \otimes \ldots \otimes e_{i_k}$ is fixed by σ if and only if it is of the form

$$e_{i_1}^{\otimes b_1} \otimes e_{i_2}^{\otimes b_2} \otimes \ldots \otimes e_{i_a}^{\otimes b_a},$$

giving d^a choices for the indices $i_1, i_2, \ldots, i_a$.

(2) In fact this is the trace form of the image $\Sigma_k(V)$ of $F[S_k]$ in the operators on $V^{\otimes k}$, $\dim V = d$. Since $\Sigma_k(V)$ is semisimple its trace form is non degenerate.

QED

Corollary 1.

$$I) \quad P = \sum_{\lambda \vdash k,\ ht(\lambda) \leq d} s_\lambda(d)\chi_\lambda, \quad II) \quad d^{c(\rho)} = \sum_{\lambda \vdash k,\ ht(\lambda) \leq d} s_\lambda(d)\chi_\lambda(\rho). \tag{1.14}$$

Proof. This is immediate from Formula (1.2). QED

We thus have, with $ht(\mu)$ the number of parts of μ (cf. page 73), that

$$P := \sum_{\rho \in S_k} d^{c(\rho)}\rho = \sum_{\mu \vdash k} d^{ht(\mu)} C_\mu \tag{1.15}$$

is an element of the center of the algebra $\Sigma_k(V)$ which we can thus write

$$P = \sum_{\lambda\vdash k,\ ht(\lambda)\leq d} s_\lambda(d)\chi_\lambda = \sum_{\rho\in S_k} d^{c(\rho)}\rho = \sum_{\lambda\vdash k,\ ht(\lambda)\leq d} r_\lambda(d) e_\lambda \tag{1.16}$$

and we have:

Proposition 3.

$$r_\lambda(d) = \prod_{u\in\lambda}(d + c_u). \tag{1.17}$$

Proof. By Formula (1.10) we have:

$$I)\quad e_\lambda = \frac{\chi_\lambda(1)}{k!}\sum_{\sigma\in S_k}\chi_\lambda(\sigma)\sigma, \quad II)\quad \chi_\lambda(e_\mu) = \begin{cases}\chi_\lambda(1) & \text{if } \lambda = \mu\\ 0 & \text{if } \lambda \neq \mu\end{cases}. \tag{1.18}$$

One has thus, from Formulas (1.14) I) and (1.18) II) and denoting by (χ_λ, P) the usual scalar product of characters:

$$r_\lambda(d) = \frac{\sum_\rho d^{c(\rho)}\chi_\lambda(\rho)}{\chi_\lambda(1)} = \frac{k!(P,\chi_\lambda)}{\chi_\lambda(1)} = \frac{k!\, s_\lambda(d)}{\chi_\lambda(1)} \overset{(1.4)}{=} \prod_{u\in\lambda}(d + c_u).$$

QED

Corollary 2. *The element* $\sum_\rho d^{c(\rho)}\rho$ *is invertible in* $\Sigma_k(V)$ *with inverse*

$$(\sum_{\rho\in S_k} d^{c(\rho)}\rho)^{-1} = \sum_{\lambda\vdash k,\ ht(\lambda)\leq d}(\prod_{u\in\lambda}(d + c_u))^{-1}e_\lambda. \tag{1.19}$$

As we shall see in §2.1, it is interesting to study $(\sum_{\rho\in S_k} d^{c(\rho)}\rho)^{-1}$ where k is fixed and d is a parameter. We can thus use formula (1.19) for $d \geq k$ and following Collins [3] we write

$$(\sum_{\rho\in S_k} d^{c(\rho)}\rho)^{-1} = \sum_{\rho\in S_k} Wg(d,\rho)\rho := Wg(d,k) \tag{1.20}$$

Since $Wg(d,\rho)$ is a class function it depends only on the cycle partition $\mu = c(\rho)$ of ρ, so we may denote it by $Wg(d,\mu)$. We call the function $Wg(d,\rho)$ the *Formanek–Weingarten function*, since it was already introduced by Formanek in [7].

From definition (1.13) $C_\mu = \sum_{c(\rho)=\mu}\rho$ we can rewrite, $d \geq k$

$$C_\mu = \sum_{\rho\in S_k | c(\rho)=\mu}\rho, \qquad Wg(d,k) = (\sum_{\rho\in S_k} d^{c(\rho)}\rho)^{-1} = \sum_{\mu\vdash k} Wg(d,\mu)C_\mu. \tag{1.21}$$

Substituting e_λ in formula (1.19) with its expression of Formula (1.18)

$$e_\lambda = \frac{\chi_\lambda(1)}{k!}\sum_{\sigma\in S_k}\chi_\lambda(\sigma)\sigma = \prod_{u\in\lambda} h_u^{-1}\sum_{\sigma\in S_k}\chi_\lambda(\sigma)\sigma$$

$$Wg(d,k) := \sum_{\rho\in S_k} Wg(d,\rho)\rho = \sum_{\lambda\vdash k}\prod_{u\in\lambda}\frac{1}{h_u(d+c_u)}\sum_\tau \chi_\lambda(\tau)\tau \tag{1.22}$$

Theorem 2.

$$Wg(d,\sigma) = \sum_{\lambda\vdash k}\prod_{u\in\lambda}\frac{1}{h_u(d+c_u)}\chi_\lambda(\sigma) = \sum_{\lambda\vdash k}\frac{\chi_\lambda(1)^2\chi_\lambda(\sigma)}{k!^2 s_\lambda(d)}. \tag{1.23}$$

In particular $Wg(d,\sigma)$ is a rational function of d with poles at the integers $-k+1\le i\le k-1$ of order p at i, $p(p+|i|)\le k$.

Proof. We only need to prove the last estimate. By symmetry we may assume that $i\ge 0$ then the p^{th} entry of i is placed at the lower right corner of a rectangle of height p and length $i+p$ (cf. Figure at page 71). Hence if $\lambda\vdash k$, we have $i(p+i)\le k$ and the claim. QED

1.3.1 A more explicit formula

Formula (1.23), although explicit, is a sum with alternating signs so that it is not easy to estimate a given value or even to show that it is nonzero.

For $\sigma_0 = (1,2,\dots,k)$ a full cycle a better Formula is available. First Formula (1.24) by Formanek when $k=d$, and then Collins Formula (1.25) in general.

When $k=d$ we write $Wg(d,\sigma)=a_\sigma$ and then:

$$d!^2 a_{\sigma_0} = (-1)^{d+1}\frac{d}{2d-1}\neq 0. \tag{1.24}$$

Collins extends Formula (1.24) to the case $Wg(d,\sigma_0)$getting:

$$Wg(d,\sigma_0) = (-1)^{k-1}\mathtt{C}_{k-1}\prod_{-k+1\le j\le k-1}(d-j)^{-1} \tag{1.25}$$

with $\mathtt{C}_i := \frac{(2i)!}{(i+1)!i!} = \frac{1}{i+1}\binom{2i}{i}$ the i^{th} Catalan number. Which, since

$$\mathtt{C}_{d-1} = \frac{(2d-2)!}{d!(d-1)!},\qquad \prod_{-d+1\le j\le d-1}(d-j) = (2d-1)!$$

agrees, when $k=d$, with Formanek.

In order to prove Formula (1.25) we need the fact that $\chi_\lambda(\sigma_0) = 0$ except when $\lambda = (a, 1^{k-a})$ is a *hook partition*, with the first row of some length a, $1 \leq a \leq k$ and then the remaining $k - a$ rows of length 1.

This is an easy consequence of the Murnaghan–Nakayama formula, see [21].

In this case we have $\chi_\lambda(\sigma_0) = (-1)^{k-a}$. We thus need to make explicit the integers $s_\lambda(d), \chi_\lambda(1)$ for such a hook partition.

For $\lambda = (a, 1^{k-a})$, we get that the boxes are

$$u = (1, j),\ j = 1, \dots, a,\ c_u = j - 1,\ h_u = \begin{cases} k & \text{if} \quad j = 1 \\ a - j + 1 & \text{if} \quad j \neq 1 \end{cases}$$

$$u = (i + 1, 1),\ i = 1, \dots, k - a, \quad c_u = -i,\ h_u = k - a - i + 1.$$

$$\prod_u h_u = k \prod_{j=2}^{a} (a - j + 1) \prod_{i=1}^{k-a} (k - a - i + 1) = k(a - 1)!(k - a)!.$$

Example 2. $a = 8,\ k = 11,\ (8, 1^3) \vdash 11$ in coordinates

1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8
2,1							
3,1							
4,1							

Hooks and content:

11	7	6	5	4	3	2	1
3							
2							
1							

,

0	1	2	3	4	5	6	7
-1							
-2							
-3							

Thus we finally have, substituting in Formula (1.23), that

$$Wg(\sigma_0, d) = \sum_{a=1}^{k} (-1)^{k-a} \frac{1}{k(a-1)!(k-a)!} \prod_{i=1-a}^{k-a} (d - i)^{-1}$$

$$= \sum_{a=1}^{k} (-1)^{k-a} \frac{\prod_{i=k-a+1}^{k-1} (d - i) \prod_{i=-k+1}^{-a} (d - i)}{k(a-1)!(k-a)!} \prod_{-k+1 \leq j \leq k-1} (d - j)^{-1}. \quad (1.26)$$

One needs to show that

$$\sum_{a=1}^{k}(-1)^a\frac{\prod_{i=k-a+1}^{k-1}(d-i)\prod_{i=-k+1}^{-a}(d-i)}{k(a-1)!(k-a)!}=$$
$$\frac{\sum_{a=1}^{k}(-1)^a\prod_{i=k-a+1}^{k-1}i(d-i)\prod_{i=a}^{k-1}i(d+i)}{k!(k-1)!}.$$
$$=P_k(d):=\frac{1}{k!}\sum_{b=0}^{k-1}(-1)^{b+1}\binom{k-1}{b}\prod_{i=k-b}^{k-1}(d-i)\prod_{i=b+1}^{k-1}(d+i)=(-1)^{k-1}\mathtt{C}_{k-1}. \tag{1.27}$$

By partial fraction decomposition we have that

$$\prod_{i=1-a}^{k-a}(d-i)^{-1}=\sum_{i=1-a}^{k-a}\frac{b_j}{d-j},$$

$$b_0=\prod_{i=1-a,\ i\neq 0}^{k-a}(-i)^{-1}=[(-1)^{k-a}(a-1)!(k-a)!]^{-1}.$$

Therefore the partial fraction decomposition of $Wg(\sigma_0,d)$, from Formula (1.26), is

$$\sum_{a=1}^{k}\frac{1}{k[(a-1)!(k-a)!]^2}\frac{1}{d}+\sum_{-k+1\leq j\leq k-1,\ j\neq 0}\frac{c_j}{d-j}.$$

On the other hand the partial fraction decomposition of the product of Formula (1.26),

$$\prod_{-k+1\leq j\leq k-1}(d-j)^{-1}=\frac{(-1)^{k-1}}{(k-1)!^2}\frac{1}{d}+\sum_{-k+1\leq j\leq k-1,\ j\neq 0}\frac{e_j}{d-j}.$$

It follows that the polynomial $P_k(d)$ of Formula (1.27) is a constant C with

$$C\frac{(-1)^{k-1}}{(k-1)!^2}=\sum_{a=1}^{k}\frac{1}{k[(a-1)!(k-a)!]^2}\implies C=(-1)^{k-1}\sum_{a=1}^{k}\frac{(k-1)!^2}{k[(a-1)!(k-a)!]^2}.$$

So finally we need to observe that

$$\sum_{a=1}^{k}\frac{(k-1)!^2}{k[(a-1)!(k-a)!]^2}=\frac{1}{k}\sum_{a=0}^{k-1}\binom{k-1}{a}^2=\frac{1}{k}\binom{2k-2}{k-1}=\mathtt{C}_{k-1}.$$

In fact

$$\sum_{a=0}^{n} \binom{n}{a}^2 = \binom{2n}{n}$$

as one can see simply noticing that a subset of n elements in $1, 2, \ldots, 2n$ distributes into a numbers $\leq n$ and the remaining $n - a$ which are $> n$.

$\boxed{QED}$

1.3.2 A Theorem of Collins, [3] Theorem 2.2

For a partition $\mu \vdash k$ we have defined, in Formula (1.13) $C_\mu := \sum_{\sigma|\pi(\sigma)=\mu} \sigma$. Clearly we have for a sequence of partitions $\mu_1, \mu_2, \ldots, \mu_i$

$$C_{\mu_1} C_{\mu_2} \ldots C_{\mu_i} = \sum_{\mu \vdash k} A[\mu; \mu_1, \mu_2, \ldots, \mu_i] C_\mu \tag{1.28}$$

where $A[\mu; \mu_1, \mu_2, \ldots, \mu_i] \in \mathbb{N}$ counts the number of times that a product of i permutations $\sigma_1, \sigma_2, \ldots, \sigma_i$ of types $\mu_1, \mu_2, \ldots, \mu_i$ give a permutation σ of type μ. These numbers are classically called *connection coefficients*.

Remark 2. Notice that this number depends only on μ and not on σ.

Set, for $i, h \in \mathbb{N}$:

$$A[\mu, i, h] := \sum_{\substack{\mu_1, \mu_2, \ldots, \mu_i | \mu_j \neq 1^k \\ \sum_{j=1}^{i} (k - ht(\mu_j)) = h}} A[\mu; \mu_1, \mu_2, \ldots, \mu_i] \tag{1.29}$$

$$A[\mu, h] := \sum_{i=1}^{h} (-1)^i A[\mu, i, h].$$

Remark 3. For a permutation $\sigma \in S_k$ with $\pi(\sigma) = \mu$ we will write

$$|\sigma| = |\mu| := k - ht(\mu). \tag{1.30}$$

This is the minimum number of transpositions with product σ (see for this Proposition 4). A minimal product of transpositions will also be called *reduced*.

We have $|\sigma\tau| \leq |\sigma| + |\tau|$, see Stanley [26] p.446 for a poset interpretation.

From Formula (1.23) we know that each $Wg(\sigma, d)$ is a rational function of d with poles in $0, \pm 1, \pm 2, \ldots, \pm(k-1)$ of order $< k$, so we can expand it in a power series in d^{-1} converging for $d > k - 1$ as in Formula (1.31):

Theorem 3 ([3] Theorem 2.2). *We have an expansion for* $(\sum_{\rho\in S_k} d^{c(\rho)}\rho)^{-1}$ *as power series in* d^{-1}*:*

$$= d^{-k}(1+\sum_{\mu\vdash k}\left(\sum_{h=|\mu|}^{\infty} d^{-h}A[\mu,h]\right)C_\mu) \tag{1.31}$$

Proof. Recall that we denote by $|\mu| := k - ht(\mu)$, (1.30).

$$P = \sum_{\rho\in S_k} d^{c(\rho)}\rho = d^k(1+\sum_{\mu\vdash k|\mu\neq 1^k} d^{-(k-ht(\mu))}C_\mu) = d^k(1+\sum_{\mu\vdash k|\mu\neq 1^k} d^{-|\mu|}C_\mu)$$

$$\text{so}\quad P^{-1} = d^{-k}(1+\sum_{i=1}^{\infty}(-1)^i(\sum_{\mu\vdash k|\mu\neq 1^k} d^{-|\mu|}C_\mu)^i)$$

$$= d^{-k}(1+\sum_{i=1}^{\infty}(-1)^i(\sum_{\mu_1,\mu_2,\dots,\mu_i|\mu_j\neq 1^k} d^{-\sum_{j=1}^{i}|\mu_j|}C_{\mu_1}C_{\mu_2}\dots C_{\mu_i})$$

$$= d^{-k}(1+\sum_{\mu\vdash k}(\sum_{i=1}^{\infty}(-1)^i\sum_{\mu_1,\mu_2,\dots,\mu_i|\mu_j\neq 1^k} d^{-\sum_{j=1}^{i}|\mu_j|}A[\mu;\mu_1,\mu_2,\dots,\mu_i])C_\mu)$$

$$= d^{-k}(1+\sum_{\mu\vdash k}\left(\sum_{h=|\mu|}^{\infty} d^{-h}A[\mu,h]\right)C_\mu)$$

since $\mu_1+\mu_2+\dots+\mu_i=\mu$ implies $|\mu|\leq\sum_{j=1}^{i}|\mu_j|$. QED

Remark 4. We want to see now that the series $\sum_{h=|\mu|}^{\infty} d^{-h}A[\mu,h]$ starts with $h=|\mu|$, i.e. $A[\mu,|\mu|]\neq 0$. Thus we compute the leading coefficient $A[\mu,|\mu|]$ which gives the asymptotic behaviour of $Wg(\sigma,d)$.

Let us denote by

$$C[\mu] := A[\mu,|\mu|] \implies \lim_{d\to\infty} d^{k+|\sigma|}Wg(\sigma,d) = C[\mu]. \tag{1.32}$$

From Formula (1.24) we have $C[(k)] = (-1)^{k-1}\mathtt{C}_{k-1}$ (Catalan number) and a further and more difficult Theorem of Collins states

Theorem 4. *[[3] Theorem 2.12 (ii)]* [2]

$$C[(k)] = (-1)^{k-1}\mathtt{C}_{k-1}, \qquad C[(a_1,a_2,\dots,a_i)] = \prod_{j=1}^{i} C[(a_j)]. \tag{1.33}$$

[2] I have made a considerable effort trying to understand, and hence verify, the proof of this Theorem in [3], to no avail. To me it looks not correct. Fortunately there is a proof in [16], I will show presently a simple natural proof.

Fixing $\sigma \in S_k$ with $\pi(\sigma) = \mu$ we have that $A[\mu; \mu_1, \mu_2, \dots, \mu_i]$ is also the number of sequences of permutations $\sigma_j,\ \pi(\sigma_j) = \mu_j$ with $\sigma = \sigma_1\sigma_2 \dots \sigma_i$.

So we shall also use the notation, for $\pi(\sigma) = \mu$:

$$A[\sigma; \mu_1, \mu_2, \dots, \mu_i] = A[\mu; \mu_1, \mu_2, \dots, \mu_i], \quad C[\sigma] := A[\sigma, |\sigma|].$$

Thus

$$C[\mu] = A[\mu, |\mu|] = \sum_{i=1} (-1)^i \sum_{\substack{\mu_1,\mu_2,\dots,\mu_i | \mu_j \neq 1^k \\ \sum_{j=1}^i |\mu_j| = |\mu|}} A[\mu; \mu_1, \mu_2, \dots, \mu_i] \tag{1.34}$$

We call a coefficient $A[\mu; \mu_1, \mu_2, \dots, \mu_i]$ with $\mu_1, \mu_2, \dots, \mu_i \mid \mu_j \neq 1^k$, and $\sum_{j=1}^i |\mu_j| = |\mu|$ a *top coefficient.*

1.3.3 Top coefficients and a degeneration of $\mathbb{Q}[S_k]$

The study of $C[\mu]$ can be formulated in terms of a *degeneration:* $\mathbb{Q}[\tilde{S}_k]$ of the multiplication in the group algebra whose elements now denote by $\tilde{\sigma}$.

Define a new (still associative) multiplication on $\mathbb{Q}[S_k][q]$, q a commuting variable by

$$\mathbb{Q}[\tilde{S}_k] := \oplus_{\sigma \in S_k} \mathbb{Q}\tilde{\sigma}, \quad \tilde{\sigma}_1\tilde{\sigma}_2 := q^{|\sigma_1|+|\sigma_2|-|\sigma_1\sigma_2|}\widetilde{\sigma_1\sigma_2}. \tag{1.35}$$

$$(\tilde{\sigma}_1\tilde{\sigma}_2)\tilde{\sigma}_3 = q^{|\sigma_1|+|\sigma_2|-|\sigma_1\sigma_2|} q^{|\sigma_1\sigma_2|+|\sigma_3|-|\sigma_1\sigma_2\sigma_3|}\widetilde{\sigma_1\sigma_2\sigma_3}$$
$$= q^{|\sigma_1|+|\sigma_2|+|\sigma_3|-|\sigma_1\sigma_2\sigma_3|}\widetilde{\sigma_1\sigma_2\sigma_3} = \tilde{\sigma}_1(\tilde{\sigma}_2\tilde{\sigma}_3), \quad \text{associativity.}$$

When $q = 1$ we recover the group algebra and when $q = 0$ we have

$$\mathbb{Q}[\tilde{S}_k] := \oplus_{\sigma \in S_k} \mathbb{Q}\tilde{\sigma}, \quad \tilde{\sigma}_1\tilde{\sigma}_2 := \begin{cases} \widetilde{\sigma_1\sigma_2} \text{ if } |\sigma_1\sigma_2| = |\sigma_1| + |\sigma_2| \\ 0 \quad \text{otherwise} \end{cases}. \tag{1.36}$$

Notice that, since S_k is generated by transpositions and $\tilde{\tau}^2 = q^2$ for a transposition, we have the algebra $\mathbb{Q}[S_k][q^2]$.

Further the product is compatible with the inclusions $S_k \subset S_{k+1} \subset \dots$ so it defines an algebra on $\mathbb{Q}[\mathcal{S}][q^2]$ where $\mathcal{S} = \cup_k S_k$.

Contrary to the semisimple algebra $\mathbb{Q}[S_k]$ the algebra $\mathbb{Q}[\tilde{S}_k]$ is a graded algebra, with $\mathbb{Q}[\tilde{S}_k]_h = \oplus_{\sigma \in S_k \mid |\sigma| = h} \mathbb{Q}\tilde{\sigma}$ and has

$$I := \oplus_{\sigma \in S_k | \sigma \neq 1} \mathbb{Q}\tilde{\sigma} = \oplus_{h=1}^{k-1} \mathbb{Q}[\tilde{S}_k]_h$$

as a nilpotent ideal, $I^k = 0$, its nilpotent radical. Observe that

$$|\sigma_1\sigma_2| = |\sigma_1| + |\sigma_2| \iff c(\sigma_1\sigma_2) = c(\sigma_1) + c(\sigma_2) - k$$

so if $c(\sigma_1) + c(\sigma_2) \leq k$ we know a priori that the product $\tilde{\sigma}_1\tilde{\sigma}_2 = 0$.

In this algebra the multiplication of two elements $\tilde{C}_{\mu_1}, \tilde{C}_{\mu_2}$ associated to conjugacy classes as in (1.13) involves only the top coefficients and is:

$$\tilde{C}_{\mu_1}\tilde{C}_{\mu_2} = \sum_{|\mu|=|\mu_1|+|\mu_2|} A[\mu;\mu_1,\mu_2]\tilde{C}_\mu. \tag{1.37}$$

We then have

$$(\sum_{\rho\in S_k} d^{c(\rho)}\tilde{\rho})^{-1} = d^{-k}(1 + \sum_{\mu\vdash k|\mu\neq 1^k} d^{-|\mu|}\tilde{C}_\mu)^{-1} = d^{-k}(1 + \sum_{\mu\vdash k} d^{-|\mu|}C[\mu]\tilde{C}_\mu)$$

$$= d^{-k}(1 + \sum_{h=1}^{k-1} d^{-h}(\sum_{\mu\vdash k||\mu|=h} C[\mu]\tilde{C}_\mu)). \tag{1.38}$$

Notice that if $h = k-1$ the only partition μ with $|\mu| = k-1$ is $\mu = (k)$ the partition of the full cycle.

Hence in Formula (1.38) the lowest term is $d^{-2k+1}C[(k)]\tilde{C}_{(k)}$.

An example,which the reader can skip, the connection coefficients for S_4, in box the top ones (write the elements C_μ with lowercase):

	$c_{1,1,2}$	$c_{1,3}$	$c_{2,2}$	c_4
$c_{1,1,2}$	$6c_{1,1,1,1} + \boxed{3c_{1,3} + 2c_{2,2}}$	$4c_{1,1,2} + \boxed{4c_4}$	$c_{1,1,2} + \boxed{2c_4}$	$3c_{1,3} + 4c_{2,2}$
$c_{1,3}$	$4c_{1,1,2} + \boxed{4c_4}$	$8c_{1,1,1,1} + 4c_{1,3} + 8c_{2,2}$	$3c_{1,3}$	$4c_{1,1,2} + 4c_4$
$c_{2,2}$	$c_{1,1,2} + \boxed{2c_4}$	$3c_{1,3}$	$3c_{1,1,1,1} + 2c_{2,2}$	$2c_{1,1,2} + c_4$
c_4	$3c_{1,3} + 4c_{2,2}$	$4c_{1,1,2} + 4c_4$	$2c_{1,1,2} + c_4$	$6c_{1,1,1,1} + 3c_{1,3} + 2c_{2,2}$

Setting $a = c_{1,1,2},\ b = c_{1,3},\ c = c_{2,2},\ d = c_4$ compute Formula (1.38)

$$a^2 = 3b + 2c,\ ab = 4d,\ ac = 2d$$
$$P = 1 + T,\ T = x^{-1}a + x^{-2}(b+c) + x^{-3}d,\ (1+T)^{-1} = 1 - T + T^2 - T^3$$
$$T^2 = x^{-2}a^2 + 2x^{-3}a(b+c) = x^{-2}(3b+2c) + x^{-3}12d,$$
$$T^3 = x^{-3}a(3b+2c) = x^{-3}(12+4)d = x^{-3}16d$$
$$-T + T^2 - T^3 = -x^{-1}a - x^{-2}(b+c) - x^{-3}d + x^{-2}(3b+2c) + x^{-3}12d - x^{-3}16d$$
$$= -x^{-1}a + x^{-2}(2b+c) - x^{-3}5d$$

The conjugacy classes and their cardinality in S_5:

$$\begin{pmatrix}1, c_{1,1,1,1,1} & 10, c_{1,1,1,2} & 20, c_{1,1,3} & 15, c_{1,2,2} & 30, c_{1,4} & 20, c_{2,3} & 24, c_5\end{pmatrix}$$

Here is a table of the top connection coefficients for S_5. The numbers to the right are the degrees $|\mu|$:

$$a = c_{1,1,1,2}, 1 \quad b = c_{1,1,3}, 2 \quad c = c_{1,2,2}, 2 \quad d = c_{1,4}, 3 \quad e = c_{2,3}, 3 \quad f = c_5, 4$$

	$c_{1,1,1,2}$	$c_{1,1,3}$	$c_{1,2,2}$	$c_{1,4}$	$c_{2,3}$	c_5
$c_{1,1,1,2}$	$3c_{1,1,3} + 2c_{1,2,2}$	$4c_{1,4} + c_{2.3}$	$2c_{1,4} + 3c_{2.3}$	$5c_5$	$5c_5$	0
$c_{1,1,3}$	$4c_{1,4} + c_{2.3}$	$5c_5$	$5c_5$	0	0	0
$c_{1,2,2}$	$2c_{1,4} + 3c_{2.3}$	$5c_5$	$5c_5$	0	0	0
$c_{1,4}$	$5c_5$	0	0	0	0	0
$c_{2,3}$	$5c_5$	0	0	0	0	0
c_5	0	0	0	0	0	0

Compute Formula (1.38)

$$a^2 = 3b + 2c,\ ab = 4d + e,\ ac = 2d + 3e,\ ad = 5f,\ ae = 5f,$$

$$b^2 = 5f,\ bc = 5f,\ c^2 = 5f,$$

$$1 + T,\ T = x^{-1}a + x^{-2}(b + c) + x^{-3}(d + e) + x^{-4}f$$

$$T^2 = x^{-2}a^2 + x^{-4}(b + c)^2 + 2x^{-3}a(b + c) + 2x^{-4}a(d + e)$$

$$= x^{-2}(3b + 2c) + 2x^{-3}(6d + 4e) + 40x^{-4}f$$

$$T^3 = x^{-3}a(3b + 2c) + 2x^{-4}a(6d + 4e) + x^{-4}(b + c)(3b + 2c)$$

$$= x^{-3}(12d + 3e + 4d + 6e) + x^{-4}(100 + 15 + 10 + 15 + 10)f$$

$$= x^{-3}(16d + 9e) + x^{-4}150f$$

$$T^4 = x^{-4}a(16d + 9e) = x^{-4}(16 \cdot 5 + 45)f = x^{-4}125f$$

$$125 - 150 + 40 - 1 = 14$$

$\mathtt{C}_i$=Catalan(i): 1, 2, 5, 14, 42,... Catalan(4)=14.

$$-T + T^2 - T^3 + T^4 =$$

$$-(x^{-1}a+x^{-2}(b+c)+x^{-3}(d+e))+x^{-2}(3b+2c)+2x^{-3}(6d+4e)-x^{-3}(16d+9e)+14f$$

$$= -x^{-1}a-x^{-2}(b+c)-x^{-3}(d+e)+x^{-2}(3b+2c)+2x^{-3}(6d+4e)-x^{-3}(16d+9e)$$

$$= -x^{-1}a + x^{-2}(3b + 2c - b - c) + x^{-3}(12d + 8e - 16d - 9e - d - e)$$

$$= -x^{-1}a + x^{-2}(2b + c) + x^{-3}(-5d - 2e) + 14f.$$

1.3.4 Young subgroups

Let $\Pi := \{A_1, A_2, \dots, A_j\},\ |A_i| = a_i$ be a decomposition of the set $[1, 2, \dots, k]$:

$$\text{i.e.} \quad A_1 \cup A_2 \cup \ldots \cup A_j = [1, 2, \dots, k],\ A_i \cap A_j = \emptyset,\ \forall i \neq j.$$

Definition 3. (1) The subgroup of S_k fixing this decomposition is the product $\prod_{i=1}^{j} S_{A_i} = \prod_{i=1}^{j} S_{a_i}$ of the symmetric groups S_{a_i}. It is usually called a *Young subgroup* and will be denoted by Y_Π.

(2) Given two decompositions of $[1, 2, \dots, k]$, $\Pi_1 := \{A_1, A_2, \dots, A_j\}$, and $\Pi_2 := \{B_1, B_2, \dots, B_h\}$ we say that $\Pi_1 \leq \Pi_2$ if each set A_i is contained in one of the sets B_d. This is equivalent to the condition $Y_{\Pi_1} \subset Y_{\Pi_2}$.

(3) In particular, if $\sigma \in S_k$ we denote by Π_σ the decomposition of $[1, 2, \dots, k]$ induced by its cycles and denote $Y_\sigma := Y_{\Pi_\sigma}$.

Remark 5. Observe that $\tau \in Y_\Pi$ if and only if $\Pi_\tau \leq \Pi$. The conjugacy classes of Y_Π are the products of the conjugacy classes in the blocks A_i.

Then we have for the group algebra and $\tau = (\tau_1, \tau_2, \dots, \tau_j) \in Y_\Pi$:

$$\mathbb{Q}[Y_\Pi] = \otimes_{i=1}^{j} \mathbb{Q}[S_{a_i}] \subset \mathbb{Q}[S_k], \quad (\tau_1, \tau_2, \dots, \tau_j) = \tau_1 \otimes \tau_2 \otimes \ldots \otimes \tau_j. \tag{1.39}$$

We denote by c_τ the sum of the elements of the conjugacy class of τ in Y_Π in order to distinguish it from C_τ the sum over the conjugacy class in S_k. We have:

$$\tau = (\tau_1, \tau_2, \dots, \tau_j) \in Y_\Pi,\ \mathsf{c}_\tau \stackrel{(1.13)}{=} C_{\tau_1} \otimes C_{\tau_2} \otimes \ldots \otimes C_{\tau_j}. \tag{1.40}$$

The first remark is:

Remark 6. If $\tau = (\tau_1, \tau_2, \dots, \tau_j) \in Y_\Pi$ then for the number $c(\tau)$ of cycles of τ we have

$$c(\tau) = c(\tau_1) + c(\tau_2) + \dots + c(\tau_j),$$

$$\implies |\tau| = \sum_i a_i - c(\tau) = \sum_i (a_i - c(\tau_i)) = |\tau_1| + |\tau_2| + \dots + |\tau_j|. \tag{1.41}$$

As a consequence if $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_j), \tau = (\tau_1, \tau_2, \dots, \tau_j) \in Y_\Pi$ we have

$$|\gamma\tau| = |\gamma| + |\tau| \iff |\gamma_i \tau_i| = |\gamma_i| + |\tau_i|,\ \forall i. \tag{1.42}$$

If we then consider the associated discrete algebras, From Formulas (1.42) and (1.39) we deduce an analogous of Formula (1.39) for the discrete algebras:

$$\mathbb{Q}[\tilde{Y}_\Pi] = \otimes_{i=1}^{j} \mathbb{Q}[\tilde{S}_{a_i}] \subset \mathbb{Q}[\tilde{S}_k], \quad \tau = (\tau_1, \tau_2, \dots, \tau_j), \; \tilde{\tau} = \tilde{\tau}_1 \otimes \tilde{\tau}_2 \otimes \cdots \otimes \tilde{\tau}_j. \tag{1.43}$$

Formula (1.41) tells us that $\mathbb{Q}[\tilde{Y}_\Pi] = \otimes_{i=1}^{j} \mathbb{Q}[\tilde{S}_{a_i}]$ as graded tensor product and the inclusion in $\mathbb{Q}[\tilde{S}_k]$ preserves the degrees.

1.3.5 A proof of Theorem 4

In particular let $\sigma \in S_k$ and $\sigma = c_1 c_2 \dots c_j$ its cycle decomposition.

Let A_i be the support of the cycle c_i of σ and a_i its cardinality, so that $\Pi_\sigma = \{A_1, \dots, A_j\}$ and $Y_\sigma = Y_{\Pi_\sigma}$. We have $\sigma \in Y_\sigma$ and its conjugacy class in Y_σ is the product of the conjugacy classes of the cycles $(a_i) \subset S_{a_i}$, (1.13). We denote, as before, by c_σ the sum of the elements of this conjugacy class.

We have now a very simple but crucial fact;

Proposition 4. *(1) Let $(i, i_1, \dots, i_a)$, $(j, j_1, \dots, j_b)$ be two disjoint cycles, $a, b \geq 0$, and take the transposition (i, j) then:*

$$(i, i_1, \dots, i_a)(j, j_1, \dots, j_b)(i, j) = (i, j_1, \dots, j_b, j, i_1, \dots, i_a) \tag{1.44}$$

$$(i, j)(i, i_1, \dots, i_a)(j, j_1, \dots, j_b) = (j, j_1, \dots, j_b, i, i_1, \dots, i_a) \tag{1.45}$$

(2) Let $\sigma \in S_k$ and $\tau = (i, j)$ a transposition. Then $|\sigma\tau| = |\tau\sigma| = |\sigma| \pm 1$ and $|\sigma\tau| = |\tau\sigma| = |\sigma| - 1$ if and only if the two indices i, j both belong to one of the sets of the partition of σ, i.e. $\tau = (i, j) \in Y_\sigma$.

Proof. 1) is clear and 2) follows immediately from 1). In fact either i, j belong to the same cycle of σ and then in $\sigma\tau$ this cycle is split into two and $c(\sigma\tau) = c(\sigma) + 1$ or i, j belong to two different cycles of σ which are joined in $\sigma\tau$ and $c(\sigma\tau) = c(\sigma) - 1$.

Notice that, if $|\sigma\tau| = |\tau\sigma| = |\sigma| - 1$, $\Pi_{\sigma\tau} < \Pi_\sigma$ and is obtained from Π_σ by replacing the support of the cycle in which i, j appear with two subsets support of the 2 cycles in which this splits. Similarly for $\Pi_{\tau\sigma}$. QED

From this we deduce the essential result of this section:

Corollary 3. *Let $\sigma \in S_k$. Consider a decomposition $\sigma = \sigma_1 \sigma_2 \dots, \sigma_h$, $\sigma_i \in S_k$, $\sigma_i \neq 1, \forall i$ with $|\sigma| = |\sigma_1| + |\sigma_2| + \ldots + |\sigma_h|$. Then for all i we have $\sigma_i \in Y_{\Pi_\sigma} = Y_\sigma$ (Definition 3).*

Proof. By induction on h, if $h = 1$ there is nothing to prove.

If $\sigma_1 = (i, j)$ is a transposition $|\sigma_1| = 1$, then the the claim follows by induction on $\sigma_1\sigma = \bar{\sigma} = \sigma_2 \ldots, \sigma_h$, since $|\sigma_1\sigma| = |\sigma| - 1$ and Proposition 4.

If $|\sigma_1| > 1$ we split $\sigma_1 = \tau\bar{\sigma}_1$ with $|\bar{\sigma}_1| = |\sigma_1| - 1$ and τ a transposition and we are reduced to the previous case. QED

We are now ready to prove the Theorem of Collins, Formula (1.33).

Let $\sigma \in S_k$ and $\sigma = c_1 c_2 \ldots c_j$ its cycle decomposition. Let A_i be the support of the cycle c_i and a_i its cardinality, so that $\Pi_\sigma = \{A_1, \ldots, A_j\}$.

By the previous Corollary 3 and Remark 2 the contribution to σ in the terms of Formula (1.29) are all in the subgroup Y_σ so that finally

$$\boxed{C[\sigma] = C[\tilde{\sigma}]} \quad \text{with } C[\tilde{\sigma}] \text{ computed in } \mathbb{Q}[\tilde{Y}_\sigma].$$

In order to compute $C[\tilde{\sigma}]$ we observe that $d^{-k-|\tilde{\sigma}|}C[\tilde{\sigma}]\mathsf{c}_{\tilde{\sigma}} = d^{-k-|\sigma|}C[\tilde{\sigma}]\mathsf{c}_{\tilde{\sigma}}$ is the lowest term in d^{-1} in

$$(\sum_{\rho \in Y_\sigma} d^{c(\rho)}\tilde{\rho})^{-1} = \bigotimes_{i=1}^{j}(\sum_{\rho \in S_{a_i}} d^{c(\rho)}\tilde{\rho})^{-1}. \tag{1.46}$$

From Formula (1.38) applied to the various full cycles $c_i \in S_{a_i}$ we have that the lowest term in $(\sum_{\rho \in S_{a_i}} d^{c(\rho)}\tilde{\rho})^{-1}$ is $d^{-2a_i+1}C[(a_i)]C_{(a_i)}$ so that we have finally that the lowest term in Formula (1.46) is

$$d^{-k-|\sigma|}C[\tilde{\sigma}]\mathsf{c}_{\tilde{\sigma}} \overset{(1.40)}{=} \prod_{i=1}^{j} d^{-2a_i+1}C[(a_i)]C_{(a_1)} \otimes \ldots \otimes C_{(a_j)},$$

$$\implies C[\sigma] = C[\tilde{\sigma}] = \prod_{i=1}^{j} C[(a_i)] \overset{(1.24)}{=} \prod_{i=1}^{j}(-1)^{a_i-1}\mathsf{C}_{a_i-1}. \tag{1.47}$$

We have proved, Formula (1.24) that $(-1)^{a_i-1}C[(a_i)]$ is the Catalan number C_{a_i-1} and the proof of Theorem 4 is complete. QED

1.3.6 A table

The case $k = d$ is of special interest, see §2.3. We write $Wg(d, \mu) = a_\mu$ so that $\sum_{\mu \vdash d} Wg(d, \mu)c_\mu = \sum_\mu a_\mu c_\mu$ in Formula (1.21).

A computation using Mathematica gives $d \leq 8$ the list $d!^2 \sum_{\mu \vdash d} a_\mu c_\mu$:

$$\frac{4}{3}c_{1,1} - \frac{2}{3}c_2 = \frac{1}{3}(4c_{1,1} - 2c_2)$$

$$\frac{21}{10}c_{1^3}-\frac{9}{10}c_{1,2}+\frac{3}{5}c_3=\frac{1}{10}(21c_{1^3}-9c_{1,2}+6c_3)$$

$$\frac{134}{35}c_{1^4}-\frac{48}{35}c_{1^2,2}+\frac{29}{35}c_{1,3}+\frac{22}{35}c_{2^2}-\frac{4}{7}c_4.$$

$$\frac{1}{35}(134c_{1^4}-48c_{1^2,2}+\ 29c_{1,3}+35c_{2^2}-20c_4).$$

The case $d=5$:

$$\frac{145}{18}c_{1^5}-\frac{299}{126}c_{1^3,2}+\frac{115}{126}c_{1,2^2}+\frac{80}{63}c_{1^2,3}-\frac{101}{126}c_{1,4}-\frac{37}{63}c_{2,3}+\frac{5}{9}c_5$$

$$\frac{1}{126}(1015c_{1^5}-299c_{1^3,2}+160c_{1^2,3}+115c_{1,2^2}-101c_{1,4}-74c_{2,3}+70c_5)$$

The case $d=6$:

$$\frac{10508}{539}c_{1^6}-\frac{2538}{539}c_{1^4,2}+\frac{1180}{539}c_{1^3,3}+\frac{2396}{1617}c_{1^2,2^2}-\frac{668}{539}c_{1^2,4}-\frac{459}{539}c_{1,2,3}+\frac{26}{33}c_{1,5}-\frac{338}{539}c_{2^3}$$
$$+\frac{922}{1617}c_{2,4}+\frac{300}{539}c_{3,3}-\frac{6}{11}c_6$$

$$\frac{1}{1617}(31524c_{1^6}-7614c_{1^4,2}+3540c_{1^3,3}+2396c_{1^2,2^2}-2004c_{1^2,4}$$
$$-1377c_{1,2,3}+1274c_{1,5}-1014c_{2^3}+922c_{2,4}+900c_{3,3}-882c_6)$$

The case $d=7$:

$$\frac{184849}{3432}c_{1^7}-\frac{12319}{1144}c_{1^5,2}+\frac{7385}{1716}c_{1^4,3}+\frac{9401}{3432}c_{1^3,2^2}-\frac{7369}{3432}c_{1^3,4}-\frac{196}{143}c_{1^2,2,3}+\frac{2107}{1716}c_{1^2,5}$$
$$-\frac{1087}{1144}c_{1,2^3}+\frac{259}{312}c_{1,2,4}+\frac{1379}{1716}c_{1,3^2}-\frac{223}{286}c_{1,6}+\frac{1015}{1716}c_{2^2,3}-\frac{961}{1716}c_{2,5}-\frac{85}{156}c_{3,4}+\frac{7}{13}c_7$$

The biggest denominator 3432 is also a multiple of all denominators:

$$\frac{1}{3432}(184849c_{1^7}-36957c_{1^5,2}+14770c_{1^4,3}+9401c_{1^3,2^2}-7369c_{1^3,4}$$
$$-4704c_{1^2,2,3}+4214c_{1^2,5}-3261c_{1,2^3}+2849c_{1,2,4}+2758c_{1,3^2}-2676c_{1,6}$$
$$+2030c_{2^2,3}-1922c_{2,5}-1870c_{3,4}+1848c_7) \quad (1.48)$$

The case $d = 8$:

$$\frac{3245092}{19305}c_{1^8} - \frac{546368}{19305}c_{1^7,2} + \frac{14434}{1485}c_{1^5,3} + \frac{112828}{19305}c_{1^4,2,2} - \frac{16336}{3861}c_{1^4,4} - \frac{4384}{1755}c_{1^3,2,3} + \frac{41332}{19305}c_{1^3,5} - \frac{10432}{6435}c_{1^2,2^3} + \frac{8608}{6435}c_{1^2\,2,4} + \frac{24718}{19305}c_{1^2,3^2} - \frac{2624}{2145}c_{1^2,6} + \frac{17122}{19305}c_{1,2^2,3} - \frac{1216}{1485}c_{1,2,5} - \frac{1384}{1755}c_{1,3,4} + \frac{151}{195}c_{1,7} + \frac{124}{195}c_{2^4} - \frac{11152}{19305}c_{2^2,4} - \frac{2176}{3861}c_{2,3^2} + \frac{1186}{2145}c_{2,6} + \frac{799}{1485}c_{3,5} + \frac{796}{1485}c_{4^2} - \frac{8}{15}c_8 \quad (1.49)$$

The biggest denominator 19305 is also a multiple of all denominators:

$$\frac{1}{19305}(3245092c_{1^8} - 546368c_{1^7,2} + 187642c_{1^5,3} + 112828c_{1^4,2,2} - 81680c_{1^4,4} - 48224c_{1^3,2,3} + 41332c_{1^3,5} - 31296c_{1^2,2^3} + 25824c_{1^2\,2,4} + 24718c_{1^2,3^2} - 23616c_{1^2,6} + 17122c_{1,2^2,3} - 15808c_{1,2,5} - 15224c_{1,3,4} + 14949c_{1,7} + 12276c_{2^4} - 11152c_{2^2,4} - 10880c_{2,3^2} + 10674c_{2,6} + 10387c_{3,5} + 10348c_{4^2} - 10296c_8). \quad (1.50)$$

The reader will notice certain peculiar properties of these sequences.

First $Wg(\sigma)$ is positive (resp. negative) if σ is an even (resp. odd) permutation. This is a special case of a Theorem of Novak [18], Theorem 5.

Conjecture The absolute values are strictly decreasing in the lexicographic order of partitions written in increasing order. The biggest denominator is also a multiple of all denominators.

I verified this up to $d = 14$.

1.4 The results of Jucys Murphy and Novak

These conjectures deserve further investigation, maybe the factorization of Jucys:

$$\sum_{\rho\in S_k} d^{c(\rho|}\rho = d\prod_{i=2}^{k}(d+J_i), \quad J_i = (1,i)+(2,i)+\ldots+(i-1,i),\ i = 2,\ldots,k \quad (1.51)$$

see [12] [17] and the approach of Novak [18] can be used.

Let me give a quick exposition of these results:

Proposition 5. *The elements J_i commute between each other.*

Proof. This follows easily from the following fact, if $i < j < k$ then:

$$(i,j)[(i,k)+(j,k)] = (i,j,k)+(j,i,k) = [(i,k)+(j,k)](i,j). \tag{1.52}$$

QED

As for Formula (1.51) for $k = 2$ it is clear and then it follows by induction using the simple

Lemma 1. *If $\sigma \in S_k \setminus S_{k-1}$ then $\sigma = \tau(i,k)$ where $\sigma(i) = k,\ i < k$ and $\tau \in S_{k-1},\ |\sigma| = |\tau| + 1$ (from Proposition 4 2.).*

Proof of Formula (1.51). Remark that, if $\rho \in S_{k-1}$, the number of cycles of ρ, thought of as element of S_k, is 1 more than if thought of as element of S_{k-1} so, by induction:

$$d\prod_{i=2}^{k}(d+J_i) = (\sum_{\rho\in S_{k-1}} d^{c(\rho)}\rho)(d+\sum_{i=1}^{k-1}(i,k)) =$$
$$(\sum_{\rho\in S_{k-1}\subset S_k} d^{c(\rho)}\rho) + (\sum_{\rho\in S_k\setminus S_{k-1}} d^{c(\rho)}\rho)$$
$$= \sum_{\rho\in S_k} d^{c(\rho)}\rho = \sum_{j=1}^{k} d^j C_j,\ C_j := \sum_{\rho\in S_k,\ c(\rho)=j} \rho. \tag{1.53}$$

QED

Given this Novak observes that in the Theory of symmetric functions, in the $k-1$ variables $x_2,\dots,x_k$ we have

$$d\prod_{i=2}^{k}(d+x_i) = d^k + \sum_{i=1}^{k-1} d^{k-i} e_i(x_2,\dots,x_k); \quad \prod_{i=2}^{k}(1-x_i)^{-1} = \sum_{j=0}^{\infty} h_j(x_2,\dots,x_k)$$

where the $e_i(x_2,\dots,x_k)$ are the elementary symmetric functions while the $h_j(x_2, \dots,x_k)$ are the *total symmetric functions*; that is $h_j(x_2,\dots,x_k)$ is the sum of all monomials in the variables $x_2,\dots,x_k$ of degree j. In particular

$$c(\rho) = k - |\rho| \implies e_i(J_2,\dots,J_k) = \sum_{\mu\vdash k\,|\,|\mu|=i} C_\mu.$$

Given this one has for $d \geq k$

$$(\sum_{\rho\in S_k} d^{c(\rho)}\rho)^{-1} = d^{-1}\prod_{i=2}^{k}(d+J_i)^{-1} = d^{-k}\sum_{j=0}^{\infty} h_j(-\frac{J_2}{d},\dots,-\frac{J_k}{d})$$

$$= d^{-k}\sum_{j=0}^{\infty}\frac{(-1)^j}{d^j}h_j(J_2,\dots,J_k) \tag{1.54}$$

a convergent series for $d \geq k$. This follows by remarking that setting

$$||\sum_\sigma a_\sigma\sigma||_\infty := \max|a_\sigma|,\ \ ||AJ_i||_\infty \leq (k-1)||A||_\infty$$

$$\implies ||J_i^j||_\infty \leq (k-1)^j. \tag{1.55}$$

This series in fact coincides with that given by Formula (1.31), but it is in many ways much better.

Observe that $h_j(J_2,\dots,J_k)$ is a sum of permutations all with sign $(-1)^j$. Moreover since it is a symmetric function conjugate permutations appear with the same coefficient so it is a sum of $C_\mu$for μ corresponding to permutations of sign $(-1)^j$ with non negative integer coefficients.

$$h_j(J_2,\dots,J_k) = \sum_{\mu\vdash k\,|\,\epsilon(\mu)=(-1)^j}\alpha_{j,\mu}C_\mu,\ \ \alpha_{j,\mu}\in\mathbb{N}.$$

Split Formula (1.20) as

$$\sum_{\rho\in S_k|\epsilon(\rho)=1} Wg(d,\rho)\rho = Wg(d,k)_+;\qquad \sum_{\rho\in S_k|\epsilon(\rho)=-1} Wg(d,\rho)\rho = Wg(d,k)_-$$

$$\implies Wg(d,k)_+ = d^{-k}\sum_{j=0}^{\infty}\frac{1}{d^{2j}}h_{2j}(J_2,\dots,J_k);$$

$$Wg(d,k)_- = -d^{-k}\sum_{j=0}^{\infty}\frac{1}{d^{2j+1}}h_{2j+1}(J_2,\dots,J_k). \tag{1.56}$$

Theorem 5. *[Novak [18]]* $Wg(d,\rho) > 0$ *if* $\epsilon(\rho) = 1$ *and* $Wg(d,\rho) < 0$ *if* $\epsilon(\rho) = -1$.

Proof. Let us give the argument for ρ even and $\pi(\rho) = \mu$. By Remark 4:

$$Wg(d,\rho) = Wg(d,\mu) = d^{-k}\sum_{j=0}^{\infty}\frac{1}{d^{2j}}\alpha_{2j,\mu} = d^{-k-|\mu|}\sum_{j=0}^{\infty}\frac{1}{d^{2j}}\alpha_{2j+|\mu|,\mu}$$

the series $\sum_{j=0}^{\infty}\frac{1}{d^{2j}}\alpha_{2j+|\mu|,\mu}$ has the initial term $\alpha_{|\mu|,\mu} = C[\mu]$ and all positive terms so $Wg(\mu,d) \geq d^{-k-|\mu|}C[\mu]$. QED

Inequalities Let us describe some inequalities satisfied by the function $Wg(\sigma, d)$, let us write for given k, d by $Wg(d,k) = \sum_\sigma Wg(d,\sigma)\sigma = \Phi(1)^{-1}$. From Formula (1.17) since $ht(\lambda) \leq d$ we have $r_\lambda(d) = \prod_{u\in\lambda}(d + c_u) > 0$. So P and $P^{-1} = Wg(d,k)$ are both positive symmetric operators. We start with

Proposition 6.

$$Wg(\sigma, d) = \operatorname{tr}(\sigma^{-1}Wg(d,k)^2). \tag{1.57}$$

Proof.

$$\sum_\sigma \operatorname{tr}(\sigma^{-1}Wg(d,k)^2)\sigma = \Phi(Wg(d,k)^2) = \Phi(1)\Phi(1)^{-2} = \Phi(1)^{-1}.$$

QED

Now in the space $V = \mathbb{R}^d$ consider the usual scalar product under which the basis e_i is orthonormal. Remark that in the algebra of operators $\Sigma_k(V)$ we have, for $\sigma \in S_k$ that the transpose of σ is σ^{-1}, by Formula (1.58).

$$(u_1 \otimes \cdots \otimes u_k, \sigma \circ v_1 \otimes \cdots \otimes v_k) = \prod_{i=1}^k (u_i, v_{\sigma^{-1}(i)} = \prod_{i=1}^k (\sigma(u_i), v_i). \tag{1.58}$$

Next we have that $Wg(d,k)$ and $Wg(d,k)^2$ are positive symmetric operators.

In the algebra $\Sigma_k(V)$, a sum of matrix algebras over $\mathbb{R}$, the nonnegative symmetric elements are of the form $aa^t,\ a \in \Sigma_k(V)$ so that we have

Proposition 7.

$$\operatorname{tr}(aa^tWg(d,k)^2) \geq 0,\ \forall a \in \Sigma_k(V). \tag{1.59}$$

This implies that, given any element $0 \neq \sum_{\sigma\in S_k} a_\sigma$ *setting*

$$\sum_{\sigma\in S_k} b_\sigma\sigma := (\sum_{\gamma\in S_k} a_\gamma\gamma)(\sum_{\tau\in S_k} a_\tau\tau^{-1}),\ b_\sigma = \sum_{\gamma,\tau|\gamma\tau^{-1}=\sigma} a_\gamma a_\tau$$

$$\implies \sum_{\sigma\in S_k} b_\sigma Wg(\sigma, d) > 0,$$

Example 3. $(1 \pm \sigma)(1 \pm \sigma^{-1}) = 2 \pm (\sigma + \sigma^{-1})$ gives

$$Wg(1,d) > Wg(\sigma,d) > -Wg(1,d),\ \forall\sigma \neq 1.$$

1.5 The algebra $(\bigwedge M_d^*)^G$

Preliminary to the next step we need to recall the theory of antisymmetric conjugation invariant functions on M_d. This is a classical theory over a field of characteristic 0 which one may take as $\mathbb{Q}$.

First, let U be a vector space. A polynomial $g(x_1,\dots,x_m)$ in m variables $x_i \in U$ is *antisymmetric* or *alternating* in the variables $X := \{x_1,\dots,x_m\}$ if for all permutations $\sigma \in S_m$ we have

$$g(x_{\sigma(1)},\dots,x_{\sigma(m)}) = \epsilon_\sigma g(x_1,\dots,x_m),\ \epsilon_\sigma \text{ the sign of } \sigma.$$

A simple way of forming an antisymmetric polynomial from a given one $g(x_1,\dots,x_m)$ is the process of *alternation*[3]

$$Alt_X g(x_1,\dots,x_m) := \sum_{\sigma\in S_m} \epsilon_\sigma g(x_{\sigma(1)},\dots,x_{\sigma(m)}). \tag{1.60}$$

Recall that the exterior algebra $\bigwedge U^*$, with U a vector space, can be thought of as the space of multilinear alternating functions on U. Then exterior multiplication as functions is given by the Formula:

$$f(x_1,\dots,x_h) \in \bigwedge^h U^*; \quad g(x_1,\dots,x_k) \in \bigwedge^k U^*,$$

$$f\wedge g(x_1,\dots,x_{h+k}) = \frac{1}{h!k!}\sum_{\sigma\in S_{h+k}} f(x_{\sigma(1)},\dots,x_{\sigma(h)})g(x_{\sigma(h+1)},\dots,x_{\sigma(h+k)}) \tag{1.61}$$

$$= \frac{1}{h!k!} Alt_{x_1,\dots,x_{h+k}} f(x_1,\dots,x_h)g(x_{h+1},\dots,x_{h+k}) \in \bigwedge^{h+k} U^*. \tag{1.62}$$

It is well known that:

Proposition 8. *A multilinear and antisymmetric polynomial $g(x_1,\dots,x_m)$ in m variables $x_i \in \mathbb{C}^m$ is a multiple, $a\det(x_1,\dots,x_m)$, of the determinant.*

In fact if the polynomial has integer coefficients $a \in \mathbb{Z}$.

For a multilinear and antisymmetric polynomial map $g(x_1,\dots,x_m) \in U$ to a vector space, each coordinate has the same property so

$$g(x_1,\dots,x_m) = \det(x_1,\dots,x_m)a,\ a \in U.$$

[3] we avoid on purpose multiplying by $1/m!$

We apply this to $U = M_d$. Let us identify $M_d = \mathbb{C}^{d^2}$ using the canonical basis of elementary matrices $e_{i,j}$ ordered lexicographically e.g.:

$$d = 2, \quad e_{1,1},\ e_{1,2},\ e_{2,1},\ e_{2,2}.$$

Given d^2 matrices $Y_1, \dots, Y_{d^2} \in M_d$ we may consider them as elements of $\mathbb{C}^{d^2}$ and then form the determinant $\det(Y_1, \dots, Y_{d^2})$.

By Proposition 8 the 1 dimensional space $\bigwedge^{d^2} M_d^*$ has as generator the determinant $\det(Y_1, \dots, Y_{d^2})$ which, since the conjugation action by $G := GL(d, \mathbb{Q})$ on M_d is by transformations of determinant 1, is thus an invariant under the action by G.

The theory of G invariant antisymmetric multilinear G invariant functions on M_d is well known and related to the cohomology of G.

The antisymmetric multilinear G invariant functions on M_d form the algebra $(\bigwedge M_d^*)^G$. This is a subalgebra of the exterior algebra $\bigwedge M_d^*$ and can be identified to the cohomology of the unitary group. As all such cohomology algebras it is a Hopf algebra and by Hopf's Theorem it is the exterior algebra generated by the primitive elements.

The primitive elements of $(\bigwedge M_d^*)^G$ are, see [14]:

$$T_{2i-1} = T_{2i-1}(Y_1, \dots, Y_{2i-1}) := \operatorname{tr}(St_{2i-1}(Y_1, \dots, Y_{2i-1})) \tag{1.63}$$

$$St_{2i-1}(Y_1, \dots, Y_{2i-1}) = \sum_{\sigma \in S_{2i-1}} \epsilon_\sigma Y_{\sigma(1)} \dots Y_{\sigma(2i-1)}$$

with $i = 1, \dots, d$. In particular, since these elements generate an exterior algebra we have:

Remark 7. A product of elements T_i is non zero if and only if the T_i involved are all distinct, and then it depends on the order only up to a sign.

The 2^n different products form a basis of $(\bigwedge M_d^*)^G$. The non zero product of all these elements $T_{2i-1}(Y_1, \dots, Y_{2i-1})$ is in dimension d^2. We denote

$$\mathcal{T}_d(Y_1, Y_2, \dots, Y_{d^2}) = T_1 \wedge T_3 \wedge T_5 \wedge \dots \wedge T_{2d-1}. \tag{1.64}$$

Proposition 9. *A multilinear antisymmetric function of $Y_1, \dots, Y_{d^2}$ is a multiple of $T_1 \wedge T_3 \wedge T_5 \wedge \dots \wedge T_{2d-1}$.*

Remark 8. The function $\det(Y_1, \dots, Y_{d^2})$ is an invariant of matrices so it must have an expression as in Formula (1.6). In fact up to a computable integer constant [7] this equals the exterior product of Formula (1.64).

The constant of the change of basis when we take as basis the matrix units can be computed up to a sign, see [7]:

$$\mathcal{T}_d(Y) = \mathcal{C}_d \det(Y_1, \dots, Y_{d^2}), \quad \mathcal{C}_d := \pm \frac{1!3!5! \cdots (2d-1)!}{1!2! \cdots (d-1)!}. \tag{1.65}$$

2 Comparing Formanek, [7] and Collins [3]

Rather than following the historical route we shall first discuss the paper of Collins, since this will allow us to introduce some notations useful for the discussion of Formanek's results.

2.1 The work of Collins

In the paper [3], Collins introduces the Weingarten function in the following context. He is interested in computing integrals of the form

$$\int_{U(d)} \prod_{\ell=1}^{k_1} u_{j_\ell, h_\ell} \prod_{m=1}^{k_2} \bar{u}_{i_m, p_m} du \tag{2.66}$$

where $U(d)$ is the unitary group of $d \times d$ matrices and the elements $u_{i,j}$ the entries of a matrix $X \in U(d)$ while $\bar{u}_{j,i}$ the entries of $X^{-1} = U^* = \bar{U}^t$. Here du is the normalized Haar measure. If one translates by a scalar matrix α, $|\alpha| = 1$ then the integrand is multiplied by $\alpha^{k_1} \bar{\alpha}^{k_2}$, on the other hand Haar measure is invariant under multiplication so that this integral vanishes unless we have $k_1 = k_2$. In this case the computation will be algebraic based on the following considerations.

Let us first make some general remarks. A finite dimensional representation R of a compact group G (with the dual denoted by R^*), decomposes into the direct sum of irreducible representations. In particular if R^G denotes the subspace of G invariant vectors there is a canonical G equivariant projection $E : R \to R^G$. The projection E can be written as integral

$$E(v) := \int_G g \cdot v \, dg, \quad dg \quad \text{normalized Haar measure.} \tag{2.67}$$

In turn the integral $E(v) = \int_G g \cdot v \, dg$ is defined in dual coordinates by

$$\langle \varphi \mid E(v) \rangle = \langle \varphi \mid \int_G g \cdot v \, dg \rangle := \int_G \langle \varphi \mid g \cdot v \rangle dg, \ \forall \varphi \in R^*. \tag{2.68}$$

The functions, of $g \in G$, $\langle \varphi \mid g \cdot v \rangle$, $\varphi \in R^*$, $v \in R$ are called *representative functions*; therefore an explicit formula for E is equivalent to the knowledge of integration of representative functions. In fact usually the integral is computed by some algebraic method of computation of E.

Consider the space $V = \mathbb{C}^d$ with natural basis e_i and dual basis e^j.

We take $R = End(V)$ with the conjugation action of $GL(V)$ or of its compact subgroup $U(d)$ of unitary $d \times d$ matrices:

$$Xe_{h,p}X^{-1} = \sum_{i,j} u_{i,h}\bar{u}_{j,p}e_{i,j}, \quad X = \sum_{i,j} u_{i,j}e_{i,j} \in U(d), \; X^{-1} = \sum_{i,j} \bar{u}_{j,i}e_{i,j}.$$

A basis of representative functions for $R = End(V)$ is

$$\operatorname{tr}(e_{i,j}Xe_{h,p}X^{-1}) = \operatorname{tr}(e_{i,j}\sum_{a,b} u_{a,h}\bar{u}_{b,p}e_{a,b}) = u_{j,h}\bar{u}_{i,p}, \quad i,j,h,p = 1,\dots,d. \tag{2.69}$$

Since a duality between $End(V)^{\otimes k}$ and itself is the non degenerate pairing:

$$\langle A \mid B \rangle := \operatorname{tr}(A \cdot B)$$

a basis of representative functions of $End(V)^{\otimes k}$ is formed by the products

$$\operatorname{tr}(e_{i_1,j_1} \otimes e_{i_2,j_2} \ldots \otimes e_{i_k,j_k} \cdot Xe_{h_1,p_1}X^{-1} \otimes Xe_{h_2,p_2}X^{-1} \ldots \otimes Xe_{h_k,p_k}X^{-1}) =$$

$$\operatorname{tr}\left(\mathsf{e}_{\underline{i},\underline{j}} \cdot X\mathsf{e}_{\underline{h},\underline{p}}X^{-1}\right) = \prod_{\ell=1}^{k} \operatorname{tr}(e_{i_\ell,j_\ell} \cdot Xe_{h_\ell,p_\ell}X^{-1}) = \prod_{\ell=1}^{k} u_{j_\ell,h_\ell}\bar{u}_{i_\ell,p_\ell}, \tag{2.70}$$

where in order to have compact notations we write

$$\underline{i} := (i_1, i_2, \dots, i_k), \quad \mathsf{e}_{\underline{i},\underline{j}} = e_{i_1,j_1} \otimes e_{i_2,j_2} \ldots \otimes e_{i_k,j_k}. \tag{2.71}$$

$$\mathtt{u}_{\underline{a},\underline{b}} = \prod_{\ell=1}^{k} u_{a_\ell,b_\ell}. \tag{2.72}$$

Therefore every integral in Formula (2.66) for $k_1 = k_2 = k$ is the integral of a representative function.

Of course the expression of a representative function as $\operatorname{tr}\left(\mathsf{e}_{\underline{i},\underline{j}} \cdot X\mathsf{e}_{\underline{h},\underline{p}}X^{-1}\right)$ is not unique.

Collins writes the explicit Formula (2.79) for

$$\int_{U(d)} \prod_{\ell=1}^{k} u_{j_\ell,h_\ell} \bar{u}_{i_\ell,p_\ell} du = \int_{U(d)} \mathtt{u}_{\underline{j},\underline{h}} \bar{\mathtt{u}}_{\underline{i},\underline{p}}\, du$$

$$= \int_{U(d)} \operatorname{tr}\left(\mathsf{e}_{\underline{i},\underline{j}} \cdot X \mathsf{e}_{\underline{h},\underline{p}} X^{-1}\right) dX = \operatorname{tr}\left(\mathsf{e}_{\underline{i},\underline{j}} \cdot E(\mathsf{e}_{\underline{h},\underline{p}})\right) \tag{2.73}$$

In order to do this, it is enough to have an explicit formula for the equivariant projection E of $End(V)^{\otimes k}$ to the $GL(V)$ (or $U(d)$) invariants $\Sigma_k(V)$, the algebra generated by the permutation operators $\sigma \in S_k$ acting on $V^{\otimes k}$.

His idea is to consider first the map

$$\Phi : End(V)^{\otimes k} \to \Sigma_k(V), \quad \Phi(A) := \sum_{\sigma} \operatorname{tr}(A \circ \sigma^{-1})\sigma. \tag{2.74}$$

This map is a $GL(V)$ equivariant map to $\Sigma_k(V)$, but it is not a projection. In fact restricted to $\Sigma_k(V)$, we have

$$\Phi : \Sigma_k(V) \to \Sigma_k(V), \quad \Phi(\tau) := \sum_{\sigma \in S_k} \operatorname{tr}(\tau \circ \sigma^{-1})\sigma.$$

Setting $\quad \sigma = \gamma\tau, \quad \tau\sigma^{-1} = \gamma^{-1}$ we have:

$$\Phi(\tau) = \sum_{\gamma \in S_k} \operatorname{tr}(\gamma^{-1})\gamma\,\tau = \Phi(1)\tau = \tau\Phi(1) = \tau \sum_{\gamma \in S_k} \operatorname{tr}(\gamma^{-1})\gamma. \tag{2.75}$$

We have seen, in Corollary 2, that

$$\Phi(1) = \sum_{\gamma \in S_k} \operatorname{tr}(\gamma^{-1})\gamma = \sum_{\gamma \in S_k} d^{c(\gamma)}\gamma$$

is a central invertible element of $\Sigma_k(V)$. So the equivariant projection E is Φ composed with multiplication by the inverse $Wg(d,k)$ of the element $\Phi(1) = \sum_{\gamma \in S_k} \operatorname{tr}(\gamma^{-1})\gamma$ given by Formula (1.23) or (1.19).

$$E = (\sum_{\gamma \in S_k} \operatorname{tr}(\gamma^{-1})\gamma)^{-1} \circ \Phi = \Phi(1)^{-1} \circ \Phi = Wg(d,k) \circ \Phi. \tag{2.76}$$

Of course

$$\Phi(\mathsf{e}_{\underline{h},\underline{p}}) = \sum_{\sigma} \operatorname{tr}(\mathsf{e}_{\underline{h},\underline{p}} \circ \sigma^{-1})\sigma$$

$$\Longrightarrow E(\mathsf{e}_{\underline{h},\underline{p}}) = \sum_{\gamma\in S_k} Wg(d,\gamma)\gamma \sum_{\sigma} \operatorname{tr}(\mathsf{e}_{\underline{h},\underline{p}} \circ \sigma^{-1})\sigma$$

and Formula (2.73) becomes

$$\operatorname{tr}(\mathsf{e}_{\underline{i},\underline{j}} \circ \sum_{\gamma\in S_k} Wg(d,\gamma)\gamma \sum_{\sigma} \operatorname{tr}(\mathsf{e}_{\underline{h},\underline{p}} \circ \sigma^{-1})\sigma) \tag{2.77}$$

$$= \sum_{\gamma,\sigma\in S_k} \operatorname{tr}(\mathsf{e}_{\underline{i},\underline{j}} \circ \gamma) \operatorname{tr}(\mathsf{e}_{\underline{h},\underline{p}} \circ \sigma^{-1}) Wg(d,\gamma\sigma^{-1}) \tag{2.78}$$

From Formulas (1.7) and (1.8) since $e_{i,j} = e_i \otimes e^j$ we have

$$\operatorname{tr}(e_{i_1,j_1} \otimes e_{i_2,j_2} \ldots \otimes e_{i_k,j_k} \circ \gamma) = \prod_h \langle e_{i_{\gamma(h)}} \mid e^{j_h}\rangle = \prod_h \delta^{j_h}_{i_{\gamma(h)}}$$

$$(2.78) = \sum_{\gamma,\sigma\in S_k} \prod_{\ell} \delta^{j_\ell}_{i_{\gamma(\ell)}} \prod_{\ell} \delta^{p_{\sigma(\ell)}}_{h_\ell} Wg(d,\gamma\sigma^{-1})$$

$$\Longrightarrow \int_{U(d)} \mathsf{u}_{\underline{j},\underline{h}} \bar{\mathsf{u}}_{\underline{i},\underline{p}}\, du = \boxed{\sum_{\gamma,\sigma\in S_k} \delta^{\underline{j}}_{\gamma(\underline{i})} \delta^{\sigma(\underline{p})}_{\underline{h}} Wg(d,\gamma\sigma^{-1})}. \tag{2.79}$$

Remark 9. In particular for $i_\ell = h_\ell = p_\ell = \ell$ and $j_\ell = \tau(\ell),\ 1 \le \ell \le k$, Formula (2.79) gives $Wg(d,\tau)$.

Collins then goes several steps ahead since he is interested in the asymptotic behaviour of this function as $d \to \infty$ and proves an asymptotic expression for any σ in term of its cycle decomposition, Theorem 4. $\boxed{QED}$

2.2 Tensor polynomials

In the forthcoming paper with Felix Huber, [11], we consider the problem of understanding k–tensor valued polynomials of n, $d \times d$ matrices.

That is maps from n tuples of $d \times d$ matrices $x_1, \ldots, x_n \in End(V)$ to tensor space $End(V)^{\otimes k}$ of the form

$$G(x_1,\ldots,x_n) = \sum_i \alpha_i m_{1,i} \otimes m_{2,i} \otimes \ldots \otimes m_{k,i},\ \alpha_i \in \mathbb{C} \quad m_{j,i} \quad \text{monomials in the } x_i.$$

A particularly interesting case is when the polynomial is multilinear and alternating in $n = d^2$ matrix variables.

In this case, by Proposition 8 we have

Theorem 6. *(1)*

$$G(x_1,\ldots,x_{d^2}) = \det(x_1,\ldots,x_{d^2})\bar{J}_G.$$

(2) *Moreover we have the explicit formula*

$$G(e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, \dots, e_{d,d}) = \bar{J}_G.$$

(3) *The element $\bar{J}_G \in M_d^{\otimes k}$ is $GL(k)$ invariant and so $\bar{J}_G \in \Sigma_k(V)$ is a linear combinations of the elements of the symmetric group $S_n \subset M_d^{\otimes k}$ given by the permutations.*

For theoretical reasons instead of computing $\bar{J}_G$ it is better to compute its multiple, as in Formula (1.65):

$$G(x_1, \dots, x_{d^2}) = \mathcal{T}_d(X) J_G, \quad \bar{J}_G = \mathcal{C}_d J_G. \tag{2.80}$$

Using Formula (2.74) we may first compute

$$\Phi(G(x_1, \dots, x_{d^2})) = \sum_{\sigma \in S_k} \operatorname{tr}(\sigma^{-1} \circ G(x_1, \dots, x_{d^2})) = \mathcal{T}_d(X)\Phi(J_G).$$

Consider the special case

$$G_d(Y_1, \dots, Y_{d^2}) := Alt_Y(m_1(Y) \otimes \dots \otimes m_d(Y)), \quad m_i(Y) = Y_{(i-1)^2+1} \dots Y_{i^2}. \tag{2.81}$$

Lemma 2.

$$Alt_Y \operatorname{tr}(\sigma^{-1} \circ m_1(Y) \otimes \dots \otimes m_d(Y)) = \begin{cases} \mathcal{T}_d(Y) & \textit{if} \quad \sigma = 1 \\ 0 & \textit{otherwise} \end{cases} \tag{2.82}$$

Proof.

$$\operatorname{tr}(\sigma^{-1} \circ m_1(Y) \otimes \dots \otimes m_d(Y)) = \prod_{i=1}^{j} \operatorname{tr}(N_i)$$

with N_i the product of the monomials m_j for j in the i^{th}cycle of σ, cf. Formula (1.6). The previous invariant gives by alternation the invariant

$$Alt_Y \prod_{i=1}^{j} \operatorname{tr}(N_i) = T_{a_1} \wedge T_{a_2} \wedge \dots \wedge T_{a_j}, \quad a_i = \text{degree of } N_i$$

in degree d^2. If $\sigma \neq 1$ we have $j < d$ hence the product is 0, since the only invariant alternating in this degree is $T_1 \wedge T_3 \wedge T_5 \wedge \ldots \wedge T_{2d-1}$.

On the other hand if $\sigma = 1$ we have $N_i = m_i$ and the claim follows. QED

Proposition 10. *We have*

$$G_d(Y_1, \dots, Y_{d^2}) := Alt_Y(m_1(Y) \otimes \dots \otimes m_d(Y)) = \mathcal{T}_d(Y) Wg(d, d). \tag{2.83}$$

Proof. The previous Lemma in fact implies that $\Phi(G_d(Y_1, \dots, Y_{d^2})) = \mathcal{T}_d(Y) 1_d$ therefore $\Phi(J_{G_d}) \overset{(2.75)}{=} \Phi(1) J_{G_d} = 1$ so that $J_{G_d} = \Phi(1)^{-1} = Wg(d, d)$. QED

2.3 The construction of Formanek

Let us now discuss a theorem of Formanek relative to a conjecture of Regev, see [7] or [1]. This states that, a certain explicit central polynomial $F(X,Y)$ in d^2, $d \times d$ matrix variables $X = \{X_1, \dots, X_{d^2}\}$ and another d^2, $d \times d$ matrix variables $Y = \{Y_1, \dots, Y_{d^2}\}$, is non zero. This polynomial plays an important role in the theory of polynomial identities, see [1].

The definition of $F(X,Y)$ is this, decompose $d^2 = 1 + 3 + 5 + \dots + (2d-1)$ and accordingly decompose the d^2 variables X and the d^2 variables Y in the two lists. Construct the monomials $m_i(X), i = 1, \dots, d$ and similarly $m_i(Y)$ as product in the given order of the given $2i-1$ variables X_i of the i^{th} list as for instance

$$m_1(X) = X_1, m_2(X) = X_2X_3X_4, m_3(X) = X_5X_6X_7X_8X_9, \dots.$$

$$m_i(X) = X_{(i-1)^2+1} \dots X_{i^2}, \quad m_i(Y) = Y_{(i-1)^2+1} \dots Y_{i^2}.$$

We finally define

$$F(X,Y) := Alt_X Alt_Y(m_1(X)m_1(Y)m_2(X)m_2(Y) \dots m_d(X)m_d(Y)), \quad (2.84)$$

where Alt_X (resp. Alt_Y) is the operator of alternation, Formula (1.60), in the variables X (resp. Y). By Theorem 6 it takes scalar values, a multiple of $\mathcal{T}_d(X)\mathcal{T}_d(Y)$, but it could be identically 0.

Theorem 7.

$$F(X,Y) = (-1)^{d-1}\frac{1}{(d!)^2(2d-1)}\mathcal{T}_d(X)\mathcal{T}_d(Y)Id_d \quad (2.85)$$

$$\stackrel{(1.65)}{=} (-1)^{d-1}\frac{\mathcal{C}_d^2}{(d!)^2(2d-1)}\Delta(X)\Delta(Y)Id_d; \quad \Delta(X) = \det(X_1, \dots, X_{d^2}).$$

Notice that by Formula (1.65) the coefficient is an integer (as predicted).

Thus $F(X,Y)$ is a central polynomial. In fact it has also the property of being in the *conductor* of the ring of polynomials in generic matrices inside the trace ring. In other words by multiplying $F(X,Y)$ by any invariant we still can write this as a non commutative polynomial. This follows by polarizing in z the identity, cf. [1] Proposition 10.4.9 page 286.

$$\det(z)^d F(X,Y) = F(zX,Y) = F(X,zY) = F(Xz,Y) = F(X,Yz).$$

Let us follow Formanek's proof. First, since $F(X,Y)$ is a central polynomial Formula (2.85) is equivalent to:

$$\mathrm{tr}(F(X,Y)) = (-1)^{d-1}\frac{d}{(d!)^2(2d-1)}\mathcal{T}_d(X)\mathcal{T}_d(Y). \tag{2.86}$$

Now we have, with $\sigma_0 = (1,2\ldots,d)$ the cycle:

$$\begin{aligned}&\mathrm{tr}(F(X,Y)) = \\ &\mathrm{tr}(\sigma_0^{-1}\circ Alt_X Alt_Y(m_1(X)m_1(Y)\otimes m_2(X)m_2(Y)\otimes\ldots\otimes m_d(X)m_d(Y)),\\ &\overset{(2.83)}{=} \mathrm{tr}(\sigma_0^{-1}\circ Alt_X(m_1(X)\otimes m_2(X)\otimes\ldots\otimes m_d(X)\cdot Wg(d,d))\mathcal{T}_d(Y).\end{aligned} \tag{2.87}$$

Denote $Wg(d,d) = \sum_{\tau\in S_d} a_\tau\tau$, we have

$$\begin{aligned}&\mathrm{tr}(\sigma_0^{-1}\circ Alt_X(m_1(X)\otimes m_2(X)\otimes\ldots\otimes m_d(X)\cdot Wg(d,d))\\ &= \sum_\tau a_\tau\,\mathrm{tr}(\sigma_0^{-1}\tau\circ Alt_X(m_1(X)\otimes m_2(X)\otimes\ldots\otimes m_d(X))\end{aligned}$$

which, by Lemma 2 equals $a_{\sigma_0}\mathcal{T}_d(X)$. Therefore the main Formula (2.85) follows from Formula (1.24).

3 Appendix

If $k > d$ of course there is still an expression as in Formula (1.20) but it is not unique.

It can be made unique by a choice of a basis of $\Sigma_k(V)$. This may be done as follows.

Definition 4. Let $0 < d$ be an integer and let $\sigma \in S_n$.

Then σ is called *d–bad* if σ has a descending subsequence of length d, namely, if there exists a sequence $1 \le i_1 < i_2 < \cdots < i_d \le n$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_d)$. Otherwise σ is called *d–good.*

Remark 10. σ is d–good if any descending sub–sequence of σ is of length $\le d-1$. If σ is d-good then σ is d'-good for any $d' \ge d$.

Every permutation is 1-bad.

Theorem 8. *If* $\dim(V) = d$ *the* $d+1$*–good permutations form a basis of* $\Sigma_k(V)$.

Proof. Let us first prove that the $d+1$–good permutations span $\Sigma_{k,d}$.

So let σ be $d+1$–bad so that there exist $1 \le i_1 < i_2 < \cdots < i_{d+1} \le n$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_d+1)$. If A is the antisymmetrizer on the $d+1$

elements $\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_d+1)$ we have that $A\sigma = 0$ in $\Sigma_k(V)$, that is, in $\Sigma_k(V)$, σ is a linear combination of permutations obtained from the permutation σ with some proper rearrangement of the indices $\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_d+1)$.These permutations are all lexicographically $< \sigma$. One applies the same algorithm to any of these permutations which is still $d+1$–bad. This gives an explicit algorithm which stops when σ is expressed as a linear combination of $d+1$–good permutations (with integer coefficients so that the algorithm works in all characteristics).

In order to prove that the $d+1$–good permutations form a basis, it is enough to show that their number equals the dimension of $\Sigma_{k,d}$. This is insured by a classical result of Schensted which we now recall. $\boxed{QED}$

3.1.1 The RSK and d-good permutations

The RSK correspondence[4], see [13], [25], is a combinatorially defined bijection $\sigma \longleftrightarrow (P_\lambda, Q_\lambda)$ between permutations $\sigma \in S_n$ and pairs P_λ, Q_λ of standard tableaux of same shape λ, where $\lambda \vdash n$.

In fact more generally it associates to a word, in the free monoid, a pair of tableaux, one standard and the other semistandard filled with the letters of the word. This correspondence may be viewed as a combinatorial counterpart to the Schur–Weyl and Young theory.

The correspondence is based on a simple *game* of inserting a letter.

We have some letters piled up so that lower letters appear below higher letters and we want to insert a new letter x. If x fits on top of the pile we place it there otherwise we go down the pile, until we find a first place where we can replace the existing letter with x. We do this and expel that letter, first creating a new pile or, if we have a second pile of letters then we try to place that letter there and so on.

So let us pile inductively the word *strange*.

$$e \mapsto e,\ g \mapsto \begin{matrix} g \\ e \end{matrix},\ n \mapsto \begin{matrix} n \\ g \\ e \end{matrix},\ a \mapsto \begin{matrix} n & \\ g & \\ a & e \end{matrix},\ r \mapsto \begin{matrix} r & \\ n & \\ g & \\ a & e \end{matrix},\ t \mapsto \begin{matrix} t & \\ r & \\ n & \\ g & \\ a & e \end{matrix},\ s \mapsto \begin{matrix} s & \\ r & \\ n & \\ g & t \\ a & e \end{matrix}.$$

Notice that, as we proceed, we can keep track of where we have placed the

[4]Robinson, Schensted, Knuth

new letter, we do this by filling a corresponding tableau.

$$\begin{matrix} 6 & & \\ 5 & & \\ 3 & & \\ 2 & 7 & \\ 1 & 4 & \end{matrix} \; , \quad \begin{matrix} s & \\ r & \\ n & \\ g & t \\ a & e \end{matrix} \; .$$

It is not hard to see that from the two tableaux one can *decrypt* the word we started from giving the bijective correspondence.

Assume now that $\sigma \longleftrightarrow (P_\lambda, Q_\lambda)$, where P_λ, Q_λ are standard tableaux, given by the RSK correspondence. By a classical theorem of Schensted [23], $ht(\lambda)$ equals the length of a longest decreasing subsequence in the permutation σ. Hence σ is $d+1$-good if and only if $ht(\lambda) \leq d$.

Now M_λ has a basis indexed by standard tableaux of shape λ, see [21]. Thus the algebra $\Sigma_k(V)$ has a basis indexed by pairs of tableaux of shape λ. $ht(\lambda) \leq d$ and the claim follows. QED

Therefore one may define the Weingarten function for all k as a function on the $d+1$–good permutations in S_k.

3.1.2 Cayley's Ω process

It may be interesting to compare the method of computing the integrals of Formula (2.76) with a very classical approach used by the 19^{th} century invariant theorists.

Let me recall this for the modern readers. Recall first that, given a $d \times d$ matrix $X = (x_{i,j})$, its *adjugate* is $\bigwedge^{d-1}(X) = (y_{i,j})$ with $y_{i,j}$ the *cofactor* of $x_{j,i}$ that is $(-1)^{i+j}$ times the determinant of the minor of X obtained by removing the j row and i column. Then the inverse of X equals $\det(X)^{-1} \bigwedge^{d-1}(X)$.

It is then easy to see that, substituting to $u_{i,j}$ the variables $x_{i,j}$ and to $\bar{u}_{i,j}$ the polynomial $y_{i,j}$ one transforms a monomial $M = \prod_{\ell=1}^k u_{j_\ell,h_\ell} \bar{u}_{i_\ell,p_\ell}$ into a polynomial $\pi_d(M)$ in the variables $x_{i,j}$ homogeneous of degree dk, the invariants under U_d become powers $\det(X)^k$. Denote by $S^{kd}(x_{i,j})$ the space of these polynomials which, under the action of $GL(d) \times GL(d)$, decomposes by Cauchy formula, cf. Formula 6.18, page 178, of [1]. Then we have also an equivariant projection from these polynomials to the 1–dimensional space spanned by $\det(X)^k$, it is given through the Cayley Ω process used by Hilbert in his famous work on invariant theory. The Ω process is the differential operator given by the determinant of the matrix of derivatives:

$$X = (x_{i,j}), \quad Y = (\frac{\partial}{\partial x_{i,j}}), \quad \Omega := \det(Y). \tag{3.88}$$

We have that Ω^k is equivariant under the action by $SL(n)$ so it maps to 0 all the irreducible representations different from the 1–dimensional space spanned by $\det(X)^k$ while

$$\Omega \det(X)^k = k(k+1)\dots(k+d-1)\det(X)^{k-1}.$$

Both statements follow from the Capelli identity, see [21] §4.1 and [2].

$$\boxed{\det(X)\Omega = \det(a_{i,j})},\ a_{i,i} = \Delta_{i,i} + n - i,\ a_{i,j} = \Delta_{i,j},\ i \neq j$$

$$\text{the polarizations}\quad \Delta_{i,j} = \sum_{h=1}^{d} x_{i,h}\frac{\partial}{\partial x_{h,j}}.$$

If we denote by $\underline{x}_i := (x_{i,1},\dots,x_{i,n})$ we have the Taylor series for a function $f(\underline{x}_1,\dots,\underline{x}_n)$ of the vector coordinates $\underline{x}_i$.

$$f(\underline{x}_1,\dots,\underline{x}_j + \lambda\underline{x}_i,\dots,\underline{x}_n) = \sum_{k=0}^{\infty}\frac{(\lambda\Delta_{i,j})^k}{k!}f(\underline{x}_1,\dots,\underline{x}_n).$$

Thus

$$\int_U M\,du = \frac{\Omega^k\pi_d(M)}{\prod_{i=1}^k(i(i+1)\dots(i+d-1))}. \tag{3.89}$$

We can use Remark 9 to give a possibly useful formula:

$$Wg(d,\gamma) = \frac{\Omega^k\pi_d(M)}{\prod_{i=1}^k(i(i+1)\dots(i+d-1))},\ M = \prod_{i=1}^k u_{i,i}\bar{u}_{i,\gamma(i)}. \tag{3.90}$$

Let me discuss a bit some calculus with these operators.

Lemma 3. *If $i \neq j$ then Δ_{ij} commutes with Ω and with* $\det(X)$ *while*

$$[\Delta_{ii},\det(X)] = \det(X),\quad [\Delta_{ii},\Omega] = -\Omega. \tag{3.91}$$

Proof. The operator Δ_{ij} commutes with all of the columns of Ω except the i^{th} column ω_i with entries $\frac{\partial}{\partial x_{it}}$. Now $[\Delta_{ij},\frac{\partial}{\partial x_{it}}] = -\frac{\partial}{\partial x_{jt}}$, from which $[\Delta_{ij},\omega_i] = -\omega_j$. The result follows immediately. QED

Let us introduce a more general determinant, analogous to a characteristic polynomial. We denote it by $C_m(\rho) = C(\rho)$ and define it as:

$$\begin{pmatrix} \Delta_{1,1} + m - 1 + \rho & \Delta_{1,2} & \dots & \Delta_{1,m} \\ \Delta_{2,1} & \Delta_{2,2} + m - 2 + \rho & \dots & \Delta_{2,m} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \Delta_{m-1,1} & \Delta_{m-1,2} & \dots & \Delta_{m-1,m} \\ \Delta_{m,1} & \Delta_{m,2} & \dots & \Delta_{m,m} + \rho \end{pmatrix}.$$

We have now a generalization of the Capelli identity:

Proposition 11.

$$\Omega C(k) = C(k+1)\Omega, \qquad \det(X) C(k) = C(k-1)\det(X)$$

$$\det(X)^k \Omega^k = C(-(k-1))C(-(k-2))\dots C(-1)C,$$

$$\Omega^k \det(X)^k = C(k)C(k-1)\dots C(1).$$

Proof. We may apply directly Formulas (3.91) and then proceed by induction. QED

Develop now $C_m(\rho)$ as a polynomial in ρ obtaining an expression

$$C_m(\rho) = \rho^m + \sum_{i=1}^{m} K_i \rho^{m-i}.$$

Capelli proved, [2], that, as the elementary symmetric functions generate the algebra of symmetric functions so the elements K_i generate the center of the enveloping algebra of the Lie algebra of matrices.

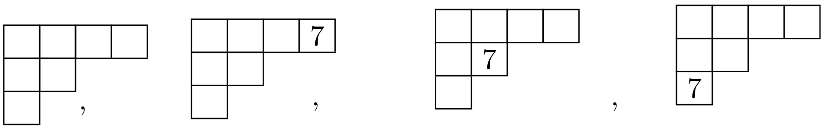
In [21] Chapter 3, §5 it is also given the explicit formula, also due to Capelli, of the action of $C_m(\rho)$ (as a scalar) on the irreducible representations which classically appear as *primary covariants*.

3.2 A quick look at the symmetric group

3.2.1 The branching rule and Young basis

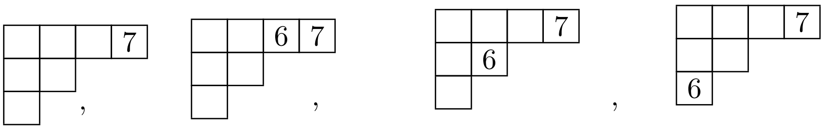
Recall that the irreducible representations of S_n over $\mathbb{Q}$ are indexed by partitions of n usually displayed as *Young diagrams*.

The Branching rules, see [22], [15] or [21], tell us how the representation M_λ decomposes once we restrict to S_{k-1}. The irreducible representation M_λ becomes the direct sum $\oplus_{\mu \subset \lambda,\ \mu \vdash k-1} M_\mu$. The various μ are obtained from λ by marking one corner box with k and removing this box.



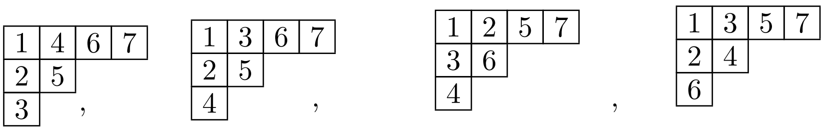
$$M_{4,2,1} = M_{3,2,1} \oplus M_{4,1,1} \oplus M_{4,2}$$

This can be repeated on each summand decomposed into irreducible representations of S_{n-2}



$$M_{3,2,1} = M_{2,2,1} \oplus M_{3,1,1} \oplus M_{3,2}$$

After $k-1$ steps we have a list of *skew standard tableaux* filled with the numbers $n, n-1, \ldots, n-k+1$ so that removing the boxes occupied by these numbers we still have a Young diagram and these tableaux index a combinatorially defined decomposition of M_λ into irreducinle representations of S_{n-k}. Getting, after n steps a decomposition of M_λ into one dimensional subspaces indexed by *standard tableaux*, as out of a total of 35:



$$M_\lambda = \oplus_{T \in \text{ standard tableaux}} M_T, \ \dim_{\mathbb{Q}} M_T = 1. \tag{3.92}$$

In fact there is a scalar product on M_λ invariant under S_n and unique up to scale for this property. The decomposition is then into orthogonal one dimensional subspaces. One then may choose a basis element v_T for the one dimensional subspace indexed by T with $|v_T| = 1$ but allowing to work on some real algebraic extension of $\mathbb{Q}$. This is then unique up to sign.

Remark 11. Observe that, given a standard tableau T and a number $k \leq n$ the space M_T lies in the irreducible representation of S_k associated to the skew tableau obtained form T by emptying all the boxes with the numbers $i \leq k$. Its Young diagram is the diagram containing the indices from $1, \ldots, k$ in T. As example the first tableau of the previous list lies in an irreducible representation of S_5 of partition $2, 2, 1$ and one of S_4 of partition $2, 1, 1$; while the third $3, 1, 1$ and again $2, 1, 1$ but different from the previous one since they are associated to different skew tableaux.

3.2.2 A maximal commutative subalgebra

Denote by $\mathcal{Z}_n$ the center of the group algebra $\mathbb{Z}[S_n]$ it is the free abelian group with basis the class functions. A basic Theorem of Higman and Farahat [5], states that the elements C_j generate (over $\mathbb{Z}$) as algebra the center $\mathcal{Z}_n$ of $\mathbb{Z}[S_n]$.

Now consider the inclusions $S_1 \subset S_2 \subset \ldots \subset S_{n-1} \subset S_n$ which induces inclusions $\mathcal{Z}_j \subset \mathbb{Z}[S_n],\ j = 1, \ldots, n$.

Definition 5. We define $\mathfrak{Z}_n$ to be the (commutative) algebra generated by all the algebras $\mathcal{Z}_j$.

Corollary 4. *The 1–dimensional subspaces M_T associated to standard tableaux are eigenspaces for $\mathfrak{Z}_n$.*

Proof. Take one such 1–dimensional subspace M_T associated to a standard tableau T. Given any $k \leq n$ the space M_T by construction is contained in an irreducible representation of S_k where the elements of $\mathcal{Z}_k$ act as scalars. QED

By the Theorem of Jucys–Murphy and the Theorem of Farahat–Higman the subalgebra of $\mathbb{Z}[S_n]$ generated by the elements $J_2, \ldots, J_k$ contains the class algebra $\mathcal{Z}_k$ (and conversely). in the next Theorem 9 we will see that in fact this subalgebra is maximal semisimple.

The final analysis is to understand the eigenvalues of the operators J_i which generate $\mathfrak{Z}_n$ on M_T. Given a standard Tableau T and a number $i \leq n$ this number appears in one specific box of the diagram of T and then we define $c_T(i)$ to be the content of this box as in Formula (1.3).

As example for the first tableau of the list before Formula (3.92)

$$c_T(1) = 0,\ c_T(2) = -1,\ c_T(3) = -2,\ c_T(4) = 1,$$
$$c_T(5) = 0,\ c_T(6) = 2,\ c_T(7) = 3.$$

Let us start with the following fact. Denote by $c_2(k)$ the sum of all transpositions of S_k. It is a central element so it acts as a scalar on each irreducible representation and one has, see Frobenius [8] or Macdonald [15]

Proposition 12. *The action of $c_2(k)$ on an irreducible representation associated to a partition $\lambda = \lambda_1, \ldots, \lambda_k$ is*

$$\frac{1}{2}\sum_{i=1}^{k}(\lambda_i^2 - (2i-1)\lambda_i) \tag{3.93}$$

If we consider $S_{k-1} \subset S_k$ we have $J_k = c_2(k) - c_2(k-1)$.

Theorem 9.

$$J_i v_T = c_T(i) v_T,\ \forall i = 2, \ldots, n,\ \forall T. \tag{3.94}$$

Proof. We follow Okounkov [19] who makes reference to Olshanski [20].

We need to compute $(c_2(i) - c_2(i-1))v_T$. Now v_T belongs to the irreducible representation of S_i whose diagram is the subdiagram D_i of the diagram of T containing the indices $1, \dots, i$ and let (a, b) be the coordinates of the box where i is placed.

In the same way v_T belongs to the irreducible representation of S_{i-1} whose diagram is the subdiagram of D_i obtained removing the box (a, b).

Applying Formula (3.93) to the two elements $c_2(i), c_2(i-1)$ we see that the two diagrams coincide except for the a row which in one case has length b in the other $b-1$ so the difference of the two values is

$$\frac{1}{2}[(b^2 - (2a-1)b) - ((b-1)^2 - (2a-1(b-1))] = b - a.$$

QED

Proposition 13. *The function $c_T(i),\ i = 1, \dots, n$ determines the standard tableau T.*

Proof. By induction the function $c_T(i),\ i = 1, \dots, n-1$ determines the part T' of the tableau T except the box occupied by n.

As for this box we know its content, $c_T(n)$. Now the boxes with a given content form a *diagonal* and then the box for T must be the first in this diagonal which is not in T'. QED

This shows that the algebra generated by the elements J_i separates all the vectors of all Young bases so:

Corollary 5. *The elements $J_i,\ i = 2, \dots$. generate the maximal semisimple commutative subalgebra $\mathcal{S}$ of $\mathbb{Q}[S_n]$ of all elements which are diagonal on all Young bases..*

Proof. By Theorem 9 and Proposition 13 the subalgebra $\mathcal{S}$ maps surjectively to the subalgebra of $\mathbb{Q}[S_n]$ of all elements which are diagonal on all Young bases. But this map is also injective since an element of $\mathbb{Q}[S_n]$ which vanishes on all irreducible representations equals to 0. Hence $\mathcal{S}$ is the direct sum of the diagonal matrices (in this basis) for all matrix algebras in which $\mathbb{Q}[S_n]$ decomposes and this is a maximal commutative semisimple subalgebra hence the claim. QED

3.3 Stanley hook–content formula

Let us finally show that the Jucys factorization, Formula (1.51), can be viewed as a refinement of Stanley hook–content formula (1.4).

In fact consider the scalar value of the central operator

$$P=\sum_{\rho\in S_k} d^{c(\rho|}\rho = d\prod_{i=2}^{k}(d+J_i)$$

on an irreducible representation M_μ. It can be evaluated, from Formula (1.14) as

$$\chi_\mu(1)^{-1}\operatorname{tr}(P)=\chi_\mu(1)^{-1}\sum_{\sigma}\sum_{\lambda\vdash k,\ ht(\lambda)\le d} s_\lambda(d)\chi_\lambda(\sigma)\chi_\mu(\sigma)$$

$$=\chi_\mu(1)^{-1}k!s_\mu(d)=\prod_{u\in\mu}h_u s_\mu(d). \tag{3.95}$$

On the other hand this scalar is also the value obtained by applying the operator $P=d\prod_{i=2}^{k}(d+J_i)$ on any standard tableau of the Young basis of M_μ giving, by Formula (3.94), the value

$$d\prod_{i=2}^{k}(d+c_T(i))=\prod_{u\in\mu}(d+c_u). \tag{3.96}$$

Comparing Formulas (3.95) and (3.96) one finally has Stanley hook–content formula (1.4).

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