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SOLVABLE EXTENSIONS OF FLAG-TRANSITIVE PLANES

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Abstract. *The translation planes of order q^n for $n \neq 3$ which are solvable extensions of a flag-transitive affine plane of order q are completely classified.*

1 Introduction

A natural analysis of a translation plane by its collineation groups normally focuses either on a particular type of group acting on the plane or by the nature and action of its collineation group on the affine points or infinite points.

In this article, we provide a general study of 'solvable extensions of flag-transitive planes'.

An affine plane π is said to be an 'extension of a flag-transitive plane' if and only if π contains a subplane π_o and a collineation group G which leaves π_o invariant and acts transitively on the sets of affine and infinite points of π_o and on the infinite points of $\pi - \pi_o$. Note that, in the finite case, this makes the subplane π_o into a translation plane.

Is it possible to provide a complete classification of extensions of flag-transitive planes if we assume that the plane is a finite translation plane? In this situation, one may relax the group condition to asking if the group acts transitively on the infinite points of π_o and on the infinite points of $\pi - \pi_o$ and leaves π_o invariant.

The Desarguesian, Hall, Hering, and Ott-Schaeffer planes are affine translation planes of order q^2, with spreads in $PG(3,q)$, each of which admit collineation groups that have an infinite point orbit of length $q+1$ and i infinite point orbits of length $(q^2-q)/i$ for $i = 1$ or 2. In the Desarguesian and Hall planes, there is an invariant subplane of order q within the net of degree $q+1$. In all of these classes, there are corresponding groups isomorphic to $SL(2,q)$. However, in the Hall case and some of the Desarguesian cases, there are also solvable groups which admit these orbit lengths.

In Jha-Johnson [9], [10], [11], a classification is given of a large subclass of translation planes, called generalized Desarguesian planes, of order q^3 that admit $GL(2,q)$. There are vast numbers of mutually nonisomorphic planes of this type and where the kernel of the plane may be chosen in a variety of ways. In these planes, the associated vector space is a standard $GF(q)$ $GL(2,q)$ module. This means that the group $SL(2,q)$ is generated by elation groups and that $GL(2,q)$ leaves invariant each subplane of order q incident with the zero vector in the associated $GF(q)$-regulus net defined by the elation axes of $SL(2,q)$. Furthermore, there are always infinite point orbits of lengths $q+1$ and q^3-q and we thus have a variety of cubic extensions of a flag-transitive plane.

In Jha-Johnson [7] and [8], regular parallelisms and associated translation planes are considered. Recently, Pentilla and Williams [16] have constructed an infinite class of cyclic regular parallelism in $PG(2,q)$ for $q \equiv 2 \bmod 3$. The previously known regular parallelisms are also cyclic and lie in $PG(3,2)$, $PG(3,5)$ and $PG(3,8)$.

In general, there are corresponding translation planes of order q^4 admitting a collineation group isomorphic to $SL(2,q) \times Z_{1+q+q^2}$ when the parallelism lies in $PG(3,q)$. In this case, there are $(q^4-q)/(q^2-q) = 1+q+q^2$ derivable nets containing a net R of degree $1+q$ and the group Z_{1+q+q^2} acts regularly on the set of these derivable nets. The group $SL(2,q)$ fixes a derivable net and acts transitively on the q^2-q component not in R and transitively on the components of R. Moreover, R is a K-regulus net for some field K isomorphic to $GF(q)$ so there are $1+q+q^2+q^3$ subplanes. It follows that Z_{1+q+q^2} leaves invariant a subplane and $SL(2,q)$ leaves invariant all of the translation planes π of order q^4 that contain a subplane π_o of order q such that π is a transitive extension of a flag-transitive plane.

Hence, the Johnson-Walker and Lorimer-Rahilly translation planes of order 2^4 admit collineation groups with orbits of length $2+1$ and 2^4-2 and also there is an invariant subplane of order 2 within the net of degree 3. The planes of Prince [17] of order 5^4, the unpublished plane of Denniston of order 8^4 as well as all of the translation planes arising from the parallelisms of Pentilla and Williams [16] then are examples of translation planes of order q^4 that admit a collineation group with infinite point orbits of lengths $q+1$ and q^4-q that leave invariant a subplane of order q.

As mentioned, in these cases, the collineation group is $SL(2,q) \times Z_{(1+q+q^2)}$ and, in fact, the existence of such a group forces the plane to be an extansion of a flag-transitive plane. However, we may consider this more generally.

That is:

Theorem 1.1 *Let π be any translation plane of order q^{2r} for $r > 1$ admitting a collineation group in the translation complement isomorphic to $SL(2,q) \times Z_{(q^{2r-1}-1)/(q-1)}$.*

(1) *Then the kernel is isomorphic to $GF(q)$, the p-elements are eleations where $q = p^t$ and there is a regular partial 2-parallelism induced on any elation axis.*

(2) *π is an extension of a flag-transitive plane.*

Proof. (1) is simply the theorem (3.2) of Jha-Johnson [8].

If $q^n = 4^4$ then there exists a cyclic group C of order 21. Futhermore, there are $1+4+4^2+4^3$ subplanes of order 4 incident with the zero vector of the eleation net. It follows that the group of order 7 in C fixes a subplane of order 4 pointwise and the full fixed point set is exactly this subplene. Hence, C leaves invariant a subplane of order 4.

If $q^n \neq 4^4$, then $2r > 2$, there is a p-primitive divisor u of $(q^{2r-1}-1)$ so it follows that there is a planar u-collineation σ and a fixed point subplane of order q that resides within the net R of elation axes. Hence, $SL(2,q)$ and $Z_{(q^{2r-1}-1)/(q-1)}$ both fix the subplane $Fix\sigma = \pi_o$. Clearly, $SL(2,q)$ acts transitively on the $q+1$ elation axes corresponding to the $q+1$ Sylow p-subgroups. There are $\frac{q^{2r}-q}{q^2-q} = \frac{q^{2r-1}-1}{q-1}$ Desarguesian partial spreads of degree q^2+1 containing the net of $q+1$ elation axes which are permuted transitively by the cyclic group of the same order. Furthermore, $SL(2,q)$ is transitive on the q^2-q infinite points not in the elation net of each Desarguesian partial spread of degree q^2+1. Hence, the group is transitive on the $q^{2r}-q$ infinite points not in the elation net so that the plane is a transitive extension of a flag-transitive plane. □

So, we see that the complete determination of the translation planes of order q^n which are extensions of flag-transitive planes of order q is probably not possible when n is 3 due to the existence of the generalized Desarguesian planes and is made further difficult by the possibility of planes which may be constructed from regular partial 2-parallelism.

In contrast with these difficulties for extensions that are at least cubic, the authors recently resolved the problem for quadratic extensions.

Theorem 1.2 *(Hiramine, Jha and Johnson [5]).*

Let π be a finite translation plane which is a quadratic extension of a flag-transitive plane π_o.

Then π is either Desarguesian or Hall.

In particular,

(1) if the associated collineation group is non-solvable then π is Desarguesian and

(2) if the associated collineation group is solvable then π is Hall or Desarguesian of order 4 or 9.

Note that, in these above known situations for extensions that are at least cubic, the collineation group is always non-solvable unless $q = 2$ or 3. Hence, if we assume that we have a 'solvable extension of a flag-transitive plane'; that is, if we assume that the group is solvable then there may be a chance to obtain a complete classification.

1.1 The Examples of Solvable Extensions

To see that a solvable group occurs in the Hall planes, consider the Hall plane π of order q constructible from a Desarguesian plane Σ of order q^2 by the replacement of a regulus net R. There exists a central collineation group C of order $q(q-1)$ which fixes a component L of R pointwise and which leaves R invariant and acts transitively on the points at infinity not in R. So, C fixes all Baer subplanes of R incident with the zero vector. Let H^* denote the homology group of order q^2-1 of Σ. H^* acts transitively on the set of $q+1$ Baer subplanes of R which are incident with the zero vector. Let π^* denote the Hall plane obtained by the replacement of the net R and let R^* denote the derived net and $\pi_o^* = L$ be a Baer subplane of R^*. Hence, CH^* acts transitively on the infinite points of R^* and transitively on the remaining infinite points and leaves $L = \pi_o^*$ invariant. That is, CH^* is a solvable group which fixes a subplane of order q and has two points orbits at infinity of lengths $q+1$ and q^2-q.

Hence, we shall find the Hall planes in any concluding statement on the classification of solvable extensions of flag-transitive planes. Are there other solvable extensions?

We have seen that the Lorimer-Rahilly and Johnson-Walker planes of order 16 admit a collineation group isomorphic to $SL(2,2) \times Z_7$ and hence are solvable extensions of a flag-transitive plane of order 2. Furthermore, the possible translation planes corresponding to regular partial 2-parallelisms of order 2^{2n} admitting a group isomorphic to $SL(2,2) \times Z_{2^{2n-1}-1}$ are also solvable extensions of a flag-transitive plane of order 2.

In a similar manner, one might expect that the translation planes of order 3^{2n} corresponding to regular partial 2-parallelisms might occur as solvable extensions of a flag-transitive plane of order 3, although there are no concrete examples of such planes at present. However, we are able to show that any solvable extensions of order 3^k forces k to be even and such extensions do correspond to regular partial 2-parallelisms.

The translation planes of order 16 of Johnson-Walker and Lorimer-Rahilly are unusual in that the planes considered as order 2^4 are solvable extensions of planes of order 2, and the planes considered as order 4^2 are tangentially transitive in that there is a group fixing a Baer subplane which acts transitive on the tangents incident with a given affine point. Although, the plane considered in the form 4^2 is not a solvable extension, it appears that the usual character is plane of order 4^n might provide instances where there is a solvable extension.

Hence, we see that it is conceivable that there might be non-Hall solvable extensions when $q = 2,3$ or 4. In fact, our main results shows that this are the only possibilities.

For convenience, we now formalize some definitions.

Definition 1.3 *If an affine plane π of order q^n admits a collineation group G which has infinite point orbits of lengths $q+1$ and q^n-1, we shall call π a '$(q+1, q^n-q)$-transitive plane' and G and '$(q+1, q^n-q)$-transitive group'.*

If G leaves a subplane π_o of order q invariant within the net of length $q+1$ and there is a collineation group transitive on the sets of affine and infinite points of π_o and transitive on the infinite points of $\pi - \pi_o$ then π_o is a flag-transitive affine plane and we shall call π and 'extension of π_o'.

If the group of an extension is solvable, we shall call the plane a 'solvable extension of a flag-transitive plane'.

Our main general result on solvable extensions is as follows:

Theorem 1.4 *Let π be a finite translation plane of order q^n which is a solvable extension of a proper flag-transitive plane π_o of order q. Let G denote the corresponding group.*

Then one of the following occur: (1) π is Desarguesian and (q,n) is in
$\{(2,2),(2,3),(3,2),(3,3),(2,5)\}$.

(a) For $(2,2),(2,3)$, the group $SL(2,2)$ is a $(3,2)$- or $(3,6)$-transitive group respectively.

(b) For $(3,2),(3,3)$, the group $SL(2,3)$ is a $(4,6)$- or $(4,24)$-transitive group respectively.

(c) For $(2,5)$, the group $SL(2,2)\times Z_5$ is a $(3,30)$-transitive group.

(2) π is Hall and $n = 2$.

(3) $n = 3$.

(4) $n > 3$ and $q = 2,3$, or 4.

Furthermore, one of the following occurs:

(a) $q = 2$ and there is an normal subgroup generated by elations isomorphic to $SL(2,2)$ which acts doubly-transitively on the infinite points of π_o. Also, the Sylow 2-subgroups have order 2 and the full group $G_{[\pi_o]}$ which fixes π_o pointwise has index 6 so that $SL(2,2)\ G_{[\pi_o]}$ is the full translation complement.

In addition, if n is even then the spread is a union of Desarguesian nets of degree 5 containing π_o and there is a regular partial 2-parallelism of $2^{n-1}-1$ 2-spreads in $PG(2n-1,2)$,

(b) $q = 3$ and n is even. Furthermore, there is a normal subgroup generated by 3-elements such that the restriction to π_o is isomorphic to $SL(2,3)$ and which acts doubly transitively on the infinite points of π_o. The Sylow 3-subgroups are non-planar groups of order 3 and the full groups $G_{[\pi_o]}$ which fixes π_o pointwise has index 24 so $SL(2,3)G_{[\pi_o]}$ is the full translation complement.

If the 3-elements elements are elations, the spread is a union of Desarguesian nets of degree 10 containing π_o and there is a regular partial 2-parallelism of $(3^{n-1}-1)/2$ 2-spreads in $PG(2n-1,3)$. Furthermore, if the 3-elements are not elations then $n \geq 20$.

(c) $q = 4$ *and* $n = 4$.

(d) $q = 4$ *and* $n > 4$. *Then all involutions are elations and there is a normal subgroup generated by elations that acts doubly transitively on the infinite points of* π_o.

Furthermore, the Sylow 2*-subgroups are cyclic of order* 4 *and there is a normal* 2*-complement. If* τ *is a collineation of order* 4 *then* π *may be decomposed into a direct sum of* n *cyclic* $\tau GF(2)$*-submodules of dimension* 4 *and each Sylow* 2*-group pointwise fixed subspace has cardinality* 2^n.

Remark 1 *We note that there are examples in case* 4*(a) of the Lorimer-Rahilly and Johnson-Walker planes of order* 16. *For the other possibilities, it is not known whether non-Desarguesia non-Hall examples exist.*

Thus, we have:

Corollary 1.5 *Let* π *be a finite translation plane of order* q^n *which is a solvable extension of a proper flag-transitive plane* π_o *of order* q.

If $q > 4$ *and* $n \neq 3$ *then* π *is the Hall plane of order* q^2.

Remark 2 *The main result on solvable extensions does not provide any information on what occurs when* $n = 3$. *As we have seen, there exists infinite families of such planes admitting* $SL(2,q)$ *which are cubic extensions of flag-transitive planes and conceivably cubic extension planes always admit non-solvable groups when* $q > 3$.

2 Desarguesian Extensions

In this section, we assume that we have a Desarguesian extension of a flag-transitive plane and provide a complete determination of the possibilities.

Theorem 2.1 *A Desarguesian plane* π *of order* q^n *for* $n \geq 2$ *with subplane* π_o *of order* q *is a transitive extension of the flag-transitive plane* π_o *if and only if*

(1) $n = 2$,

(2) $n = 3$ *or*

(3) $(q,n) = (2,5)$.

Furghermore, the group is solvable if and only if (q,n) *is in*

$\{(2,2),(2,3),(3,2),(3,3),(2,5)\}$.

(a) For $(2,2)$, $(2,3)$, *the group* $SL(2,2)$ *is a* $(3,2)$*- or* $(3,6)$*-transitive group respectively.*

(b) For $(3,2),(3,3)$, *the group* $SL(2,3)$ *is a* $(4,6)$*- or* $4,24)$*-transitive group respectively.*

(c) For $(2,5)$, *the group* $SL(2,2) \times Z_5$ *is a* $(3,30)$*-transitive group.*

Proof. The group is a subgroup of $\Gamma L(2,q^n)$ and the subgroup which fixes π_o pointwise has order dividing n where $q = p^r$ and p is a prime. Furthermore, the group induced on π_o is a subgroup of $\Gamma L(2,q)$ and contains the subkernel group of order $q-1$ induced from the kernel group of order $q^n - 1$ of π. Hence, it follows that $q(q^2-1)\ rn \geq q^n - q$. Moreover, the full collineation group induced on the line at infinity is divisible by $q(q^2-1)rn$ and so $q^n - q$ divides $q(q^2-1)rn$.

Thus, $(q^2-1)rn \geq q^{n-1}-1$.

Assume that $n > 3$ then $rn \geq (q^{n-1}-1)/(q^2-1) > q^{n-3} = p^{r(n-3)}$.

If $p=2$ the only possible solutions are $n=4$ and $r=1,2,3$ or $n=5$ and $r=1$.

Now we check the divisibility condition for the various values of (p,r,n):

For $(p,r,n)=(2,1,4)$, $(q(q^2-1)rn,\ (q^n-q)) = (24,14) \neq 14$, for $(p,r,n)=(2,2,4)$, $(q(q^2-1)rn,\ (q^n-q)) = (32\cdot 15, 4\cdot 63) \neq 4\cdot 63$ and for

$(p,r,n)=(2,3,4)$, $(q(q^2-1)rn,\ (q^n-q)) = (3\cdot 32\cdot 63, 56\cdot 73) \neq 56\cdot 73$.

If $p=3$ the only possible solutions is $r=1$. Clearly, there are no other solutions. For $(p,r,n)=(3,1,4)$, $(q(q^2-1)rn,\ (q^n-q)) = (12\cdot 8, 3\cdot 26) \neq 3\cdot 26$.

Hence, we have possible solutions when $q^n = 2^5$ or $n=2$ or 3.

Now assume that the group is solvable. Note that if there exists an element inducing an elation on π_o then $SL(2,p)$ or $SL(2,5)$ is generated on the subplane by the set of elations of the subplane.

First assume that p is odd and larger than 3. Then, there can be no nontrivial linear p-elements that act on π_o since otherwise a nonsolvable group would be generated. Thus, there is a p-group of order divisible by q which acts on π_o perhaps not faithfully. Let p^a be the order of the subgroup which acts faithfylly on π_o and p^b the order of the p-subgroup which fixes π_o pointwise. Hence, p^a divides n so that q divides p^{a+b} which divides nr. Hence, p^r divides nr and by the above remarks, $n=2$ or 3. However, in all cases $5^r > 3r$ so we must have either $p=2$ or 3.

We now establish a fundamental lemma to complete the analysis.

Lemma 2.2 *As G is solvable, let $G_{[\pi_o]}$ denote the pointwise stabilizer of the set of infinite points of π_o.*

The either $G/G_{[\pi_o]} \leq \Gamma L(1,q^2)$ or $q=3$.

Proof. Since G is transitive on the infinite points of π_o, it arises as the stabilizer of the zero vector of a solvable flag-transitive translation plane. Now we apply the results of Foulser [3] to complete the proof of the lemma, noting that π_o is Desarguesian of order $q=p^r$ where $p=2$ or 3. □

Let f denote the order of the subgroup which fixes π_o pointwise and note when the plane is Desarguesian f must divine n.

Lemma 2.3 *Let Δ denote the set of infinite points of π_o.*

Then we either have

(i) the order of $G|\Delta$ (modulo the kernel) divides $2r(q+1)$ or

(ii) $q=3$ and order of $G|\Delta$ (modulo the kernel) divides $4!$.

Proof. Note that we are assuming that G contains the kernel subgroup. □

Hence, in case (ii) above, we have $(q,n)=(3,2)$ or $(3,3)$.

So, we may assume that $q=2^r$.

Lemma 2.4 *If $n=2$ then $q=2$.*

Proof. We may apply the previously listed result of Hiramine, Jha and Johnson. □

Thus, we may assume that $n = 3$.

Lemma 2.5 *Assume $f = 1$ and $n = 3$. Then $q = 2$ or 4.*

Proof. If $f = 1$ then q divides rp by (i) above. Thus $q = 2$ or 4. □

Now assume $f \neq 1$. Then $f = 3$ and as (i) occurs, $(q^n - q)(q+1)/3$ $(q+1, q^n - q)$ divides $2r(q+1)$ and $q = 2^r$. From this, we have $(q,n) = (2,2), (2,3), (4,3), (3,3)$.

When we have the case $(p,r,n) = (2,1,5)$ then the group induced on the subplane has order divisible by 30 and is in $GL(2,2)$. It follows that the group of order 5 which fixes π_o pointwise arises from the Frobenius automorphism $z \longmapsto z^2$ of order 5 within $\Gamma L(2,2^5)$. The remaining parts of the theorem now follows as $SL(2,2)$ is transitive on the infinite points not in the elation net on the Desarguesian affine planes of orders 2^2 and 2^3 and the analogous statement holds for $SL(2,3)$ on Desarguesian affine planes of orders 3^2 and 3^3. □

2.1 The Main Result

We shall make use of results of Jha [6] which we list for convenience.

Theorem 2.6 *(Jha [6] Lemma 1, p. 774).*

Let V be an elementary Abelian group of order q^n for $n \geq 2$ and suppose that U is any nontrivial u is a prime p-primitive divisor of $q^{n-1} - 1$ and U is in $Aut(V,+)$.

Then the following are valid:

(a) The fixed point subspace $FixU$ under U has order q.

(b) U acts semiregularly on $V - FixU$.

(c) U is cyclic.

(d) If $n > 2$ then $V = FixU \oplus C_U$ where C_U is the only U submodule of V disjoint from $FixU$.

(e) If $n > 2$ and W is a U-submodule then either $W \subset FixU$ or $|W| \geq q^{n-1}$.

Theorem 2.7 *(Jha [6] Theorem B, p. 774 part (iii)).*

Let π be a translation plane of order q^n, $n > 2$, which admits a planar collineation group H of order $u^{\alpha}\, p^{\beta}$ where $q = p^r$ for p a prime and u is a prime p-primitive divisor of $q^{n-1} - 1$ for $\alpha \geq 1$, $\beta \geq 1$.

Then a Sylow u-subgroup is normal in H.

We now give the proof of the main result. The reader is refered to the introduction for the statement of the theorem. The proof shall be given as a series of lemmas.

Lemma 2.8 *If $n = 2$ then the plane is Hall or Desarguesian and the situations where the plane is Dearguesian are given in the previous section and are those of case (1).*

Proof. We merely apply the previously noted theorem of Hiramine, Jha and Johnson. □

Let Δ denote the set of infinite points of π_o and Γ denote the remaining infinite points. Hence, the group is transitive on Δ and Γ.

Let S denote a Sylow p-subgroup. Clearly, S fixes a point P of Δ.

Lemma 2.9 *If $q+1$ is not a prime power then S does not act transitively on $\Delta - \{P\}$.*

Proof. If S does act transitively then there is a doubly transitive group action on Δ. However, this implies that the group acting on Δ is non-solvable since otherwise there is a solvable socle requiring $q+1$ to be a prime power. □

Lemma 2.10 *Assume that $q^n \neq 2^7, 4^4$. If S does not act transitively on $\Delta - \{P\}$ then $n = 2$ or $n = 3$.*

Proof. Hence, by order, there exists a element h of prime order p in S which fixes two components of π_o and hence, h is a planar p-element. Since h leaves π_o invariant, there is a subplane ω_o of π_o which h fixes pointwise.

If $n > 3$ then either there is a p-primitive divisor u of $(q^{n-1}-1)$ as $q^n \neq 2^7$ or 4^4 and note that $((q+1),(q^{n-1}-1))$ divides $(q^2-1, q^{n-1}-1) = (q^{(2,n-1)}-1)$ or $q^{n-1}-1 = 2^6-1$ and $q^n = 2^7$ or 4^4.

Hence, u cannot divide $((q+1,(q^{n-1}-1))$ when $n-1 > 2$.

Thus, u divides $(q^{n-1}-1)/((q+1),(q^{n-1}-1))$.

So, there exists an element g of the group G which has order u. Also, g leaves the subplane π_o of order q invariant and u does not divide q^2-1. We may assume that g leaves an infinite point of π_o invariant since the group is transitive on Δ. Hence, g must fix a second infinite point. Since u does not divide q^2-1, then u fixes a third infinite point and must fix non-zero points on any fixed component $\ell \cap \pi_o$ as the cardinality of this latter set is q.

So, it follows that g must be planar and actually must fix π_o pointwise.

That is, there exists a planar u-element provided there is a p-primitive element u which divides $(q^{n-1}-1)/((q+1),(q^{n-1}-1))$.

Since the group G is solvable, use Hall's extension of the Sylow theorem to obtain that there must be a planar group H of order $u^{\alpha}p^{\beta}$, for $\alpha \geq 1$, $\beta \geq 1$, whose fixed point subspace contains ω_o.

Now let the order of a Sylow p-subgroup S_p of G be qp^a. Now, as the group is not transitive on the infinite points of $\pi_o - \{P\}$, it follows that there is a planar group of order p^{β} where $\beta \geq a+1$ and we assume that p^{β} is the order of the largest planar group within S_p.

Let ℓ be a component of ω_o so that ℓ is invariant under H. By the results of Jha mentioned above [6] considered on ℓ, there is a unique Maschke complement C on ℓ for the fixed point subspace on ℓ of a Sylow u-subgroup S_u of order u^{α} and furthermore, S_u is normal in H. Hence, again by the results of Jha, $\text{Fix}S_u = \pi_o$ so that $C \oplus (\ell \cap \text{Fix}S_u) = \ell$.

Thus, S_p normalizes S_u so leaves $\text{Fix}S_u \cap \ell$ invariant and permutes the Maschke complements and so must leave C invariant and hence has non-zero fixed points on C.

Let the order of ω_o be p^b where $b \leq z$ and $q = p^z$ and let Γ denote the orbit of infinite points of length $q^n - q$. Hence, S_p fixes a subplane Σ_o pointwise of order p^c where $c > b$. If Σ_o is contained within the net N_{Δ} containing π_o then S_p would fix p^c+1 components of π_o which would imply that ω_o has order $> p^b$. Hence, part of Σ_o intersects components of Γ nontrivially. This says that a component M of Γ is fixed by a group of order p^{β} so that the order of the Sylow p-subgroup is at least $pq^{\beta} > qp^a$, a contradiction. This completes the proof of the lemma. □

So, we obtain:

Lemma 2.11 *(1) If $q+1$ is not a prime power then $n = 2$ or 3.*

(2) Assume that $q^n \neq 4^4$. If $q+1$ is a prime power and $n > 3$ then there are no planar p-elements.

Proof. If $q^n \neq 2^7$, we may apply the two previous lemmas and their arguments. Since 7 is odd, it follows that there are no planar 2-elements in the case when $q^n = 2^7$. □

Lemma 2.12 *If q is odd and $q+1$ is a prime power then $q = 3$.*

Proof. If q is odd then $q+1 = 2^e$ which implies that q is a prime p as q^2-1 does not admit a p-primitive divisor. Thus, the subplane π_o is Desarguesian.

In this case, the Sylow p-group induces a faithful subgroup of $\Gamma L(2,p)$ on π_o so that the group is an elation group on π_o.

Hence, the group acting on π_o generated by the elations of π_o is $SL(2,p)$ in this case (acting on the subplane). Since the group is assumed solvable, it follows that $p = 3 = q$. □

Lemma 2.13 *If $n > 3$ then the subplane π_o is either Desarguesian or Lüneburg-Tits.*

Proof. The group is doubly transitive on Δ. The affine translation planes admitting collineation groups acting doubly-transitive on the points at infinity are either Desarguesian or Lüneburg-Tits (see e.g. Buekenhout et al [1]). □

Lemma 2.14 *If $n > 3$, q is even, $q^n \neq 2^7$ or 4^4 and $q+1$ is a prime power then $q = 2$ or 4.*

Proof. If q is even, first assume that the subplane is Lüneburg-Tits. Then $S_z(\sqrt{q})$ is the full subgroup on the subplane which is generated by the elations (of the subplane). As there are no planar 2-elements, any Sylow 2-subgroup S_2 induces a faithful group on π_o so that S_2 must normalize $S_z(\sqrt{q})$.

Since the outer automorphism group has odd order ([2]), it follows that S_2 is in $S_z(\sqrt{q})$. It then follows that the group generated by the Sylow 2-subgroups acting on the plane π contains a group isomorphic to $S_z(\sqrt{q})$ which is contary to the assumption that the group is solvable.

Hence, we must have that π_o is Desarguesian and, again noting there are no planar 2-elements, S_2 induces a faithful group on $\Gamma L(2,q)$. Let $q = 2^{2^z t}$ where $(2,t) = 1$. If $q = 2^{2^z t}/2^z > 2$ then the elation groups acting on π_o must generate a nonsolvable group isomorphic to $SL(2,2^c)$ for $c \geq 2$. The inequality does not hold only if either q is 2 or 4. This completes the proof of the lemma. □

Lemma 2.15 *If u is a p-primitive divisor of $q^{n-1}-1$ and u^α is the largest divisor then the order of a Sylow u-subgroup is u^α.*

Proof. A Sylow u-subgroup U fixes exactly π_o pointwise.

Lemma 2.16 *(1) If $n > 3$ and $q = 2$ then the involutions are elations and generate a normal subgroup isomorphic to $SL(2,2)$ acting 2-transitively on Δ. If $G_{[\pi_o]}$ is the subgroup of G fixing π_o pointwise then $G = SL(2,2)G_{[\pi_o]}$.*

(2) If n is even then the translation plane is the union of a set of $2^n - 2$ Desarguesian partial spreads of degree 5 containing the net N_Δ and there is a regular partial 2-parallelism of $2^n - 2$ spreads induced on any elation axis.

Proof. We have seen that the involutions cannot be planar. Hence, the involutions are elations and since the group is transitive on Δ, it follows that $SL(2,2)$ is generated and clearly the group generated by all elations is normal. We note that $SL(2,2)$ is transitive on the nonzero points of π_o. This proves (1).

Now assume that n is even. By Johnson [14] (1.2), there is a rational net of degree 2^2+1 coordinatized by a field isomorphic to $GF(4)$ containing the elation net N_Δ. Moreover, the group $SL(2,2)$ fixes this net and acts transitively on the components outside of N_Δ. Since $SL(2,2)$ is normal, it follows that there is a set of $2^n - 2$ such rational nets of degree 2^2+1 containing N_Δ. Now applying the results of Jha and Johnson [8], it follows that such a spread of rational nets imply that there is a regular partial 2-parallelism on any elation axis. □

Lemma 2.17 *(1) If $q = 3$ and $n > 3$ then the 3-elements are non-planar and acting on π_o, the group is isomorphic to $SL(2,3)$.*

If $G_{[\pi_o]}$ is the subgroup fixing π_o pointwise then $SL(2,3)G_{[\pi_o]} = G$.

(2) n is even.

(3) Either the 3-elements are elations or $3|(u-1)$ for every 3-primitive divisor of $3^{n-1}-1$.

Furthermore, either the 3-elements are elations or if U is a Sylow u-subgroup where u is a 3-primitive divisor of $3^{n-1}-1$ of order u^α then there are exactly u^α Sylow 3-subgroups of $U\langle\tau\rangle$ where τ is any collineation of order 3.

(4) Let τ be an element of order 3 and let the number of 3, 2, and 1 dimensional $\tau GF(3)$-modules on the component ℓ containing $Fix\tau$ be a,b,c respectively. Let the number of 3, 2, 1 dimensional modules on π be a^*, b^*, and c^* respectively.

Then

$$\begin{aligned} 3a+2b+c &= n, c>0 \\ 3a^*+2b^*+c^* &= 2n, \\ c^* &= 0, \text{ and} \\ a+b+c &= a^*+b^* \text{ so that} \\ a^*+c-n &= a \text{ and} \\ b^*+n-2c &= b, a^* \text{ and } b^* > 0. \end{aligned}$$

Thus, if τ is not an elation then $n > 20$.

More generally, let u be the largest 3-primitive divisor of $3^{n-1}-1$ and u^α the largest u-factor. If $(3^{\min(a^*+b^*)-1}-1)u^\alpha > (3^{n-1}-1)$ where $3a^*+2b^* = 2n$, then τ is an elation.

(5) If τ is an elation then there is a set of $(3^n-3)/3$ Desarguesian nets of degree 3^2+1 containing the elation net N_Δ and inducing a regular partial 2-parallelism of $(3^n-3)/3$ 2-spreads on any elation axis.

Proof. We note that when n is odd then 8 divides $3^{n-1}-1$. Furthermore, there is a kernel involution which is not accounted for as we must have an orbit of length 3^n-3. Hence, the order of a Sylow 2-subgroup must be at least 16. Since this group must act on π_o, and the order of $\Gamma L(2,3)$ is 24, then there is an element of even order which fixes π_o pointwise from which it follows that there is a Baer involution. But, this then implies that n is even. Hence, n must be even.

Now a Sylow u-subgroup U is cyclic of order dividing $3^{n-1}-1$ for u a 3-primitive divisor. Note that U fixes π_o pointwise and $FixU = \pi_o$ by the above results of Jha.

A Sylow 3-subgroup S_3 has order 3 and fixes a unique component ℓ of π_o as otherwise, there would be a planar 3-element.

Let τ be a collineation of order 3. Since the group is solvable, there exists a subgroup of order $u^\alpha \cdot 3$ where u^α is the order of U. Since $1+ku>3$, we may assume that U is a normal subgroup within the group of order $u^\alpha \cdot 3$, or rather, that τ normalizes U.

There is a unique u-complement C of dimension $n-1$ on the component ℓ containing $Fix\tau$. First assume that τ centralizes U. Then, there are τ-fixed points in C on ℓ and U permutes the fixes point semi-regularly. Since U cannot fix a proper subspace of C, as u is 3-primitive, it follows that τ fixes C pointwise. Hence, τ fixes $C \oplus \pi_o \cap \ell$ pointwise so that τ is an elation.

Hence, either τ is an elation or τ permutes semi-regularly the generators of the cyclic group U. Hence, 3 divides $u^\alpha - u^{\alpha-1} = u^{\alpha-1}\,(u-1)$ so 3 divides $u-1$.

Thus, u divides $3^{n-1}-1$ and 3 divides $u-1$.

Now either τ is an elation or the minimal polynomial for τ is $(x-1)^3 = (x^3-1)$ (see e.g. Lüneburg [15] (47.2) and (47.5)). Let the unique component fixed by τ by M. Then M may be considered a direct sum of cyclic $\tau GF(3)$-modules of dimensions 3,2 or 1.

Hence,

$$n = 3a+2b+c$$

where a,b,c are the numbers of cyclic submodules of dimensions 3,2 or 1 respectively. Note that τ will fix exactly a 1-dimensional subspace pointwise in each cyclic submodule. Hence, $Fix\tau$ has dimension $a+b+c$ as τ is not planar.

Similarly, the vector space of dimension $2n$ is a direct sum of cyclic submodules of dimensions 3,2,1 respectively. Assume that there are a^*, b^*, and c^* cyclic submodules of dimensions 3,2,1 respectively.

Hence, we obtain

$$2n = 3a^* + 2b^* + c^*.$$

Also, we must have $a^* + b^* + c^* = a+b+c$.

In the decomposition of the vector space of dimension $2n$, assume that there is a submodule of dimension 1. In this case, there is a submodule S of dimension $2n-1$. This submodule is a hyperplane and since the spread is a dual spread, we must have that S contains a unique component of π which then must be left invariant by τ. Hence, S contains the unique component containing $Fix\tau$. However, this is a contradiction as there is a submodule of dimension 1 disjoint from S. Hence, there are no submodules of dimension 1 and we must have $c^*=0$.

The interrelationships between the a,b,c's and a^* and b^* follows directly. Note that $c>0$ since τ fixes $\pi_o \cap \ell$ pointwise and there is a unique τ invariant Maschke complement C of U and U is normalized by τ. Since there must be non-trivial 3-dimensional submodules then $a^*>0$. Since π_o is left invariant, there must be non-trivial 2-dimensional submodules so that $b^*>0$.

Now assume that $n = 4$. Then $3a+2b+c = 4$ and $3a^* + 2b^* = 8$ which implies that $a^* = 2$ and $b^* = 1$. Hence, $a^* + b^* = 3 = a+b+c$ which implies that $2a+b = 1$ so that $a = 0$ and $b = 1$ so that $c = 2$. Hence, τ fixes exactly 3^3 points on ℓ and exactly 3^2 points on C. But $u = 13$ in this situation and as $(3^2-1)13 > 3^3-1$, it follows that there exist overlaps of elements fixed by elements of the form τ^g where g is a collineation of order 13 in U. Thus, $\langle \tau, \tau^g \rangle$ fixes > 3 points on ℓ. This group is generated within $U\langle \tau \rangle$ so must involve elements of order 13. But, the elements of order 13 fixes exactly π_o pointwise.

Hence, $n \neq 4$. Since, 3 does not divide $11-1$, it follows that $n \neq 6$.

Assume that $n = 8$. Then $3a^* + 2b^* = 16$ and a^* cannot be $0, 1, 3, 4$ or 5. Hence, $a^* = 2$ and $b^* = 5$ which implies that τ fixes exactly 3^7 points on ℓ and 3^6 points on C. However, since $(3^7-1)/2 = 1093$ is a prime and $(3^6-1)1093 3^7 - 1$, the same argument implies that there are u elements fixing points other than π_o pointwise. If $n = 10$, then $757 = (3^9-1)/(3^3-1)$ is prime so $u = 757$. Since $3a^* + 2b^* = 20$, it follows that $(a^*, b^*) = (2,7), (4,4)$, or $(6,1)$ which implies that τ fixes $3^9, 3^8$ or 3^7 points on ℓ and so $3^8, 3^7$ or 3^6 points on C which is dimension $n-1 = 9$. Since $(3^6-1)757 > 3^9-1$, the previous argument again provides a contradiction.

Assume that $n = 12$. It follows that $(a^*, b^*) = (2,9), (4,6)$ or $(6,3)$ and since $(3^8-1)u > 3^9-1$ for $u > 3$, we have a contradiction as before.

One can carry one in this manner, and conceivably prove that τ is an elation in general if $(3^{\min(a^*+b^*)-1}-1)\, u^{\alpha} > (3^{n-1}-1)$ where $3a^* + 2b^* = 2n$.

Hence, for example, $(3^{(2n-1)/3}-1)u^{\alpha} > (3^{n-1}-1)$ whenever $u^{\alpha} > 3^{n/3}$.

If $n = 14$ then $3a^* + 2b^* = 28$. Hence, $(a^*, b^*) = (2,11), (4,8), (6,5), (8,2)$. Thus, $(3^9 - 1)\, u^{\alpha} > 3^{13} - 1$ provided $u^{\alpha} > 3^4$.

So, $3^{13}-1$ which when divided by 2 is 797161. But, direct calculation shows that there are no prime divisors less than 81.

Suppose $n = 16$ so that $3a^* + 2b^* = 32$ and (a^*, b^*) is $(2,13), (4,10), (6,7), (8,4), (10,1)$ so that C has at least $(310-1)u^{\alpha}$.

Now $3^{15}-1$ divided by 2 is 7174453. However, the odd part of 3^5-1 divides this number as does the odd part of 3^3-1. Hence, we may divide by $121 \cdot 13$ which is then 4561 which is prime.

Since $4561 > 3^5 - 1$, we have a contradiction.

Hence, n cannot be 16.

If $n = 18$ then $3a^* + 2b^* = 36$ which implies that a minimum for $a^* + b^* = 13$ with $(a^*, b^*) = (10,3)$. Hence, $(3^{12}-1)u^{\alpha} > (3^{17}-1)$ requires a prime power divisor $u^{\alpha} > 3^5$. Since $(3^{17}-1)/2 = 64570081$, assume that all prime divisors are less than $3^5 = 243$. A straightforward tedious calculation shows that are no prime divisors less than 243.

Hence, $n \neq 18$.

Part (5) now follows exactly as in the case when $q = 2$. □

Lemma 2.18 *Assume that $n > 4$ and $q = 4$.*

(1) Then there is a cyclic group of order 4 in $\Gamma L(2,4) - GL(2,4)$ generated by an element τ such that τ^2 is an elation. The Sylow 2-subgroups have order 4.

(2) There is a normal 2-complement

(3) π is a direct sum of n cyclic $\tau GF(2)$-submodules of dimension 4 and the unique component ℓ fixed by τ is the direct sum of n cyclic $\tau GF(2)$-submodules of dimension 2.

Hence, τ fixes exactly 2^n points on ℓ.

Proof. If the 2-group is cyclic, there is a normal 2-complement of a Sylow 2-subgroup S_2 e.g. by Gorenstein 7.6.1 [4]. We note that any 2-element must be an elation. Furthermore, note by Foulser [3] (4.3), we see that such a group does act or rather can act as maintained on π_o or there is an elation group of order at least 4 which means that $SL(2,4)$ is generated contrary to the solvablity assumption.

If there is a 2-group of order strictly larger than 4, then there is a planar 2-element, which does not occur.

So, there is a cyclic group C_4 of order 4 and since the group is solvable, there is a group of order $4 \cdot u^{\alpha}$ where u is a 2-primitive divisor of $4^{n-1} - 1$. Since we may take $u > 5$, it follows that there is a Sylow u-group which C_4 normalizes. The Sylow u-group fixes the subplane π_o pointwise so that there is a unique u-Maschke complement C on ℓ of dimension $2(n-1)$,

Decompose the vector space relative as a $\tau GF(2)$-module.

We assert that every 4 and 3 τ-submodules intersect ℓ in 2-dimensional submodules and every 2-submodule must be in ℓ as well as an 1-dimensional submodule.

To see this consider a cyclic module of dimension 4, written in the form $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and notice that the square is $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ must fix the vectors (x,y,x,y) pointwise; fix a 2-dimensional subspace pointwise which must lie in ℓ. But, τ maps (x,y,x,y) onto (y,x,y,x). That is, there is an induced 2-dimensional module on ℓ.

A cyclic submodule of dimension 3 may be written in the form $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ whose square in $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ which fixes (x,y,x) pointwise; fixes a 2-dimensional $\tau GF(2)$-submodule pointwise which must lie in ℓ. Furthermore, τ map (x,y,x) onto (y,x,y).

Any cyclic submodule of dimension 2 when squared is the identify so that this submodule must lie within ℓ.

Similarly, any 1-dimensional submodule must lie within ℓ.

However, we know that there are 1-dimensional submodule as τ fixes non-zero point on ℓ.

Thus, in any decomposition of π into a direct sum of cyclic submodules, the intersection with ℓ produces a direct sum of cyclic submodules of ℓ as ℓ is τ-invariant.

So, assume that there is a 1-dimensional submodule. Then there is a hyperplane disjoint from the 1-space which then contains a unique component and since τ fixes exactly one component, it follows that the component must be ℓ. However, there are no fixed points not in ℓ. Hence, there are no 1-dimensional submodules.

We know that τ is a generalized elation on ℓ and hence fixes pointwise a subspace of dimension n over $GF(2)$.

Now let the number of 2 and 1 modules $\tau GF(2)$-submodules, where τ has order 4, on ℓ be a and b so that $2a+b=2n$ as τ^2 is an elation. Let the number of 4,3,2 modules on π be a^*, b^*, c^*, respectively so that:

$$4a^* + 3b^* + 2c^* = 4n.$$

Since τ^2 is an elation, it follows that

$$2a^* + 2b^* + 2c^* = 2n.$$

Hence, we must have

$$2a^* + b^* = 2n$$

which, in turn, implies that $b^* + 2c^* = 0$ so that $b^* = c^* = 0$.

Hence, $4a^* = 4n$ so that $a^* = n$. Since ℓ intersects every 4-dimensional submodule in a 2-dimensional submodule, this implies that $a = a^*$ and every module on ℓ is 2-dimensional.

We may assume that u is a 2-primitive divisor larger than 5. Hence, any u-group is normal since the Sylow 2-subgroups have order 4.

This completes the proof of the main result. The proof of the corollary is now immediate. □

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