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On simultaneous approximation

Yusuf Karakus
University of Çukurova, Faculty of Arts and Sciences,
Department of Mathematics, 01330 Adana Turkey
yusufk@mail.cu.edu.tr

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Abstract. In this paper first we give two different definitions for best simultaneous L_p approximation to n functions and study the relation between best simultaneous approximation and best L_p approximations to the arithmetic mean of n functions. In addition we consider the definition and the theorem about the simultaneous approximation to n (n odd) functions in the "sum" norm.

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Holland, McCabe, Phillips and Sahney in [1] considered the best simultaneous L_1 approximation and studied the relation between the best simultaneous approximation and the L_1approximations to the arithmetic mean of n functions. Phillips and Sahney [4] gave results for the L_1 and L_2 norms. The problem of the simultaneous approximation to an arbitrary number of functions was discussed by Holland and Sahney [3], who generalized the results in [4] for the L_2 norm. Ling [5] has considered for two functions the simultaneous Chebyshev approximation in the "sum" norm.

We now examine two possible definitions of best simultaneous L_p approximation to n functions and explore whether, for any of these definitions the best simultaneous approximation coincides with the best L_p approximation to the arithmetic mean of the n functions.

Definition 1. Let $p \geq 1$ be a real number and $S \subset L_p$ [a,b] a non-empty set of real-valued functions. Let us assume that real valued functions $f_1, \ldots, f_n$ and $s \in S$ are L_p integrable. If there exists an element $s^* \in S$ such that

$$\inf_{s\in S} \sum_{k=1}^{n} \|f_k - s\|_p^p = \sum_{k=1}^{n} \|f_k - s^*\|_p^p \tag{1}$$

then s^* is said to be a *best simultaneous approximation to the functions* $f_1, \ldots, f_n$ *in the* L_p *norm.* If the infimum is attained in (1), then this number is called *the degree of the best simultaneous approximation.*

Remark 1. Phillips and Sahney [4] showed that the best simultaneous approximation to two functions in the sense of Definition l does coincide with the best L_2 approximation to the arithmetic mean of two function for $p = 2$.

We now take as a lemma the Theorem 16 in [2].

Lemma 1. *For $p > 1$ and positive real numbers $a_1, \dots, a_n$ there exists the inequality*

$$a_1^p + \cdots + a_n^p \geq n \left(\frac{a_1 + \cdots + a_n}{n} \right)^p \tag{2}$$

where the equality is true only for $a_1 = \cdots = a_n$.

Theorem 1. *Let s and f_i, $i = 1, \dots, n$ be as defined in the Definition 1. If $p > 1$ is a real number, then*

$$\inf_{s \in S} \sum_{k=1}^{n} \|f_k - s\|_p^p \geq \inf_{s \in S} \{n \| \frac{1}{n} \sum_{k=1}^{n} f_k - s\|_p^p\}. \tag{3}$$

Proof. If we take $a_k = |f_k(x) - s(x)|$ in the inequality (2), we obtain

$$\begin{aligned} \sum_{k=1}^{n} |f_k(x) - s(x)|^p &\geq n \left(\frac{1}{n} \sum_{k=1}^{n} |f_k(x) - s(x)| \right)^p \\ &\geq n \left| \frac{(f_1(x) - s(x)) + \cdots + (f_n(x) - s(x))}{n} \right|^p \\ &= n \left| \frac{1}{n} \sum_{k=1}^{n} f_k(x) - s(x) \right|^p \end{aligned}$$

By integrating both side of this inequality from a to b and by taking the infimum over all $s \in S$, we have the result of Theorem 1. If $f_i(x) \neq f_j(x)$ for $i \neq j$, then from (3) we have

$$\inf_{s \in S} \sum_{k=1}^{n} \|f_k - s\|_p^p \geq \inf_{s \in S} \{n \| \frac{1}{n} \sum_{k=1}^{n} f_k - s\|_p^p\}. \tag{4}$$

This completes the proof of Theorem 1. QED

Theorem 1 says that, the degree of the best L_p approximation to the arithmetic mean is bounded above by the degree of the best simultaneous approximation in the sense of Definition 1.

The following counterexample shows that, in general, the best simultaneous approximation in the sense of Definition 1 does not coincide with the best L_p approximation to the arithmetic mean.

Counterexample 1. Let $p = 3$. Choose $f_1(x) = 0$ and $f_2(x) = x$ on $[0, 1]$ and let S be the set of real numbers. The best simultaneous approximation to f_1 and f_2 from S in the sense of Definition 1 is the number $s_1^* = 0,31290841$, whereas the best L_3 approximation to $(f_1 + f_2)/2$ is the number $s_2^* = 0,25$.

Definition 2. Let $p \geq 1$ be a real number and $S \subset L_p\,[a,b]$ a non-empty set of real-valued functions.Let us assume that real-valued functions $f_1, \ldots, f_n$ and $s \in S$ are L_p integrable. If there exists an element $s^* \in S$ such that

$$\inf_{s\in S} \max_k \|f_k - s\|_p^p = \max_k \|f_k - s^*\|_p^p, \quad k = 1, \ldots, n \tag{5}$$

then s^* is called to be a *best simultaneous approximation to the functions* $f_1, \ldots, f_n$ in the L_p norm.

Theorem 2. *Let f_i, $i = 1, \ldots, n$ and s be as defined above. If $p > 1$ is a real number, then*

$$\inf_{s\in S} \|\frac{1}{n}\sum_{k=1}^{n} f_k - s\|_p^p \leq \inf_{s\in S} \max_k \|f_k - s\|_p^p. \tag{6}$$

Proof. From Lemma 1 and Theorem 1,

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=1}^{n} f_k - s\right\|_p^p &\leq \frac{1}{n}\sum_{k=1}^{n} \|f_k - s\|_p^p \\ &\leq \frac{1}{n}[n.\max_k \|f_k - s\|_p^p] \\ &= \max \|f_k - s\|_p^p\,, k = 1, \ldots, n\end{aligned}$$

and the proof is completed by taking the infimum over S. QED

The Theorem 2 says that the degree of the best L_p approximation to the arithmetic mean is bounded above by the degree of the best simultaneous approximation in the sense of Definition 2.

We now give a counterexample to show that the best simultaneous approximation in the sense of Definition 2 does not, in general, coincide with the best L_p approximation to the arithmetic mean.

Counterexample 2. Let $p = 2$. Choose f_1, f_2 and S as in Counterexample 1. The best simultaneous approximation to f_1 and f_2 in the sense of Definition 2, is the constant function $s_1^* = 1/3$, whereas the best L_2 approximation to $(f_1 + f_2)/4$ is $s_2^* = 1/4$.

Ling [5] gave the following definition and theorem.

Definition 3. Let S be a non-empty set of real-valued functions defined on the interval $[a,b]$. For two real-valued functions f_1 and f_2 if there exists an $s^* \in S$ such that

$$\inf_{s\in S} \||f_1 - s| + |f_2 - s|\| = \||f_1 - s^*| + |f_2 - s^*|\|\,,$$

we say that s^* is *a best simultaneous approximation* to f_1 and f_2 in the "sum" norm. Where

$$\|g\| = \sup_{x\in[a,b]} |g(x)| .$$

Theorem 3.

(1) if

$$\inf_{s\in S} \left\| \frac{f_1+f_2}{2} - s \right\| \geq \frac{1}{2} \|f_1 - f_2\| ,$$

then

$$\inf_{s\in S} \| |f_1 - s| + |f_2 - s| \| = 2 \inf_{s\in S} \left\| \frac{f_1+f_2}{2} - s \right\| .$$

(2) if

$$\inf_{s\in S} \left\| \frac{f_1+f_2}{2} - s \right\| < \frac{1}{2} \|f_1 - f_2\| ,$$

then

$$\inf_{s\in S} \| |f_1 - s| + |f_2 - s| \| = \|f_1 - f_2\| .$$

The Theorem 3(1) says that, the problem simultaneous approximation to f_1 and f_2 in the "sum" norm is, with one restriction, equivalent to approximating the arithmetic mean.

Definition 4. Let X be Banach space and let $S \subset X$, $S \neq \Phi$. For any $x \in X$ if there exists an element $s^* \in S$ such that

$$\inf_{s\in S} \|x - s\| = \|x - s^*\| \tag{7}$$

then we say that s^* *is the best approximation to* x by elements of S.

Definition 5. Let $B\,[a,b]$ be the set of bounded real-valued functions defined on the interval $[a,b]$. Let $S \subset B\,[a,b]$, $S \neq \Phi$ and $F = \{f_1, \ldots, f_n\} \subset B\,[a,b]$. If there exists an element $s^* \in S$ such that

$$\inf_{s\in S} \left\| \sum_{i=1}^{n} |f_i - s| \right\| = \left\| \sum_{i=1}^{n} |f_i - s^*| \right\| \tag{8}$$

then we say that s^* *is the best simultaneous approximation to the functions* $f_1, \ldots, f_n$ (or to F) in the "sum" norm by elements of S.

Theorem 4. *Let $F = \{f_1, \dots, f_n\} \subseteq B\,[a,b]$ where n is an odd integer. For $x \in [a,b]$, let $d^*(x) = (f_1^*(x), \dots, f_n^*(x))$ be the rearrangement of $d(x) = (f_1(x), \dots, f_n(x))$ such that $f_1^*(x) \le f_2^*(x) \le \dots \le f_n^*(x)$. Define $c(F)\colon [a,b] \to R$ by $c(F)(x) = f^*_{(n+1)/2}(x)$.*

Let $S = \left\{ s \in B\,[a,b] \mid f^*_{\frac{n+1}{2}-1}(x) \le s(x) \le f^*_{\frac{n+1}{2}+1}(x) \right\}$. Then an element $s^* \in S$ is a best approximation to $c(F)$ if and only if it is a best simultaneous approximation to F in the sense Definition 5.

We now take as a lemma a special case of the Lemma 3.1 in [6].

Lemma 2. *For every $d = (d_i)_{1 \le i \le n} \in R^n$, $t \in R$ and the odd natural number n let*

$$\Phi_1(d,t) = \sum_{i=1}^{n} |d_i - t| \,. \tag{9}$$

The equation

$$\Phi_1(d, t_1(d)) = \inf_{t \in R} \Phi_1(d,t). \tag{10}$$

has a unique solution $t_1\,(d)$ continuously depending on $d \in R^n$ and $\Phi_1(d,t)$ is a strictly monotone function of $|t - t_1\,(d)|$.

Proof. Proof of Theorem 4 Let $d = d(x) = (d_i)_{1 \le i \le n}$ and $d^* = d^*(x) = (d_i^*)_{1 \le i \le n}$ be as defined in the statement of Theorem 4. For the odd natural number n Milman [6] showed that

$$t_1\,(d) = d^*_{(n+1)/2}. \tag{11}$$

On the other hand the assumptions of Theorem 4 imply

$$t_1\,(d) = c(F)(x) = f^*_{(n+1)/2}(x). \tag{12}$$

Let $s_1 = s^*$ be a best approximation to $c(F)$ and suppose that s_2 any element of S. Then from Definition 4 we have

$$|s_1\,(x) - c\,(F)\,(x)| = \min_{j=1,2} \{|s_j\,(x) - c\,(F)\,(x)|\} \,. \tag{13}$$

On the other hand, using the strict monotonicity of $\Phi_1(d,t)$ as a function of $|t - t_1\,(d)|$, from (9) and (13) we obtain

$$\sum_{i=1}^{n} |f_i\,(x) - s_1\,(x)| \le \min_{j=1,2} \left\{ \sum_{i=1}^{n} |f_i\,(x) - s_j\,(x)| \right\}. \tag{14}$$

On taking the supremum both sides of this inequality over $[a,b]$, we have

$$\left\| \sum_{i=1}^{n} |f_i - s_1| \right\| \le \min_{j=1,2} \left\{ \left\| \sum_{i=1}^{n} |f_i - s_j| \right\| \right\}. \tag{15}$$

The inequality (15) says that $s^* = s_1$ is a best simultaneous approximation to the functions $f_1, \ldots, f_n$ in the "sum" norm.

Now assume that $s_1 = s^*$ is a best simultaneous approximation of F and s_2 any element of S. If we take

$$|f_i(x) - s_j(x)| = \frac{|c(F)(x) - s_j(x)|}{n}$$

in the inequality (14), we obtain

$$|s_1(x) - c(F)(x)| \leq \min_{j=1,2} \{|s_j(x) - c(F)(x)|\}. \tag{16}$$

On taking the supremum over $[a, b]$, we find

$$\|s_1 - c(F)\| \leq \min_{j=1,2} \{\|s_j - c(F)\|\}. \tag{17}$$

Hence $s^* = s_1$ is a best approximation to $c(F)$. QED

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