Note di Matematica Vol. 18 - n. 1, 43-61 (1998)

A WEIERSTRASS TYPE REPRESENTATION FOR SURFACES IN HYPERBOLIC SPACE WITH MEAN CURVATURE ONE

CÉLIA CONTIN GÓES, M. ELISA, E.L. GALVÃO

Abstract. *The subject of this paper is to give a Weierstrass type representation for mean curvature one surfaces in the hyperbolic space. This representation depends on the hyperbolic Gauss map. Some known examples are described and a new one, associated to the minimal Bonnet surface is constructed with this representation.*

INTRODUCTION

A Weierstrass type formula for surfaces of prescribed mean curvature in $\mathbf{R}^3$ was given by Kenmotsu ([K]) in 1979. In 1987, R. Bryant ([B]) studied the surfaces of mean curvature one in hyperbolic space as local projections of null curves in the space of the 2×2 Hermitian symmetric matrices with its Cartan-Killing metric. Recently, Umehara and Yamada ([UY-1], [UY-2], [RUY]) produced an explicit tool to construct examples of these surfaces. They described the null curves in terms of a meromorphic function g and a holomorphic 1-form ω obtained as solutions of two ordinary differential equations.

The subject of this paper is to describe the surfaces in $\mathbf{H}^3$ with mean curvature one in a very similar manner as the minimal surfaces in $\mathbf{R}^3$. It is already well known that these surfaces have a hyperbolic holomorphic Gauss map ([B]); in our work, the function h describes the holomorphic Gauss map. Its properties will give us a Weierstrass type representation.

From the main theorem we have the immersion $X : U \subset \mathbb{C} \longrightarrow \mathbf{H}^3$ as

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re e\, \phi_3(z), \Im m\, \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

where $\phi_j, j = 1, 2, 3$ are solutions of the system:

$$\begin{cases} \phi_1 \phi_2 = 1 + |\phi_3|^2 \\[2ex] \dfrac{\partial \phi_1}{\partial z} = h \dfrac{\partial \overline{\phi}_3}{\partial z} \\[2ex] \dfrac{\partial \phi_2}{\partial z} = \dfrac{1}{h} \dfrac{\partial \phi_3}{\partial z} \end{cases}$$

whose integrability condition is that of

$$\Im m\{ \overline{h} \Delta \phi_3 \} = 0.$$

We also have a local integral representation:

$$X = \left(\Re e \int_{z_0}^{z} \left(h \frac{\partial \overline{\phi}_3}{\partial z} + \frac{1}{h} \frac{\partial \phi_3}{\partial z} \right) dz, \Re e \phi_3, \Im m \phi_3, \Re e \int_{z_0}^{z} \left(h \frac{\partial \overline{\phi}_3}{\partial z} - \frac{1}{h} \frac{\partial \phi_3}{\partial z} \right) dz \right)$$

In the last part of the paper we exhibit local solutions of this system for all functions h.

The hyperbolic Gauss map.

We consider the Lorentz space $\text{Ł}^4 = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}$ with the inner product

$$< x, y > = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The Minkovsky model for the hyperbolic space is the submanifold

$$\mathbf{H}^3 = \{x \in \text{Ł}^4 \mid < x, x > = -1, \ x_0 > 0\}.$$

In $\mathbf{H}^3$ we will consider the induced orientation from Ł^4 for which the vectors v_1, v_2, v_3 in $T_p\mathbf{H}^3$ form a positive oriented basis iff $\{p, v_1, v_2, v_3\}$ forms a positive oriented basis of Ł^4.

Let $X : M \longrightarrow \mathbf{H}^3$ be an isometric immersion of an orientable Riemann surface M in the hyperbolic space and $N(p)$ the oriented unitary normal vector at $p \in M$. In local isothermical coordinates $z = u + iv$ we have $||X_u|| = ||X_v|| = \lambda$, $< X_u, X_v > = 0$, and N is such that $\{X(p), \frac{1}{\lambda} X_u, \frac{1}{\lambda} X_v, N(p)\}$ is a positive basis of $T_p\text{Ł}^4$.

We will consider the map

$$\Phi : \mathbf{H}^3 \longrightarrow D$$

$$(x_0, x_1, x_2, x_3) \longrightarrow (1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})$$

and the vector $\Phi_*(N(p))$ where

$$D = \{(x_0, x_1, x_2, x_3) \mid x_0 = 1, \ x_1^2 + x_2^2 + x_3^2 < 1\}.$$

This map is the natural isometry between $\mathbf{H}^3$ and the Klein model for the hyperbolic space given by unitary disc with the appropriated metric.

The boundary of D can be identified with the Riemann two sphere S^2.

Definition. The *hyperbolic Gauss map of an immersion* $X : M \longrightarrow \mathbf{H}^3$ is

$$n : M \longrightarrow \partial D$$

given by

$$n(p) = \Phi(X(p)) + t\Phi_*(N(p))$$

where $t > 0$ and $n(p) \in \partial D$.

It follows immediately:

Lemma 1. $n = \dfrac{1}{x_0 + N_0}(X + N)$.

Proof. For $X(p) = (x_0, x_1, x_2, x_3)$ and $N = (N_0, N_1, N_2, N_3)$

$$\Phi_*(N) = -\frac{N_0}{x_0^2}X + \frac{1}{x_0}N.$$

As $n(p) = \Phi(X(p)) + t\Phi_*(N(p))$ is in the cone $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ we have

$$<n, n> = -\frac{(x_0 - tN_0)^2}{X_0^4} + \frac{t^2}{X_0^2} = 0.$$

The solution t with $t > 0$ is $t = x_0/(x_0 + N_0)$. □

Remarks. 1. Since the vector $X + N$ is also in the cone there exists $\psi : U \longrightarrow \mathbb{R}$,

$$\psi(p) = -\frac{1}{<n, X>} = x_0 + N_0 = -<X + N, e_0>, \quad e_0 = (1, 0, 0, 0)$$

such that

$$\psi(p)\, n(p) = X(p) + N(p), \ \forall p \in U$$

and

$$N = -\frac{1}{<n, X>}n - X.$$

2. The coefficients of the second fundamental form for the immersion X can be calculated as

$$h_{ij} = -<\nabla_{e_i} N, e_j>, \quad i, j = 1, 2$$

with $e_1 = \dfrac{1}{\lambda}X_u$ and $e_2 = \dfrac{1}{\lambda}X_v$. We have

$$N_u = -h_{11}X_u - h_{12}X_v,$$

$$N_v = -h_{12}X_u - h_{22}X_v.$$

The mean curvature in the choosen normal direction and the gaussian curvature have, respectively, the expressions

$$H = \frac{1}{2}(h_{11} + h_{22}) \qquad \textit{and} \qquad K = h_{11}h_{22} - h_{12}^2 - 1.$$

3. In isothermical parameters

$$<X_{z\bar{z}}, N> = \frac{1}{2}\lambda^2 H$$

where

$$\frac{1}{2}\lambda^2 = <X_z, X_{\bar{z}}>.$$

The mean curvature H is equal to one if and only if

$$<X_{z\bar{z}}, N> = <X_z, X_{\bar{z}}>$$

or

$$<X_{z\bar{z}}, -\frac{1}{<n,X>}n - X> = <X_z, X_{\bar{z}}>.$$

We will have $H = 1$ if and only if

$$<X_z, n_{\bar{z}}> = 0.$$

4. Taking $z = u + iv$ isothermical parameters in $U \subset \mathbb{C}$ we have the diagram:

$$\begin{array}{ccc} M & \xrightarrow{\ n\ } & \partial D \approx S^2 \\ \Big\downarrow & & \Big\downarrow \Pi \\ U \subset \mathbb{C} & \xrightarrow{\ h\ } & \mathbb{C} \end{array}$$

with Π the stereographic projection; then

$$n(z) = \left(1, \frac{2\Re e\, h}{|h|^2+1}, \frac{2\Im m\, h}{|h|^2+1}, \frac{|h|^2-1}{|h|^2+1}\right),$$

and n is holomorphic if and only if h is holomorphic.

This hyperbolic Gauss map behaves as the classical Gauss map for minimal surfaces in an euclidean space, that is, we have the following theorem ([B]):

Theorem 1. *Let $n : M \longrightarrow \partial D$ be the hyperbolic Gauss map of a surface $X : M \longrightarrow \mathbf{H}^3$, n non constant. The map $n : M \longrightarrow \partial D$ is conformal iff the immersion X either has mean curvature H constant and equal to one (in which case n preserves the orientation) or X is totally umbilic (in which case n reverses the orientation).*

Proof. From $n = \frac{1}{x_0 + N_0}(X + N)$ we have

$$n_u = \left(\frac{1}{x_0+N_0}\right)_u (X+N) + \frac{1}{x_0+N_0}(X_u + N_u) =$$

$$= \left(\frac{1}{x_0+N_0}\right)_u (X+N) + \frac{1}{x_0+N_0}[(1-h_{11})X_u - h_{12}X_v]$$

and

$$n_v = \left(\frac{1}{x_0+N_0}\right)_v (X+N) + \frac{1}{x_0+N_0}(X_v + N_v) =$$

$$= \left(\frac{1}{x_0 + N_0}\right)_v (X + N) + \frac{1}{x_0 + N_0}[-h_{12}X_u + (1 - h_{22})X_v].$$

Consequently,

$$||n_u||^2 = \frac{\lambda^2}{(x_0 + N_0)^2}[(1 - h_{11})^2 + h_{12}^2]$$

$$||n_v||^2 = \frac{\lambda^2}{(x_0 + N_0)^2}[(1 - h_{22})^2 + h_{12}^2]$$

$$< n_u, n_v > = \frac{\lambda^2}{(x_0 + N_0)^2}(-h_{12})(2 - h_{11} - h_{22})$$

and for $H = 1$ or for umbilic immersions we will have $||n_u||^2 = ||n_v||^2 \geq 0$ and $< n_u, n_v > = 0$.

We also observe that $||n_u|| = ||n_v|| = 0$ if and only if the immersion is umbilical and $H = 1$; in this case we have a horosphere and the hyperbolic Gauss map n is constant.

Considering the complex differentiation

$$n_{\bar{z}} = \left(\frac{1}{x_0 + N_0}\right)_{\bar{z}} (X + N) + \frac{1}{x_0 + N_0}(X_{\bar{z}} + N_{\bar{z}})$$

it follows that

(1) $$< n_{\bar{z}}, n_{\bar{z}} > = \frac{1}{2}\frac{\lambda^2}{(x_0 + N_0)^2}(H - 1)[(h_{11} - h_{22}) + 2\,i\,h_{12}],$$

from

$$n(z) = (1, \frac{2\Re e\, h}{|h|^2 + 1}, \frac{2\Im m\, h}{|h|^2 + 1}, \frac{|h|^2 - 1}{|h|^2 + 1})$$

we have

$$n_{\bar{z}} = \left(\frac{1}{|h|^2 + 1}\right)_{\bar{z}} \tilde{n} + \frac{1}{|h|^2 + 1}\tilde{n}_{\bar{z}}$$

where

$$\tilde{n} = (|h|^2 + 1, h + \bar{h}, -i\,(h - \bar{h}), |h|^2 - 1)$$

and

$$\tilde{n}_{\bar{z}} = h_{\bar{z}}(\bar{h}, 1, -i, \bar{h}) + \bar{h}_{\bar{z}}(h, 1, i, h).$$

With these calculations we conclude that

(2). $$< n_{\bar{z}}, n_{\bar{z}} > = \frac{4h_{\bar{z}}\ \bar{h}_{\bar{z}}}{(|h|^2 + 1)^2}$$

From (1) and (2)

$$\frac{4h_{\bar{z}}\ \bar{h}_{\bar{z}}}{(|h|^2 + 1)^2} = \frac{1}{2}\frac{\lambda^2}{(x_0 + N_0)^2}(H - 1)[(h_{11} - h_{22}) + 2\,i\,h_{12}]$$

and the hyperbolic Gauss map is conformal iff either $H = 1$ or the immersion is umbilical. In both cases the induced metric is given by

$$<dn, dn> = <n_z\,dz + n_{\bar z}\,d\bar z, n_z\,dz + n_{\bar z}\,d\bar z> = 2<n_z, n_{\bar z}>|dz|^2$$

or

$$<dn, dn> = \frac{\lambda^2}{(x_0 + N_0)^2}[2\,H(H-1) - K]|dz|^2.$$

When $H = 1$ we have

$$<dn, dn> = \frac{\lambda^2}{(x_0 + N_0)^2}(-K)|dz|^2,$$

if the immersed surface is different from a horosphere $2\,H(H-1) - K > 0$ and $-K > 0$.

Finally we compare the orientations of $X : M \longrightarrow \mathbf{H}^3 \subset Ł^4$ and $n : M \longrightarrow \partial D \subset Ł^4$, when the immersion X is distinct from the horosphere.

The stereographic projection $\Pi : \partial D \approx S^2 \longrightarrow \{1\} \times \mathbb{R}^3 \subset Ł^4$ induces a positive orientation in S^2 in which the normal vector is the internal one.

Let $\{X(p), \frac{1}{\lambda}X_u, \frac{1}{\lambda}X_v, N\}$ and $\{e_0, n_u, n_v, e_0 - n\}$, $e_0 = (1,0,0,0)$ be orthogonal frames adaptated to $X(M)$ and $n(M)$, respectively.

These frames are related by the matrix

$$\begin{bmatrix} x_0 & \left(\frac{1}{x_0+N_0}\right)_u & \left(\frac{1}{x_0+N_0}\right)_v & x_0 - \frac{1}{x_0+N_0} \\ \frac{1}{\lambda}<X_u, e_0> & \frac{\lambda(1-h_{11})}{x_0+N_0} & \frac{-\lambda h_{12}}{x_0+N_0} & \frac{1}{\lambda}<X_u, e_0> \\ \frac{1}{\lambda}<X_v, e_0> & \frac{-\lambda h_{12}}{x_0+N_0} & \frac{\lambda(1-h_{22})}{x_0+N_0} & \frac{1}{\lambda}<X_v, e_0> \\ -N_0 & \left(\frac{1}{x_0+N_0}\right)_u & \left(\frac{1}{x_0+N_0}\right)_v & -N_0 - \frac{1}{x_0+N_0} \end{bmatrix}$$

whose determinant is

$$\begin{vmatrix} x_0 & \left(\frac{1}{x_0+N_0}\right)_u & \left(\frac{1}{x_0+N_0}\right)_v & -\frac{1}{x_0+N_0} \\ \frac{1}{\lambda}<X_u, e_0> & \frac{\lambda(1-h_{11})}{x_0+N_0} & \frac{-\lambda h_{12}}{x_0+N_0} & 0 \\ \frac{1}{\lambda}<X_v, e_0> & \frac{-\lambda h_{12}}{x_0+N_0} & \frac{\lambda(1-h_{22})}{x_0+N_0} & 0 \\ -(x_0+N_0) & 0 & 0 & 0 \end{vmatrix} =$$

$$= -\frac{\lambda^2}{(x_0+N_0)^2}[(1-h_{11})(1-h_{22}) - h_{12}^2] = \frac{\lambda^2}{(x_0+N_0)^2}[-K + 2(H-1)].$$

It is easy to see that the determinant is positive if $H = 1$ in which case n preserves the orientation (that is, n is holomorphic); in the umbilic case the determinant is negative, n reverses the orientation and is antiholomorphic. □

Remark. We observe that

$$<n_z, n_{\bar{z}}> = \frac{2}{(1+|h|^2)^2}\,[\ |h_z|^2 + |h_{\bar{z}}|^2\ ].$$

When $H = 1$

$$< dn, dn > = 2<n_z, n_{\bar{z}}>|dz|^2 = \frac{4|h_z|^2}{(|h|^2+1)^2}\,|dz|^2 = -K\frac{\lambda^2}{(x_0+N_0)^2}|dz|^2,$$

and

$$-K = \frac{4|h_z|^2}{(|h|^2+1)^2}\left(\frac{\lambda^2}{(x_0+N_0)^2}\right)^{-1}. \tag{3}$$

A Representation Theorem.

Working with a holomorphic hyperbolic Gauss map, that is, with surfaces with constant mean curvature equal to one, we have a local representation theorem similar to the Weierstrass representation for minimal surfaces in the euclidean space.

Theorem 2. *Let $X : M \longrightarrow \mathbf{H}^3$ be a non-umbilic immersion in $\mathbf{H}^3$ with mean curvature one and*

$$n(z) = \left(1, \frac{2\Re eh}{1+|h|^2}, \frac{2\Im mh}{1+|h|^2}, \frac{|h|^2-1}{|h|^2+1}\right)$$

its hyperbolic Gauss map. Denoting $X(z) = (x_0(z), x_1(z), x_2(z), x_3(z))$, the real functions $\phi_1(z) = x_0(z) + x_3(z)$ and $\phi_2(z) = x_0(z) - x_3(z)$ and the complex function $\phi_3(z) = x_1(z) + i\, x_2(z)$ satisfy

$$(*)\qquad \begin{cases} \phi_1\,\phi_2 = 1 + |\phi_3|^2 \\[2ex] \dfrac{\partial\,\phi_1}{\partial\,z} = h\dfrac{\partial\,\overline{\phi}_3}{\partial\,z} \\[2ex] \dfrac{\partial\,\phi_2}{\partial\,z} = \dfrac{1}{h}\dfrac{\partial\,\phi_3}{\partial\,z} \end{cases}$$

Conversely, given a holomorphic non-constant function $h : U \subset \mathbb{C} \longrightarrow \mathbb{C}$, two real functions ϕ_1 and ϕ_2 ($\phi_2 > 0$) and a complex function ϕ_3 satisfying $(*)$ in the simply connected domain U, then

$$X(z) = \left(\frac{\phi_1(z)+\phi_2(z)}{2},\ \Re e\,\phi_3(z),\ \Im m\,\phi_3(z),\ \frac{\phi_1(z)-\phi_2(z)}{2}\right)$$

defines a conformal immersion in $\mathbf{H}^3$ with constant mean curvature one and hyperbolic Gauss map n given by h as above.

Proof. First of all we observe that

$$X(z) = (x_0, x_1, x_2, x_3) \in \mathbf{H}^3 \iff -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \iff \phi_1\,\phi_2 = 1 + |\phi_3|^2,$$

from the first equivalence it also follows that if $\phi_2 = x_0 - x_3$ then $\phi_2 > 0$.

Given ϕ_1, ϕ_2, ϕ_3 as above we have

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2},\ \Re e\,\phi_3(z),\ \Im m\,\phi_3(z),\ \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

and $< X_z, n > = 0$ if and only if

$$\frac{1}{2}\left(-1 - \frac{|h|^2 - 1}{|h|^2 + 1}\right)\frac{\partial\,\phi_1}{\partial\,z} + \frac{1}{2}\left(-1 + \frac{|h|^2 - 1}{|h|^2 + 1}\right)\frac{\partial\,\phi_2}{\partial\,z} + \frac{1}{1 + |h|^2}\left[\Re e\,h\left(\frac{\partial\,\phi_3}{\partial\,z} + \frac{\partial\,\overline{\phi}_3}{\partial\,z}\right) + i\,\Im m\,h\left(\frac{\partial\,\overline{\phi}_3}{\partial\,z} - \frac{\partial\,\phi_3}{\partial\,z}\right)\right] = 0$$

or

$$\frac{\partial\,\phi_1}{\partial\,z} + |h|^2\frac{\partial\,\phi_2}{\partial\,z} - h\,\frac{\partial\,\overline{\phi}_3}{\partial\,z} - \overline{h}\,\frac{\partial\,\phi_3}{\partial\,z} = 0 \tag{4}$$

The assumption on the mean curvature gives us

$$H = 1 \iff < X_z, n_{\overline{z}} > = < X_z, \widetilde{n}_{\overline{z}} > = 0$$

where

$$n(z) = \left(1, \frac{2\Re e h}{1 + |h|^2}, \frac{2\Im m h}{1 + |h|^2}, \frac{|h|^2 - 1}{|h|^2 + 1}\right)$$

and

$$\widetilde{n} = (1 + |h|^2, h + \overline{h}, -i\,(h - \overline{h}), |h|^2 - 1\,).$$

We have in this case h holomorphic and therefore

$$\widetilde{n}_{\overline{z}} = (h\,\overline{h}_{\overline{z}},\ \overline{h}_{\overline{z}},\ i\,\overline{h}_{\overline{z}},\ h\,\overline{h}_{\overline{z}}\,),$$

as h is nonconstant ($h_z \neq 0$) it follows

$$H = 1 \iff < X_z, n_{\overline{z}} > = 0 \iff$$

$$-h\left(\frac{\partial\,\phi_1}{\partial\,z} + \frac{\partial\,\phi_2}{\partial\,z}\right) + \left(\frac{\partial\,\phi_3}{\partial\,z} + \frac{\partial\,\overline{\phi}_3}{\partial\,z}\right) + h\left(\frac{\partial\,\overline{\phi}_1}{\partial\,z} - \frac{\partial\,\phi_2}{\partial\,z}\right) = 0 \iff$$

$$\frac{\partial\,\phi_3}{\partial\,z} = h\,\frac{\partial\,\phi_2}{\partial\,z}. \tag{5}$$

Returning with this last equation in (4), finally we have

$$\frac{\partial \phi_1}{\partial z} = h \frac{\partial \overline{\phi}_3}{\partial z}.$$

Let $p \in M$ be a zero of h with order 1; we have from (5) that p is a zero of $\frac{\partial \phi_3}{\partial z}$ whose order is greater or equal to 1 and we can write

$$\frac{\partial \phi_2}{\partial z} = \frac{1}{h} \frac{\partial \phi_3}{\partial z}.$$

Let now be

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re e\, \phi_3(z), \Im m\, \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

with ϕ_1, ϕ_2, ϕ_3 verifying $(*)$. It is easy to see that

$$X_z = \frac{1}{2} \left[\frac{\partial \phi_3}{\partial z} \left(\frac{1}{h}, 1, -i, -\frac{1}{h} \right) + \frac{\partial \overline{\phi}_3}{\partial z} (h, 1, i, h) \right] \tag{6}$$

From the fact that $< X_z, X_z > = 0$ it follows that we have isothermical parameters.

Let now consider

$$\widehat{n}(z) = \left(1, \frac{2\Re e h}{1 + |h|^2}, \frac{2\Im m h}{1 + |h|^2}, \frac{|h|^2 - 1}{|h|^2 + 1} \right)$$

with h the holomorphic function from $(*)$. The vector

$$\widehat{N} = -\frac{1}{< \widehat{n}, X >} \widehat{n} - X$$

has norm equal to one, verifies $< X_z, \widehat{N} > = 0$, $< X, \widehat{N} > = 0$ and

$$-\frac{1}{< \widehat{n}, X >} \widehat{n} = X + \widehat{N}$$

therefore $\widehat{N}$ is exactly the normal vector N and $\widehat{n}$ the hyperbolic Gauss map n of the immersion X. With some calculations we obtain

$$< n_{\bar{z}}, X_z > = \frac{h_{\bar{z}}}{(1 + |h|^2)} \left[\frac{\partial \overline{\phi}_3}{\partial z} - \frac{\partial \phi_3}{\partial z} \frac{\overline{h}}{h} \right].$$

From the fact that h is holomorphic it follows that $< n_{\bar{z}}, X_z > = 0$ which implies $H = 1$; h non-constant gives us a non-umbilic immersion. □

Remarks.

1. The compatibility condition for the two partial differential equations in $(*)$ is the same and writes

$$\Im m\{\overline{h}\ \Delta\phi_3\} = 0. \tag{7}$$

This follows from the fact that each differential equation of $(*)$ is as

$$\frac{\partial \phi}{\partial z} = F(z)$$

or as

$$\begin{cases} \dfrac{\partial \phi}{\partial u} = 2\,F_1(u,v) \\ \dfrac{\partial \phi}{\partial v} = -2\,F_2(u,v) \end{cases}$$

with $z = u + i\,v$, $F(z) = F_1(u,v) + i\,F_2(u,v)$, $\partial / \partial z = \frac{1}{2}\,(\partial / \partial u - i\,\partial / \partial v)$.

The integrability condition for this system is:

$$\frac{\partial F_1}{\partial v} = -\frac{\partial F_2}{\partial u} \iff \Im m\{\frac{\partial F}{\partial \bar{z}}\} = 0$$

Returning to the system $(*)$, each equation will have its integrability condition respectively given by:

$$\Im m\{h\ \frac{\partial^2 \overline{\phi_3}}{\partial \bar{z}\,\partial z}\} = 0$$

and

$$\Im m\{\frac{1}{|h|^2}\ \overline{\frac{h\,\partial^2 \overline{\phi_3}}{\partial \bar{z}\,\partial z}}\} = -\frac{1}{|h|^2}\ \Im m\{h\ \frac{\partial^2 \overline{\phi_3}}{\partial \bar{z}\,\partial z}\} = 0$$

Consequently, the two compatibility conditions are verified if and only if, locally,

$$\Im m\{\overline{h}\ \Delta\phi_3\} = 0$$

2. Choosing h and ϕ_3 such as to verify (7) we will have ϕ_1 and ϕ_2 given locally by

$$\phi_1 = 2\,\Re e \int_{z_0}^{z} h\ \frac{\partial \overline{\phi_3}}{\partial z}\ dz$$

and

$$\phi_2 = 2\,\Re e \int_{z_0}^{z} \frac{1}{h}\ \frac{\partial \phi_3}{\partial z}\ dz.$$

3. An integral formula can be written from (6):

$$X = \left(\Re e \int_{z_0}^{z} \left(h\frac{\partial \overline{\phi_3}}{\partial z} + \frac{1}{h}\frac{\partial \phi_3}{\partial z}\right) dz, \Re e\phi_3, \Im m\phi_3, \Re e \int_{z_0}^{z} \left(h\frac{\partial \overline{\phi_3}}{\partial z} - \frac{1}{h}\frac{\partial \phi_3}{\partial z}\right) dz\right). \tag{I}$$

4. The metric $ds^2 = \lambda^2|dz|^2$ is such that $\lambda^2 = 2 < X_z, X_{\bar{z}} >$, from (6) we have

$$\lambda^2 = \left[\left|\frac{\partial \phi_3}{\partial z}\right|^2 + \left|\frac{\partial \phi_3}{\partial \bar{z}}\right|^2 - 2\Re e\left(\frac{\bar{h}}{h}\frac{\partial \phi_3}{\partial z}\frac{\partial \phi_3}{\partial \bar{z}}\right)\right] \tag{8}$$

and from this last expression we can conclude that p is a regular point if the derivatives $\frac{\partial \phi_3}{\partial z}$ and $\frac{\partial \phi_3}{\partial \bar{z}}$ do not vanish simultaneously at p.

We also can write:

$$\lambda^2 = \left|\frac{\partial}{\partial z}(\overline{\phi}_3 - \bar{h}\phi_2)\right|^2 = \left|\frac{\partial}{\partial \bar{z}}(\phi_3 - h\phi_2)\right|^2$$

5. From Lemma 1 we have:

$$-\frac{1}{<n, X>} = x_0 + N_0 = -<X + N, e_0>,$$

some calculations give us:

$$1 + |\phi_3 - h\phi_2|^2 = \phi_2(\phi_1 + |h|^2\phi_2 - \bar{h}\phi_3 - h\overline{\phi}_3)$$

and

$$<n, X> = \frac{1}{|h|^2 + 1}(-\phi_1 - |h|^2\phi_2 + \bar{h}\phi_3 + h\overline{\phi}_3) = -\frac{1 + |\phi_3 - h\phi_2|^2}{\phi_2\,(|h|^2 + 1)}$$

The total curvature is

$$c = \int_M K\,dA$$

and from (3) it follows that

$$K = -\frac{4|h_z|^2}{(|h|^2 + 1)^2}\left(\frac{\lambda^2}{(x_0 + N_0)^2}\right)^{-1}.$$

In local coordinates

$$c = -\int \frac{4|h_z|^2}{(\phi_1 + |h|^2\phi_2 - \bar{h}\phi_3 - h\overline{\phi}_3)^2}\,\frac{i}{2}dz \wedge \overline{dz} = \tag{9}$$

$$-\int \frac{4|h_z|^2\phi_2^2}{(1 + |\phi_3 - h\phi_2|^2)^2}\,\frac{i}{2}dz \wedge \overline{dz} = -\int \frac{4\left|\frac{\partial}{\partial \bar{z}}(\phi_3 - h\phi_2)\right|^2}{(1 + |\phi_3 - h\phi_2|^2)^2}\,\frac{i}{2}dz \wedge \overline{dz}$$

6. Given the immersion $X : U \subset \mathbb{C} \longrightarrow \mathbf{H}^3$ in isothermal coordinates, calling $F = \lambda^2 / 2$, the gaussian curvature can also be calculated as

$$K = -\frac{\partial\,\bar{\partial}\,log\,F}{F}.$$

If we denote

$$\psi = \frac{1}{2}[(h_{11} - h_{22}) - 2\,i\,h_{12}] = \frac{2}{\lambda^2} < X_{zz}, N >$$

the Gauss equation can be written as

$$|\psi|^2 = -K - \frac{4}{\lambda^2} < X_{z\bar{z}}, X_{z\bar{z}} > = -K + H^2 - 1.$$

By using (*) we get

$$|\psi|^2 = \frac{4|h_z|^2\phi_2}{\lambda^2(1 + |\phi_3 - h\phi_2|^2)^2} = -K$$

that means, we have the Gauss' equation verified.

We will call the Hopf's form ([H]) the quadratic form

$$\Psi = \psi\,\lambda^2\,dz^2.$$

As in ([H]) the Codazzi equations can be written in a complex form and we have

$$\frac{\partial(\lambda^2\psi)}{\partial\,\bar{z}} = \lambda^2 \frac{\partial\,H}{\partial\,z}$$

With some calculations we can show that (*) implies that $\lambda^2\psi$ is holomorphic (Proposition 2 in [B]) and the Codazzi equations are also verified.

Examples.

To exhibit some examples we need to get two real functions ϕ_1 and ϕ_2, $\phi_2 > 0$ and a complex function ϕ_3, solutions of the the system:

$$(*) \qquad \begin{cases} \phi_1\,\phi_2 = 1 + |\phi_3|^2 \\ \dfrac{\partial\,\phi_1}{\partial\,z} = h\dfrac{\partial\,\overline{\phi_3}}{\partial\,z} \\ \dfrac{\partial\,\phi_2}{\partial\,z} = \dfrac{1}{h}\dfrac{\partial\,\phi_3}{\partial\,z} \end{cases}$$

To find solutions, we begin with some important remarks.

1. First of all we will analyse the solutions that correspond to ϕ_3 holomorphic (or antiholomorphic); we will have that ϕ_1 (resp. ϕ_2) is constant. As $\phi_1\,\phi_2 = 1 + |\phi_3|^2$ the constant cannot be zero; it is easy to see that ϕ_1 (resp. ϕ_2) constant implies that the surface is umbilical and $x_0 + x_3$ (resp. $x_0 - x_3$) is constant; in this case the functions x_1 and x_2 will be harmonical conjugates.

2. Given the function h we can search solutions as

$$\phi_3 = h(z)F(|z|^2),$$

with F a one real variable differentiable function.

Since

$$\overline{h}\ \Delta\phi_3 = z\overline{h}\,h_z F'(|z|^2) + |h|^2 F'(|z|^2) + |z|^2\,|h|^2\,F''(|z|^2)$$

the compatibility condition is

$$\Im m\{\,\overline{h}\ \Delta\phi_3\,\} = 0 \iff \Im m\{\,z\overline{h}\,h'\} = 0.$$

The last condition is satisfied by all the functions $h(z) = z^\alpha$, for real α.

3. If $\phi_3 = h(z)F(|z|^2)$ then metric (6) will be

$$\lambda^2 = |h_z|^2\ F^2(|z|^2). \tag{10}$$

Example 1. We have an immersion with constant mean curvature one

$$X : \mathbb{C} - \{0\} \longrightarrow \mathbf{H}^3$$

solving $(*)$ with $h(z) = z$, $F_\alpha(t) = t^\alpha$ and

$$\phi_3(z) = h(z)\ [\ A\,F_\alpha(|z|^2) + B\,F_\beta(|z|^2)\ ].$$

Now, the integrability condition is satisfied (remark 2) and the solutions ϕ_1 and ϕ_2 are:

$$\phi_1(z) = \frac{\alpha}{\alpha+1}\,A\,|z|^{2\alpha+2} + \frac{\beta}{\beta+1}\,B\,|z|^{2\beta+2}$$

and

$$\phi_2(z) = \frac{\alpha+1}{\alpha}\,A\,|z|^{2\alpha} + \frac{\beta+1}{\beta}\,B\,|z|^{2\beta}$$

with α and β both distinct from zero and -1.

The condition $\phi_1\,\phi_2 = 1 + |\phi_3|^2$ is verified under the restrictions:

$$\alpha + \beta = -1$$

and

$$A\,B\left(\frac{2\alpha+1}{\alpha\,(\alpha+1)}\right)^2 = 1 \tag{11}$$

therefore

$$2\alpha + 1 \neq 0.$$

In this case the solutions of (∗) are:

$$
(12)\qquad \begin{cases}
\phi_1(z) = \dfrac{\alpha}{\alpha+1} A\,|z|^{2\alpha+2} + \dfrac{\alpha+1}{\alpha} B\,|z|^{-2\alpha} \\[2ex]
\phi_2(z) = \dfrac{\alpha+1}{\alpha} A\,|z|^{2\alpha} + \dfrac{\alpha}{\alpha+1} B\,|z|^{-2\alpha-2} \\[2ex]
\phi_3 = z\,[A\,|z|^{2\alpha} + B\,|z|^{-2\alpha-2}\,]
\end{cases}
$$

Writing $z = r\,e^{i\theta}$:

$$
\begin{cases}
\phi_1(r,\theta) = \dfrac{\alpha}{\alpha+1} A\,r^{2\alpha+2} + \dfrac{\alpha+1}{\alpha} B\,r^{-2\alpha} = f_1(r) \\[2ex]
\phi_2(r,\theta) = \dfrac{\alpha+1}{\alpha} A\,r^{2\alpha} + \dfrac{\alpha}{\alpha+1} B\,r^{-2\alpha-2} = f_2(r) \\[2ex]
\phi_3(r,\theta) = r(cos\,\theta + i\,sin\,\theta\,)(A\,r^{2\alpha} +\ B\,r^{-2\alpha-2}) = f_3(r)\,e^{i\,\theta}
\end{cases}
$$

Now it is easy to see that all these surfaces are rotational surfaces generated by the curve

$$
C(r) = (c_0(r), c_1(r), 0, c_3(r)) = \left(\frac{f_1(r)+f_2(r)}{2}, f_3(r), 0, \frac{f_1(r)-f_2(r)}{2}\right),
$$

$C(r) \subset \mathbf{H}^3\mathcal{P}^3$, with $\mathcal{P}^3 = [e_0, e_1, e_3]$. We have a spherical rotation and

$$
X(r,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\,\theta & -sin\,\theta & 0 \\ 0 & sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ 0 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1\ \cos\,\theta \\ c_1\ \sin\,\theta \\ c_3 \end{bmatrix}
$$

Using (9) and (11) we can have the total curvature

$$
c = \int_{\mathbb{C}-\{0\}} \frac{4\alpha^2(\alpha+1)^2|z|^{4\alpha}}{(A|z|^{4\alpha+2}+B)^2}\,\frac{i}{2}\,dz\overline{dz} = -4(2\alpha+1)\pi
$$

As in (10) we have

$$ds^2 = (A|z|^{2\alpha} + B|z|^{-2\alpha-2})^2|dz|^2$$

and the surface is complete.

The minimal catenoid in the euclidean space has a Weierstrass representation given by

$$g(z) = \mu z \qquad \eta = f(z)\,dz = \nu z^{-2}\,dz.$$

The k-recovering of a minimal catenoid has Weierstrass data

$$g(z) = \mu z^k \qquad \eta = f(z)\,dz = \nu z^{-k-1}\,dz$$

and its induced metric is given by

$$ds^2 = \frac{1}{4}[\nu|z|^{-k-1} + \nu\mu^2|z|^{k-1}]^2\,|dz|^2.$$

Given a rotational hyerbolic mean curvature one surface (consequently, given A, B and α), there exists a minimal catenoid isometric to this surface.

Choosing μ and ν such that $A = \nu\,\mu^2\,/\,2$, $B = \nu\,/\,2$ the hyperbolic rotational surface is isometric to the $k = (2\alpha + 1)$-recovering of the minimal catenoid.

Conversely, it is easy to see that a k-recovering of a minimal catenoid is isometric to one immersion in the family of rotational surfaces exhibited above. This immersion is such that $A = \nu\,\mu^2\,/\,2$, $B = \nu\,/\,2$,

$$\left(\frac{2\alpha + 1}{\alpha(\alpha + 1)}\right)^2 = \frac{4}{\nu^2\mu^2}$$

and $k = 2\,\alpha + 1$.

The rotational hyperbolic mean curvature one surfaces are called "the catenoid cousins".

Example 2. The system $(*)$ also admits solutions as

$$\phi_3(z) = F(z)\,\overline{G(z)}$$

with F and G holomorphic functions. In this case if

$$F'(z) = h(z)G'(z) \tag{13}$$

the integrability condition (7) is verified.

The two last equations in $(*)$ can be integrated and the solutions are

$$\phi_1(z) = |F(z)|^2$$

and

$$\phi_2(z) = |G(z)|^2.$$

We will modify these solutions to have the first equation satisfied; in this way, we will take F_1, G_1, F_2, G_2 as in (13), A and B real constants such that

$$\begin{cases} \phi_1 = A|F_1|^2 + B|F_2|^2 \\ \\ \phi_2 = A|G_1|^2 + B|G_2|^2 \\ \\ \phi_3 = A\, F_1\, \overline{G}_1 + B\, F_2\, \overline{G}_2 \end{cases}$$

with

$$AB(\overline{F}_1\overline{G}_2 - \overline{F}_2\overline{G}_1)(F_1G_2 - F_2G_1) = 1. \tag{14}$$

The surfaces called "Enneper Cousins" are corresponding to

$$h(z) = tanh\, \lambda z,$$

$$G_1'(z) = cosh\, \lambda z$$

$$G_2'(z) = z\, cosh\, \lambda z,$$

consequently, by (13) and (14),

$$F_1(z) = \frac{1}{\lambda}\, cosh\, \lambda z$$

$$F_2(z) = \frac{1}{\lambda}\left(zcosh\, \lambda z - \frac{1}{\lambda} sinh\, \lambda z\right)$$

$$G_1(z) = \frac{1}{\lambda} sinh\, \lambda z$$

$$G_2(z) = \frac{1}{\lambda}\left(zsinh\, \lambda z - \frac{1}{\lambda} cosh\, \lambda z\right)$$

and

$$AB = |\lambda|^6,\ \lambda \in \mathbb{C}.$$

The total curvature can be calculated by (9), observing that

$$\phi_1 + |h|^2\phi_2 - \overline{h}\phi_3 - h\overline{\phi}_3 = A\,|F_1 - hG_1|^2 + B\,|F_2 - hG_2|^2 = \frac{(A + B\,|z|^2)}{|\lambda|^2|cosh\, z|^2}$$

and

$$K = -\int \frac{4|\lambda|^6}{A^2\left(1 + \frac{B}{A}|z|^2\right)^2}\frac{i}{2}\, dz \wedge \overline{d}z = -\int \frac{4}{\left(1 + |w|^2\right)^2}\frac{i}{2}\, dw \wedge \overline{d}w = -4\pi.$$

It is also easy to see that the metric

$$ds^2 = \left[\frac{A}{\lambda} + \frac{B}{\lambda}|z|^2\right]^2 |dz|^2$$

is complete.

The classical Enneper surface is given by the Weierstrass data

$$g(z) = \mu z \qquad f(z)dz = \nu dz$$

and has the metric:

$$ds^2 = \frac{1}{4}[\nu + \nu\mu^2|z|^2]^2 |dz|^2.$$

The corresponding isometric Enneper cousin will be given by λ, A and B such that $\lambda^2 = \nu\mu/2$, $A = \lambda\nu/2$ and $B = \lambda\nu\mu^2/2$.

Example 3. By taking

$$h(z) = tanh\left(\frac{\sqrt{5}}{2}z\right) = \frac{sinh(\alpha_1 z) + sinh(\alpha_2 z)}{cosh(\alpha_1 z) + cosh(\alpha_2 z)},$$

with $\alpha_1 = \dfrac{\sqrt{5}-1}{2}$ and $\alpha_2 = \dfrac{\sqrt{5}+1}{2}$ and

$$\phi_3 = A\,F_1\,\overline{G}_1 + B\,F_2\,\overline{G}_2,$$

we can obtain the "Bonnet Cousins" ([GN]) corresponding to the solutions:

$$F_1(z) = \frac{1}{\alpha_1}cosh(\alpha_1 z) + \frac{1}{\alpha_2}cosh(\alpha_2 z)$$

$$G_1(z) = \frac{1}{\alpha_1}sinh(\alpha_1 z) + \frac{1}{\alpha_2}sinh(\alpha_2 z)$$

$$F_2(z) = \frac{1}{\alpha_1}sinh(\alpha_1 z) - \frac{1}{\alpha_2}sinh(\alpha_2 z)$$

$$G_2(z) = \frac{1}{\alpha_1}cosh(\alpha_1 z) - \frac{1}{\alpha_2}cosh(\alpha_2 z),$$

$$AB = \frac{1}{(\alpha_2^2 - \alpha_1^2)^2} = \frac{1}{(\alpha_1 + \alpha_2)^2}.$$

The metric in this case is given by

$$ds^2 = 4\,\left[A\,(\alpha_1 + \alpha_2)\,|\,cosh\,z\,/\,2\,|^2 + B\,(\alpha_1 + \alpha_2)\,|\,sinh\,z\,/\,2\,|^2\right]^2\,|dz|^2 =$$

$$= 4\left[A\,(\alpha_1+\alpha_2)\,|\,cosh\,z\,/\,2\,|^2 + \tfrac{1}{A\,(\alpha_1+\alpha_2)}\,|\,sinh\,z\,/\,2\,|^2\right]^2 |dz|^2$$

and the surface is regular, complete and isometric to a homothety of the classical Bonnet minimal surface given by the Weierstrass data

$$g(z) = -i\,\alpha\,tanh\left(\frac{z}{2}\right) \;\; and \;\; f(z)dz = \frac{2\,i}{\alpha}\,cosh^2\left(\frac{z}{2}\right)dz,\;\; z \in \mathbb{C}\;\;,\;\; \alpha = \sqrt{\frac{1+a}{1-a}},\;\; 0<a<1$$

and metric

$$ds^2 = \frac{1}{4}\left[\frac{1}{\alpha}|cosh\,\frac{z}{2}|^2 + \alpha\,|sinh\,\frac{z}{2}|^2\right]^2 |dz|^2.$$

To get new examples we have to find solutions of

$$\Im m\,\{\overline{h}\,\Delta\phi_3\,\} = 0$$

and a linear combination of this solutions in order to have

$$\phi_1\phi_2 = 1 + |\phi_3|^2$$

that is, in order to have the corresponding immersion in $\L^4$ contained in $\mathbf{H}^3$.

The classification of these immersions depends on the description of all the solutions of this problem.

REFERENCES

[B] Bryant, R. L. *Surfaces of mean curvature one in hyperbolic space,* Astérisque, 155 (1987), (exposé XVI), 321-347.

[GN] Galvão, M. Elisa E. L., Nelli, B., *Hyperbolic Mean Curvature One Surfaces associated to Minimal Surfaces with Planar Curvature Lines*, in preparation.

[H] Hopf, H., *Differential Gometry in the Large* , Lecture Notes in Mathematics, 1000, Springer Verlag, 1983.

[K] Kenmotsu, K. *Weierstrass Formula for Surfaces of Prescribed Mean Curvature* , Math. Ann., 245, (1979), 89-99.

[O] Oliveira, M. C. A. *Superfícies com curvatura média constante um no espaço hiperbólico,* Dissertação de Mestrado, IME-USP, 1994.

[RUY] Rossman, W., Umehara, M., Yamada, K., *Irreducible CMC-c Surfaces in* $H^3(-c^2)$ *with positive genus*, preprint, 1995.

[UY-1] Umehara, M., Yamada, K., *Complete surfaces of constant mean curvature-1 in the hyperbolic 3-space*, Annals of Math., 137, (1993), 611-638.

[UY-2] Umehara, M., Yamada, K., *Surfaces of constant mean curvature c in* $H^3(-c^2)$ *with prescribed Gauss map*, Math. Ann., 304, (1996), 203-224.

Celia C. Góes (goes@ime.usp.br)
M. Elisa E.L. Galvão (elisa@ime.usp.br)
INSTITUTO DE MATEMÁTICA E ESTAÍSTICA
UNIVERSIDADE DE SÃO PAULO
Rua do Matão 1010
Cidade Universitária
Caixa Postal 66281 - Ag. Cidade de São Paulo
CEP 05889-970
SÃO PAULO - BRASIL