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# THE SUBMANIFOLDS $X_m$ OF THE MANIFOLD $*g-MEX_n$ I. THE INDUCED CONNECTION ON $X_m$ OF $*g-MEX_n$

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**Abstract.** *An Einstein's connection which takes the form (2.33) is called an $^*g$-ME-connection. Recently, Chung and et al ([15],1993) introduced a new manifold, called an n-dimensional $^*g$-ME-manifold(denoted by $^*g$-$MEX_n$). The manifold $^*g-MEX_n$ is a generalized n-dimensional Riemannian manifold $X_n$ on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}m$ satisfying certain conditions through the $^*g-$ME-connection. In the following series of two papers, we investigate the submanifold $X_m$ of $^*g-MEX_n$:*

*I. The induced connection on $X_m$ of $^*g$-$MEX_n$*

*II. The generalized fundamental equations on $X_m$ of $^*g$-$MEX_n$*

*In this paper, Part I of the series, we present a brief introduction of n-dimensional $^*g$-unified field theory, the C-nonholonomic frame of reference in $X_n$ at points of $X_m$, and the manifold $^*g$-$MEX_n$. And then, we introduce the generalized coefficients of the second fundamental form of $X_m$ and prove a necessary and sufficient condition for the induced connection on $X_m$ of $^*g$-$MEX_n$ to be a $^*g$-ME-connection. Our subsequent paper, Part II of the series, deals with the generalized fundamental equations on $X_m$ of $^*$-$MEX_n$, such as the generalized Gauss formulae, the generalized Weingarten equations, and the Gauss-Codazzi equations.*

## 1 Introduction

In Appendix II to his last book Einstein ([18],1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates in $X_4$, Hlavatý ([19],1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical consequences of these postulates.

Generalizing $X_4$ to $n$-dimensional generalized Riemannian manifold $X_n$, $n$-dimensional generalization of this theory, so called *Einstein's n-dimensional unified field theory* (denoted by $n$-$g$-UFT hereafter), had been attempted by Wrede ([23],1958) and Mishra ([21], 1959). On the other hand, corresponding to $n-g-$UFT, Chung ([1], 1963) introduced a new unified field theory, called *Einstein's n-dimensional $^*g$-unified field theory* (denoted by $n-^*g$-UFT hereafter). This theory is more useful than $n-g$-UFT in some physical aspects. Chung and et al obtained many results concerning this theory ([2]-[5], 1968-1983; [4],1981; [9],1988; [16][17];1998), particulary proving that $n-^*g$-UFT is equivalent to $n-g$-UFT so far as the classes and indices of inertia are concerned ([6],1985).

Recently, Chung and et al ([7],1987) introduced a very interesting manifold, called $n$-dimensional SE-manifold (denoted by SE$X_n$ hereafter), imposing the semi-symmetric condition to the Einstein's connection of $X_n$, and displayed a unique representation of the $n$-

dimensional Einstein's connection in a beautiful and surveyable form in terms of $g_{\lambda\mu}$. Many results concerning SEX$_n$ have been obtained since then ([8],1988; [10]-[14],1989-1991).

An Einstein's connection which takes the form (2.33) is called a $^*g$-ME-connection. Recently, Chung and et al ([15],1993) introduced a new manifold, called an $n$-dimensional $^*g$-ME-manifold (denoted by $^*g$-MEX$_n$). The manifold $^*g$-MEX$_n$ is a generaliwed $n$ dimensional Riemannian manifold $X_n$, on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$ satisfying the present conditions through the $^*g-$MEX$_n$-connection (see above Definition (2.11) for the words "the present conditions"). In the following series of two papers, we investigate the submanifolds $X_m$ of $^*g$-MEX$_n$:

I. The induced connection on $X_m$ of $^*g$-MEX$_n$

II. The generalized fundamental equations on $X_m$ of $^*g$-MEX$_n$

In this paper, Part I of the series, we present a brief introduction of $n$-dimensional $^*g$-unified field theory, the C-nonholonomic frame of reference in $X_n$ at points of $X_m$, and the manifold $^*g$-MEX$_n$. And then, we introduce the generalized coefficients of the second fundamental form of $X_m$ and prove a necessary and sufficient condition for the induced connection on $X_m$ of $^*g$-MEX$_n$ to be a $^*g$-ME-connection. Our subsequent paper, Part II of the series, deals with the generalized fundamental equations on $X_m$ of $^*g$-MEX$_n$, such as the generalized Gauss formulae, the generalized Weingarten equations, and the Gauss-Codazzi equations.

## 2 Preliminaries

This section is a brief collection of basic concepts, notations, and results, which are needed in our subsequent considerations. It consists of three subsections; the first subsection (a) is mostly due to [1], the second subsection (b) due to [10], and the third subsection (c) due to [15].

**(a) $n$-dimensional $^*g$-unified field theory.** Corresponding to the Einstein's $n$-$g$-UFT[1], our $n-^*g$-UFT, initiated by Chung ([1], 1963), is based on the following three principles.

*Principle A.* Let $X_n$ be an $n$-dimensional generalized Riemannian manifold referred to a real coordinate system $x^\nu$, which obeys the coordinate transformation $x^\nu \to x^{\nu'}$ for which

$$\det(\frac{\partial x'}{\partial x}) \neq 0 \qquad 2.1$$

In $n$-$g$-UFT the manifold $X_n$ is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ [2]

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \qquad 2.2a$$

where

$$\mathfrak{g} = det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = det(h_{\lambda\mu}) \neq 0 \qquad 2.2b$$

[1]Hlavatý characterized Einstein's 4-dimensional unified field theory 4-$g$-UFT as a set of geometrical postulates in $X_4$ for the first time [19] and gave its mathematical foundation.

[2]Throughout the present paper, Greek indices are used for the holonomic components of tensors in $X_n$. They take the values $1,2,\ldots,$ and follow the summation convention. We also assume that $n > 1$ in this paper.

In $n$-$^*g$-UFT the algebraic structure on $X_n$ is imposed by the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}\,^*g^{\lambda\nu} \stackrel{\text{def}}{=} g_{\mu\lambda}\,^*g^{\nu\lambda} = \delta_\mu^\nu \tag{2.3}$$

It may be also decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$ :

$$^*g^{\lambda\nu} = ^* h^{\lambda\nu} + ^* k^{\lambda\nu} \tag{2.4}$$

Since $det(^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}\,^*h^{\lambda\nu} \stackrel{\text{def}}{=} \delta_\mu^\nu \tag{2.5}$$

*In $n$-$^*g$-UFT we use both $^*h^{\lambda\nu}$ and $^*h_{\lambda\mu}$ as tensors for raising and/or lowering indices of all tensors defined in $X_n$ in the usual manner.* We then have

$$^*k_{\lambda\mu} = {^*k^{\rho\sigma}}\,^*h_{\lambda\rho}\,^*h_{\mu\sigma}, \quad ^*g_{\lambda\mu} = {^*g^{\rho\sigma}}\,^*h_{\lambda\rho}\,^*h_{\mu\sigma} \tag{2.6a}$$

so that

$$^*g_{\lambda\mu} = ^* h_{\lambda\mu} + ^* k_{\lambda\mu} \tag{2.6b}$$

*Principle B.* The differential geometric structure on $X_n$ is imposed by the tensor $^*g^{\lambda\nu}$ by means of a connection $\Gamma_{\lambda\mu}^\nu$ defined by a system of equations [3]

$$D_\omega\,^*g^{\lambda\mu} = -2S_{\omega\alpha}{}^\mu\,^*g^{\lambda\alpha} \tag{2.7a}$$

Here $D_\omega$ denotes the symbol of the covariant derivative with respect to $\Gamma_{\lambda\mu}^\nu$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma_{\lambda\mu}^\nu$. Under certain conditions the system (2.7) admits a unique solution $\Gamma_{\lambda\mu}^\nu$. A connection satisfying (2.7a) is called an *Einstein's connection* in $n$-$^*g$-UFT.

*Principle C.* In order to obtain $^*g^{\lambda\nu}$ involved in the solution for $\Gamma_{\lambda\mu}^\nu$ certain conditions are imposed. These conditions may be condensed to

$$S_\lambda \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^\alpha = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu}Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0 \tag{2.8}$$

where $Y_\lambda$ is an arbitrary vector, and $R_{\omega\mu\lambda}{}^\nu$ together with $R_{\mu\lambda}$ and $V_{\omega\mu}$ are the curvature tensors of $X_n$ defined by

$$R_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]}) \tag{2.9}$$

$$R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha, \quad V_{\omega\mu} \stackrel{\text{def}}{=} R_{\omega\mu\alpha}{}^\alpha \tag{2.10}$$

In the following remark, we summarize the main differences between $n$-$g$-UFT and $n$-$^*g$-UFT.

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[3]Hlavatý ([19]) proved that system (2.7a) is equivalent to

$$D_\omega\, g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha\, g_{\lambda\alpha} \tag{2.7b}$$

which is also equivalent to the original Einstein's equations

$$\partial_\omega\, g_{\lambda\mu} - \Gamma_{\lambda\omega}^\alpha\, g_{\alpha\mu} - \Gamma_{\omega\mu}^\alpha\, g_{\lambda\alpha} = 0 \tag{2.7c}$$

**Remark 1** *In* $\left\{\begin{array}{l} n-g-UFT \\ n-{}^*g-UFT \end{array}\right.$, *the algebraic structure on* $X_n$ *is imposed by the tensor* $\left\{\begin{array}{l} g_{\lambda\mu} \\ {}^*g^{\lambda\nu} \end{array}\right.$, *and* $\left\{\begin{array}{l} \text{the tensor } h_{\lambda\mu} \text{ and its inverse tensor } h^{\lambda\nu} \\ \text{the tensor } {}^*h^{\lambda\nu} \text{ and its inverse tensor } {}^*h_{\lambda\mu} \end{array}\right.$ *are used for raising and/or lowering the indices of tensors in* $X_n$. *On the other hand, the differential geometric structure on* $X_n$ *is imposed by* $\left\{\begin{array}{l} g_{\lambda\mu} \text{ in } n-g-UFT \\ {}^*g^{\lambda\nu} \text{in } n-{}^*g-UFT \end{array}\right.$ *through the Einstein's connection* $\Gamma^\nu_{\lambda\mu}$ *satisfying* $\left\{\begin{array}{l} (2.7b) \\ (2.7a) \end{array}\right.$. *Therefore, if the system* $\left\{\begin{array}{l} (2.7b) \\ (2.7a) \end{array}\right.$ *admits a unique solution, the connection* $\Gamma^\nu_{\lambda\mu}$ *will be expressed in terms of* $\left\{\begin{array}{l} g_{\lambda\mu} \text{ in } n-g-UFT \\ {}^*g^{\lambda\nu} \text{ in } n-{}^*g-UFT \end{array}\right.$ *in virtue of* $\left\{\begin{array}{l} (2.7b) \\ (2.7a) \end{array}\right.$.

The following quantities are frequently used in our further considerations:

$$ {}^*\mathfrak{g} = det({}^*g_{\lambda\mu}), \quad {}^*\mathfrak{h} = det({}^*h_{\lambda\mu}), \quad {}^*\mathfrak{k} = det({}^*k_{\lambda\mu}) \qquad 2.11a $$

$$ {}^*g = \frac{{}^*\mathfrak{g}}{{}^*\mathfrak{h}}, \quad {}^*k = \frac{{}^*\mathfrak{k}}{{}^*\mathfrak{h}} \qquad 2.11b $$

$$ \sigma = \left\{\begin{array}{ll} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{array}\right. \qquad 2.11c $$

$$ K_p = {}^*k_{[\alpha_1}{}^{\alpha_1}\, {}^*k_{\alpha_2}{}^{\alpha_2} \cdots {}^*k_{\alpha_p]}{}^{\alpha_p}, \qquad (p = 0, 1, 2, \cdots) \qquad 2.11d $$

$$ {}^{(0)*}k_\lambda{}^\nu = \delta^\nu_\lambda, \quad {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha\ {}^{(p-1)*}k_\alpha{}^\nu, \qquad (p = 1, 2, \cdots) \qquad 2.11e $$

Using these notations we may prove the following two theorems.

**Theorem 2** *The following relations hold in* $X_n$ *([4]):*

$$ {}^{(p)*}k_{\lambda\mu} = (-1)^p\, {}^{(p)*}k_{\mu\lambda}, \qquad (p = 0, 1, 2, \cdots) \qquad 2.12a $$

$$ K_0 = 1, \quad K_n = {}^*k \text{ if n is even}, \quad \text{and} \quad K_p = 0 \quad \text{if p is odd} \qquad 2.12b $$

$$ {}^*g = \sum_{s=0}^{n-\sigma} K_s \qquad 2.12c $$

$$ \sum_{s=0}^{n-\sigma} K_s\, {}^{(n-s)*}k_\lambda{}^\mu = 0 \qquad 2.12d $$

*Here and in what follows, the index s is assumed to take the values* $0, 2, 4, \cdots$ *in the specified range.*

**Theorem 3** *If the system (2.7) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form ([1])*

$$\Gamma^{\nu}_{\lambda\mu} = {}^*\{{}^{\ \nu}_{\lambda\mu}\} + S_{\lambda\mu}{}^{\nu} + {}^*U^{\nu}{}_{\lambda\mu} \qquad 2.13$$

*where ${}^*\{{}^{\ \nu}_{\lambda\mu}\}$ are the Christoffel symbols defined by ${}^*h_{\lambda\mu}$ and*

$${}^*U^{\nu}{}_{\lambda\mu} = S_{\beta(\lambda}{}^{\nu}\, {}^*k_{\mu)}{}^{\beta} + S^{\nu}{}_{\beta(\lambda}\, {}^*k_{\mu)}{}^{\beta} - S^{\beta}{}_{(\lambda\mu)}\, {}^*k_{\beta}{}^{\nu} \qquad 2.14$$

### (b) The C-nonholonomic frame of reference in $n$-${}^*g$-UFT.

This subsection deals with a brief introduction of the concept, *the C-nonholonomic frame of reference in $X_n$ at points of its submanifold $X_m$, $m < n$*, in $n$-${}^*g$-UFT. It is based on the symbols and results of [15].

**Agreement 4** *In our further considerations in the present paper, we use the following types of indices:*

*(a) Small Greek indices $\alpha, \beta, \gamma, \cdots$, running from 1 to n and used for the holonomic components of tensors in $X_n$.*

*(b) Capital Roman indices $A, B, C, \cdots$, running from 1 to n and used for the C-nonholonomic components of tensors in $X_n$ at points of $X_m$.*

*(c) Small Roman italic indices $i, j, k, \cdots$ with the exception of $x, y$ and $z$, running from 1 to $m(< n)$.*

*(d) Small Roman italic indices $x, y$ and $z$, running from $m+1$ to $n$.*

*The summation convention is operative with respect to each set of the above indices within their range, with the exception of $x, y$ and $z$.*

Let $X_m$ be a submanifold of $X_n$ defined by a system of sufficiently differentiable equations

$$y^{\nu} = y^{\nu}(x^1, \cdots, x^m) \qquad 2.15$$

where the matrix of derivatives $B^{\nu}_i = \frac{\partial y^{\nu}}{\partial x^i}$ is of rank $m$. At each point of $X_m$ there exists *the first set* $\{B^{\nu}_i, N^{\nu}_x\}$ of $n$ linearly independent non-null vectors. The $m$ vectors $B^{\nu}_i$ are tangential to $X_m$, and $n-m$ vectors $N^{\nu}_x$ are normals to $X_m$ and mutually orthogonal. That is,

$${}^*h_{\alpha\beta}\, B^{\alpha}_i\, N^{\beta}_x = 0, \quad {}^*h_{\alpha\beta}\, N^{\alpha}_x\, N^{\beta}_y = 0 \quad \text{for} \quad x \neq y \qquad 2.16a$$

The process of determining the set $\{N^{\nu}_x\}$ is not unique unless $m = n-1$. However, we may choose their magnitudes such that

$${}^*h_{\alpha\beta}\, N^{\alpha}_x\, N^{\beta}_x = \varepsilon_x \qquad 2.16b$$

where $\varepsilon_x = +1$ or $-1$ according as the left-hand sides of (2.16b) is positive or negative. Put

$$E^{\nu}_A = \begin{cases} B^{\nu}_i, & \text{if} \quad A = i = 1, \cdots, m \\ N^{\nu}_x, & \text{if} \quad A = x = m+1, \cdots, n \end{cases} \qquad 2.17$$

Corresponding to the first set $\{E_A^\nu\}$ of $n$ linearly independent vectors, there exists a *unique second set* $\{E_\nu^A\}$ of linearly independent vectors at points of $X_m$ such that

$$E_\lambda^A E_A^\nu = \delta_\lambda^\nu, \quad E_\alpha^A E_B^\alpha = \delta_B^A \tag{2.18}$$

Putting

$$E_\lambda^A = \begin{cases} B_\lambda^i, & \text{if } A = i = 1, \cdots, m \\ N_\lambda^x, & \text{if } A = x = m+1, \cdots, n \end{cases} \tag{2.19}$$

we note that the vectors $B_\lambda^i$ and $N_\lambda^x$ are also tangential and normal respectively to $X_m$ in virtue of Theorem (2.6).

Now, we are ready to introduce the following concept of C-nonholonomic frame of reference and induced tensors.

**Definition 5** *The set $E_A^\nu$ and $E_\nu^A$ will be referred to as the C-nonholonomic frame of reference in $X_n$ at points of $X_m$. This frame gives rise to C-nonholonomic components of tensors in $X_n$: if $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor in $X_n$, then at points of $X_m$ its C-nonholonomic components $T_{B\cdots}^{A\cdots}$ are defined by*

$$T_{B\cdots}^{A\cdots} = T_{\beta\cdots}^{\alpha\cdots} E_\alpha^A \cdots E_B^\beta \cdots \tag{2.20a}$$

*In particular, the quantities*

$$T_{j\cdots}^{i\cdots} = T_{\beta\cdots}^{\alpha\cdots} B_\alpha^i \cdots B_j^\beta \cdots \tag{2.20b}$$

*are components of a tensor in $X_m$ and are called the components of the induced tensor of $T_{\lambda\cdots}^{\nu\cdots}$ on $X_m$ of $X_n$.*

In virtue of (2.18), an easy inspection shows that

$$T_{\lambda\cdots}^{\nu\cdots} = T_{B\cdots}^{A\cdots} E_A^\nu \cdots E_\lambda^B \cdots \tag{2.21}$$

The following theorems are consequences of the powerful C-nonholonomic frame of reference.

**Theorem 6** *The tensors $B_i^\nu, B_\lambda^i, N_x^\nu, N_\lambda^x$, and*

$$B_\lambda^\nu = B_\lambda^i B_i^\nu \tag{2.22}$$

*are involved in the following identities:*

$$B_\alpha^i B_j^\alpha = \delta_j^i, \quad N_\alpha^x N_y^\alpha = \delta_y^x, \quad B_\alpha^i N_x^\alpha = N_\alpha^x B_i^\alpha = 0 \tag{2.23}$$

$$B_\lambda^i = B_j^\alpha \, {}^*h_{\lambda\alpha} \, {}^*h^{ij} \tag{2.24a}$$

$$B_i^\nu = B_\alpha^j \, {}^*h^{\nu\alpha} \, {}^*h_{ij} \tag{2.24b}$$

$$^*h^{\nu\alpha}\,B^i_\alpha = {}^*h^{ij}\,B^\nu_j, \quad {}^*h_{\lambda\alpha}\,B^\alpha_i = {}^*h_{ij}\,B^j_\lambda \qquad 2.25$$

$$B^\nu_\lambda = \delta^\nu_\lambda - \sum_x N^x_\lambda\,N^\nu_x \qquad 2.26a$$

$$B^\alpha_\lambda\,N^x_\alpha = B^\nu_\alpha\,N^\alpha_x = 0 \qquad 2.26b$$

$$B^\alpha_\lambda\,B^i_\alpha = B^i_\lambda, \quad B^\nu_\alpha\,B^\alpha_i = B^\nu_i, \quad B^\nu_\alpha\,B^\alpha_\lambda = B^\nu_\lambda \qquad 2.26c$$

**Theorem 7** *At each points of $X_m$, a vector $X_\lambda$ of $X_n$ may be expressed as the sum of two vectors $X_i\,B^i_\lambda$ and $\sum_x X_x\,N^x_\lambda$, the former tangential to $X_m$ and the latter normal to $X_m$. That is*

$$X_\lambda = X_i\,B^i_\lambda + \sum_x X_x\,N^x_\lambda \qquad 2.27a$$

*or equivalently*

$$X^\nu = X^i\,B^\nu_i + \sum_x X^x\,N^\nu_x \qquad 2.27b$$

*where*

$$X_i = X_\alpha\,B^\alpha_i, \quad X_x = X_\alpha\,N^\alpha_x, \quad X_x = \varepsilon_x\,X^x \qquad 2.28a$$

$$X^i = X^\alpha\,B^i_\alpha, \quad X^x = X^\alpha\,N^x_\alpha \qquad 2.28b$$

*Furthermore, $X_i(X^i)$ are components of a tangent vector relative to the transformations of $X_m$, while $X_x(X^x)$ is invariant relative to the transformations of $X_m$ and $X_n$.*

**Theorem 8** *The induced tensor $^*g_{ij}$ of $^*g_{\lambda\mu}$ may be given by*

$$^*g_{ij} = {}^*g_{\alpha\beta}\,B^\alpha_i\,B^\beta_j \qquad 2.29a$$

*where its symmetric part $^*h_{ij}$ and skew-symmetric part $^*k_{ij}$ are*

$$^*h_{ij} = {}^*h_{\alpha\beta}\,B^\alpha_i\,B^\beta_j, \quad {}^*k_{ij} = {}^*k_{\alpha\beta}\,B^\alpha_i\,B^\beta_j \qquad 2.29b$$

*so that*

$$^*g_{ij} = {}^*h_{ij} + {}^*k_{ij} \qquad 2.30$$

*In this paper, we restrict our considerations to submanifolds for which the following condition holds:*

$$Det({}^*h_{ij}) \neq 0 \qquad 2.31$$

In virtue of the condition (2.31), we may define a unique inverse tensor $^*\bar{h}^{ik}$ of $^*h_{ij}$ by

$$^*h_{ij}\,{}^*\bar{h}^{ik} = \delta^k_j \qquad 2.32$$

It has been shown that $^*\bar{h}^{ik}$ is the induced tensor $^*h^{ik}$ of $^*h^{\lambda\nu}$. That is, $^*\bar{h}^{ik} = {}^*h^{ik}$. Therefore, *the tensors $^*h_{ij}$ and $^*h^{ij}$ may be used for raising and/or lowering indices of the induced tensors in $X_m$ in the usual manner.*

**(c) The manifold $^*g$-MEX$_n$ in $n$-$^*g$-UFT.** All results and symbols in this subsection are based on [15].

**Definition 9** *An Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ of the form*

$$\Gamma^{\nu}_{\lambda\mu} = {}^*\{{}^{\ \nu}_{\lambda\mu}\} + 2\delta^{\nu}_{\lambda} X_{\mu} - 2{}^*g_{\lambda\mu} X^{\nu} \tag{2.33}$$

*for a non-null vector $X_{\lambda}$ is called a ${}^*g$-ME-connection in n-${}^*g$-UFT, and $X_{\lambda}$ the corresponding ${}^*g$-ME-vector.*

In the following theorem, we need the tensor $A_{\lambda\mu}$ defined by

$$A_{\lambda\mu} \stackrel{\text{def}}{=} -n^*g_{\lambda\mu} + {}^*g_{\mu\lambda} \tag{2.34}$$

Since this tensor is of rank $n$, there exists a unique tensor $B^{\lambda\nu}$ satisfying

$$A_{\lambda\mu} B^{\lambda\nu} = A_{\mu\lambda} B^{\nu\lambda} = \delta^{\nu}_{\mu} \tag{2.35}$$

**Theorem 10** *(a) If $X_n$ admits a ${}^*g$-ME-connection $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form (2.13), where*

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda} X_{\mu]} - 2\,{}^*k_{\lambda\mu} X^{\nu}, \quad {}^*U^{\nu}{}_{\lambda\mu} = 2\delta^{\nu}_{(\lambda} X_{\mu)} - 2\,{}^*h_{\lambda\mu} X^{\nu} \tag{2.36}$$

*(b) A necessary and sufficient condition for the system (2.7a) to admit exactly one ${}^*g$-ME-connection $\Gamma^{\nu}_{\lambda\mu}$ of the form (2.33) is that the tensor field ${}^*g^{\lambda\nu}$ satisfies the following condition*

$$\nabla_{\omega}{}^*k_{\lambda\mu} = 2({}^*h_{\omega[\lambda}\,{}^*g_{\mu]\beta} - {}^*h_{\omega\beta}\,{}^*k_{\lambda\mu}) C_{\alpha} B^{\alpha\beta} \tag{2.37}$$

*If this condition is satisfied, then*

$$X^{\nu} = C_{\alpha} B^{\alpha\nu} \tag{2.38}$$

*where*

$$C_{\lambda} = \nabla_{\alpha}{}^*k_{\lambda}{}^{\alpha} \tag{2.39}$$

Hence, if (2.37) is satisfied, there always exists a unique ${}^*g$-ME-connection $\Gamma^{\nu}_{\lambda\mu}$ in our $n$-${}^*g$-UFT. In virtue of (2.33) and (2.38), this connection may be written as

$$\Gamma^{\nu}_{\lambda\mu} = {}^*\{{}^{\ \nu}_{\lambda\mu}\} + 2(\delta^{\nu}_{\lambda}\,{}^*h_{\mu\beta} - {}^*g_{\lambda\mu}\,\delta^{\nu}_{\beta}) C_{\alpha} B^{\alpha\beta} \tag{2.40}$$

*The situation that the conditions (2.31) and (2.37) are imposed on the unified field tensor ${}^*g^{\lambda\nu}$ are described in this paper by the words "under the present conditions".*

**Definition 11** *An n-dimensional generalized Riemannian manifold $X_n$, on which the differential geometric structure is imposed by the tensor ${}^*g^{\lambda\nu}$ under the present conditions by means of the unique ${}^*g$−ME-connection given by (2.40), is called an n-dimensional ${}^*g$-ME-manifold and denoted by ${}^*g$-MEX$_n$.*

## 3 The induced connection on $X_m$ of $^*g$-MEX$_n$

This section is devoted to the investigations of the induced connection of the $^*g$-ME-connection imposed on a submanifold $X_m$ of $^*g$-MEX$_n$ together with the generalized coefficients $\Omega^x_{ij}$ of the second fundamental form of $X_m$ with emphasis on the proof of Theorem 17, in which we prove a necessary and sufficient condition for the induced connection of $X_m$ in $^*g - MEX_n$ to be a $^*g - ME$-connection. The convenient and powerful $C$-nonholonomic frame of reference in $^*g - MEX_n$ at points of $X_m$ will be employed throughout the present section. Particularly, we note in virtue of Definition 11 that under the present conditions the $^*g - ME$-connection of a given $^*g - MEX_n$ is unique.

**Definition 12** *If $\Gamma^\nu_{\lambda\mu}$ is a connection on a general $X_n$, the connection $\Gamma^k_{ij}$ defined by*

$$\Gamma^k_{ij} = B^k_\gamma(B^\gamma_{ij} + \Gamma^\gamma_{\alpha\beta} B^\alpha_i B^\beta_j), \quad B^\gamma_{ij} = \frac{\partial B^\gamma_i}{\partial x^j} = \frac{\partial^2 y^\gamma}{\partial x^i \partial y^j} \tag{3.1}$$

*is called the induced connection of $\Gamma^\nu_{\lambda\mu}$ on $X_m$ of $X_n$.*

The following Theorem is an immediate consequence of Definition (3.1).

**Theorem 13** *(a) The torsion tensor $S_{ij}{}^k$ of the induced connection $\Gamma^k_{ij}$ is the induced tensor of the torsion tensor $S_{\lambda\mu}{}^\nu$ of $\Gamma^\nu_{\lambda\mu}$. That is*

$$S_{ij}{}^k = S_{\alpha\beta}{}^\gamma B^\alpha_i B^\beta_j B^k_\gamma \qquad 3.2$$

*(b) The induced connection $^*\{^{\,k}_{ij}\}$ of $^*\{^{\,\nu}_{\lambda\mu}\}$ is the Christoffel symbols defined by $^*h_{ij}$. That is,*

$$^*\{^{\,k}_{ij}\} = \tfrac{1}{2}{}^*h^{kp}(\partial_i{}^*h_{jp} + \partial_j{}^*h_{ip} - \partial_p{}^*h_{ij}) \qquad 3.3$$

*Proof.* The statement (a) is a direct consequence of (3.1). Using (2.5), (2.21), (2.23), (2.25), and (2.27b), the statement (b) may be proved as in the following way:

The right-hand side of (3.3)=

$$\begin{aligned}
&= \tfrac{1}{2}{}^*h^{kp}\left[B^\alpha_i\,\partial_\alpha({}^*h_{\beta\varepsilon} B^\beta_j B^\varepsilon_p) + B^\beta_j\,\partial_\beta({}^*h_{\alpha\varepsilon} B^\alpha_i B^\varepsilon_p) - B^\varepsilon_p\,\partial_\varepsilon({}^*h_{\alpha\beta} B^\alpha_i B^\beta_j)\right]\\
&= \tfrac{1}{2}({}^*h^{kp} B^\varepsilon_p)(\partial_\alpha{}^*h_{\beta\varepsilon} + \partial_\beta{}^*h_{\alpha\varepsilon} - \partial_\varepsilon{}^*h_{\alpha\beta})B^\alpha_i B^\beta_j + {}^*h_{\alpha\varepsilon} B^\alpha_{ij}(B^\varepsilon_p\,{}^*h^{kp})\\
&= B^k_\gamma({}^*\{^{\,\gamma}_{\alpha\beta}\}B^\alpha_i B^\beta_j + B^\gamma_{ij}) = {}^*\{^{\,k}_{ij}\}
\end{aligned}$$

**Theorem 14** *The vector $D^\circ_j B^\alpha_i$ in $X_n$ is normal to $X_m$ and may be given by*

$$D^\circ_j B^\alpha_i = -\sum_x \Omega^x_{ij} N^\alpha_x \qquad 3.4$$

*where $D^\circ_j$ is the symbolic vector of the generalized covariant derivative with respect to $x'$s. Hence*

$$\Omega^x_{ij} = -(D^\circ_j B^\alpha_i) N^x_\alpha \qquad 3.5$$

*Proof.* In virtue of (2.23) and (3.1), multiplication of $B^m_\alpha$ to both sides of

$$D^\circ_j B^\alpha_i = B^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} B^\beta_i B^\gamma_j - \Gamma^k_{ij} B^\alpha_k \tag{3.6}$$

gives $(D^\circ_j B^\alpha_i) B^m_\alpha = 0$. This proves that $D^\circ_j B^\alpha_i$ is normal to $X_m$, and hence we have (3.4). The relation (3.5) follows from (3.4) in virtue of (2.23).

The tensors $\Omega^x_{ij}$ will be called *the generalized coefficients of the second fundamental form of* $X_m$.

**Theorem 15** *The coefficients $\Omega^x_{ij}$ have the following representations:*
*(a) The tensor $\Omega^x_{ij}$ is the induced tensor of $D^\circ_\beta N^x_\alpha$ on $X_m$ of $X_n$. That is,*

$$\Omega^x_{ij} = (D^\circ_\beta N^x_\alpha) B^\alpha_i B^\beta_j \tag{3.7}$$

*(b) On $X_m$ of $^*g$-$MEX_n$, the coefficients $\Omega^x_{ij}$ may be given by*

$$\Omega^x_{ij} = \Lambda^x_{ij} - 2\varepsilon_x X_x {}^*g_{ij} \tag{3.8}$$

*where*

$$\Lambda^x_{ij} = (\nabla_\beta N^x_\alpha) B^\alpha_i B^\beta_j \tag{3.9}$$

*are the generalized coefficients of the second fundamental form with respect to $^*\{{}^{\nu}_{\lambda\mu}\}$. Here $\nabla_\beta$ denotes the symbolic vector of the covariant derivative with respect to $^*\{{}^{\nu}_{\lambda\mu}\}$.*

*Proof.* In virtue of (2.23), we first note that

$$0 = \partial_j (B^\alpha_i N^x_\alpha) = B^\alpha_{ij} N^x_\alpha + (\partial_\beta N^x_\alpha) B^\alpha_i B^\beta_j \tag{3.10}$$

Using (3.5), (3.6) and (3.10), our assertion (3.7) follows as in the following way:

$$\begin{aligned} \Omega^x_{ij} &= -(B^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} B^\beta_i B^\gamma_j - \Gamma^k_{ij} B^\alpha_k) N^x_\alpha \\ &= -B^\alpha_{ij} N^x_\alpha - \Gamma^\alpha_{\beta\gamma} N^x_\alpha B^\beta_i B^\gamma_j \\ &= (D_\beta N^x_\alpha) B^\alpha_i B^\beta_j \end{aligned}$$

On the other hand, making use of (2.33), (3.9), (3.4), (2.28a) and (2.29a), the representation (3.8) may be obtained from (3.7) as:

$$\begin{aligned} \Omega^x_{ij} &= (\partial_\beta N^x_\alpha + \Gamma^\gamma_{\alpha\beta} N^x_\gamma) B^\alpha_i B^\beta_j \\ &= \left[\partial_\beta N^x_\alpha + ({}^*\{{}^{\gamma}_{\alpha\beta}\} + 2\delta^\gamma_\alpha X_\beta - 2\,{}^*g_{\alpha\beta} X^\gamma) N^x_\gamma\right] B^\alpha_i B^\beta_j \\ &= \Lambda^x_{ij} - 2\,{}^*g_{ij} (X^\gamma N^x_\gamma) \\ &= \Lambda^x_{ij} - 2\varepsilon_x X_x {}^*g_{ij} \end{aligned}$$

In virtue of Theorem 15, we note that the coefficients $\Lambda^x_{ij}$ are symmetric, while the coefficients $\Omega^x_{ij}$ are not.

Now, we are ready to prove the following two important theorems.

**Theorem 16** *On $X_m$ of $^*g$-$MEX_n$, the induced connection $\Gamma^k_{ij}$ of the $^*g$- ME-connection $\Gamma^\nu_{\lambda\mu}$ is of the form*

$$\Gamma^k_{ij} = {}^*\{{}^{\,k}_{ij}\} + 2\delta^k_i X_j - 2\,{}^*g_{ij} X^k \tag{3.11}$$

*where $X_i$ is the induced vector of $X_\lambda$.*

*Proof.* Substituting (2.33) into (3.1) and making use of (2.21), the representation (3.11) may be obtained as in the following way:

$$\begin{aligned}
\Gamma^k_{ij} &= B^k_\gamma \left[ B^\gamma_{ij} + ({}^*\{{}^{\,\gamma}_{\alpha\beta}\} + 2\delta^\gamma_\alpha X_\beta - 2\,{}^*g_{\alpha\beta} X^\gamma) B^\alpha_i B^\beta_j \right] \\
&= B^k_\gamma (B^\gamma_{ij} + {}^*\{{}^{\,\gamma}_{\alpha\beta}\} B^\alpha_i B^\beta_j) + 2(\delta^\gamma_\alpha B^k_\gamma B^\alpha_i)(X_\beta B^\beta_j) - \\
&\qquad\qquad - 2({}^*g_{\alpha\beta} B^\alpha_i B^\beta_j)(X^\gamma B^k_\gamma) \\
&= {}^*\{{}^{\,k}_{ij}\} + 2\delta^k_i X_j - 2\,{}^*g_{ij} X^k
\end{aligned}$$

**Theorem 17** *On $X_m$ of $^*g$-$MEX_n$, the induced connection $\Gamma^k_{ij}$ of the unique $^*g$-ME-connection $\Gamma^\nu_{\lambda\mu}$ is a $^*g$-ME-connection if and only if the following conditions hold:*

$$\delta^{(i}_k X^{j)} - {}^*h^{ij} X_k + \delta^{(j}_k {}^*k_h{}^{i)} X^h - 2\,{}^{(2)*}k_k{}^{(i} X^{j)} = 0 \tag{3.12a}$$

$$\nabla_k {}^*k^{ij} = 2(\delta^{[j}_k {}^*k_h{}^{i]} X^h - {}^*k^{ij} X_k + \delta^{[i}_k X^{j]}) \tag{3.12b}$$

*Proof.* In virtue of Theorem 16, we first note that on $X_m$ of $^*g$-$\mathrm{MEX}_n$ the induced connection of the $^*g$-ME-connection is of the form (3.11). Suppose that it is an Einstein's connection on $X_m$. Then, in virtue of (2.7)a, we have

$$D_k {}^*g^{ij} = -2S_{kh}{}^j\,{}^*g^{ih} \tag{3.13}$$

Substituting (3.11) into the left-hand side of (3.13), we have

$$\begin{aligned}
D_k {}^*g^{ij} &= \partial_k {}^*g^{ij} + \Gamma^i_{hk}\,{}^*g^{hj} + \Gamma^j_{hk}\,{}^*g^{ih} \\
&= \partial_k {}^*g^{ij} + ({}^*\{{}^{\,i}_{hk}\} + 2\delta^i_h X_k - 2\,{}^*g_{hk} X^i)\,{}^*g^{hj} + \\
&\qquad + ({}^*\{{}^{\,j}_{hk}\} + 2\delta^j_h X_k - 2\,{}^*g_{hk} X^j)\,{}^*g^{ih} \\
&= \nabla_k {}^*k^{ij} + 4\,{}^*g^{ij} X_k - 4\delta^{(i}_k X^{j)} + 4\,{}^*k_k{}^i X^j - 4\,{}^{(2)*}k_k{}^{[i} X^{j]}
\end{aligned} \tag{3.14a}$$

In virtue of (2.36) and Theorem 13(a), the right-hand side of (3.13) may be written as

$$\begin{aligned}
-2S_{kh}{}^j\,{}^*g^{ih} &= -2(2\delta^j_{[k} X_{h]} - 2\,{}^*k_{kh} X^j)({}^*h^{ih} + {}^*k^{ih}) \\
&= 2\delta^j_k({}^*k_h{}^i X^h - X^i) + 2\,{}^*g^{ij} X_k + \\
&\qquad + 4({}^*k_k{}^i - {}^{(2)*}k_k{}^i) X^j
\end{aligned} \tag{3.14b}$$

Consequently, substitution of (3.14) into (3.13) gives

$$\begin{aligned}
\nabla_k {}^*k^{ij} &= 2\left[\delta^i_k X^j - {}^*g^{ij} X_k + \delta^j_k\,{}^*k_h{}^i X^h - 2^{(2)*}k_k{}^{(i} X^{j)}\right] \\
&= 2(\delta^{[i}_k X^{j]} - {}^*k^{ij} X_k + \delta^{[j}_k\,{}^*k_h{}^{i]} X^h) + 2(\delta^{(i}_k X^{j)} - \\
&\qquad - {}^*h^{ij} X_k + \delta^{(j}_k\,{}^*k_h{}^{i)} X^h - {}^{(2)*}k_k{}^{(i} X^{j)})
\end{aligned} \tag{3.15}$$

The conditions (3.12) immediately follow from (3.15). Conversely, suppose that the conditions (3.12) hold. Then, since $\nabla_k {}^*h^{ij} = 0$, we have (3.13) in virtue of (3.14) and (3.15). Hence, the induced connection $\Gamma^k_{ij}$ is Einstein.

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