SUFFICIENT CONDITIONS FOR OSCILLATION OF THE SOLUTIONS OF A CLASS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

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Abstract. *Sufficient conditions for oscillation of the solutions of first order linear impulsive differential equations with advanced argument are found.*

1. INTRODUCTION

The impulsive differential equations with a deviating argument are adequate mathematical models of numerous processes and phenomena studied in physics, biology, electronics, etc. In spite of the great possibilities for application, the theory of these equations is developing rather slowly due to the obstacles of technical and theoretical character arising in the investigation of the impulsive differential equations.

In the recent two decades the number of investigations of the oscillatory properties of the solutions of functional differential equations is constantly growing. The greater part of the works on this subject published by 1977 are given in [5]. In the monographs [4] and [3] published respectively in 1987 and 1991 the oscillatory and asymptotic properties of the solutions of various classes of functional differential equations are systematically studied.

The first work in which the oscillatory properties of impulsive differential equations with retarded argument are investigated is the work of Gopalsamy and Zhang [2].

In the present work sufficient conditions are found for oscillation of the solutions of the linear impulsive differential equation with advanced argument

$$x'(t) = p(t)x(t+\tau), \quad t \neq t_k$$

$$\Delta x(t_k) = b_k x(t_k). \tag{1}$$

2. PRELIMINARY NOTES

Together with equation (1) we consider the impulsive differential inequalities with advanced argument

$$x'(t) \geq p(t)x(t+\tau), \quad t \neq t_k$$

$$\Delta x(t_k) \geq b_k x(t_k) \tag{2}$$

and

$$x'(t) \leq p(t)x(t+\tau), \quad t \neq t_k$$

$$\Delta x(t_k) \leq b_k x(t_k) \tag{3}$$

provided that the following conditions are met:

A1. The constant τ is positive and the sequence $\{t_k\}$ is such that $0 < t_1 < t_2 < \dots$, $\lim_{k\to+\infty} = +\infty$.

A2. The function $p : \mathbb{R}_+ \to \mathbb{R}$ is piecewise continuous in $\mathbb{R}_+ = [0, \infty)$ with points of discontinuity $\{t_k\}$ at which it is continuous from the left.

Let $J = [\alpha, \beta) \subset \mathbb{R}_+$.

Definition 1. The function $x = \varphi(t)$ is said to be a solution of equation (1) in the interval J if:

1. $\varphi(t)$ is defined in $[\alpha, \beta + \tau)$.
2. For $t \in J, t \neq t_k$ the function $\varphi(t)$ is absolutely continuous and satisfies the relation

$$\varphi'(t) = p(t)\varphi(t + \tau).$$

3. For $t_k \in J$ the function $\varphi(t)$ satisfies the relations

$$\varphi(t_k^-) = \varphi(t_k), \quad \varphi(t_k^+) - \varphi(t_k) = b_k\varphi(t_k).$$

Analogously the solutions of inequalities (2) and (3) are defined.

Definition 2. The solution $x(t)$ of inequality (2) is said to be *eventually positive* if there exists $t_0 > 0$ such that the solution $x(t)$ is defined for $t \geq t_0$ and $x(t) > 0$, $t \geq t_0$.

Definition 3. The solution $x(t)$ of inequality (3) is said to be *eventually negative* if there exists $t_0 > 0$ such that the solution $x(t)$ is defined for $t \geq t_0$ and $x(t) < 0$, $t \geq t_0$.

Definition 4. The nonzero solution $x(t)$ of equation (1) is said to be *nonoscillating* if there exists $t_0 > 0$ such that $x(t)$ is defined for $t \geq t_0$ and does not change its sign for $t \geq t_0$.

Otherwise the solution $x(t)$ is said to *oscillate*.

3. MAIN RESULTS

Theorem 1. *Let conditions A1, A2 hold and a sequence of disjoint intervals $J_n = (\xi_n, \eta_n)$ exists with $\eta_n - \xi_n = 2\tau$ such that:*

1. For each $n \in \mathbb{N}, t \in J_n$ and $t_k \in J_n$

$$p(t) \geq 0, b_k \geq 0. \tag{4}$$

2. There exists $\nu_1 > 0$ such that for $n \geq \nu_1$

$$\int_{\xi_n}^{\xi_n+\tau} p(s) \prod_{s \leq t_k \leq \xi_n+\tau} (1 + b_k)ds \geq 1. \tag{5}$$

Then:

1. Inequality (2) has no eventually positive solution.

2. Inequality (3) has no eventually negative solution.

3. All solutions of equation (1) defined in an interval of the form $[\alpha, +\infty) \subset \mathbb{R}_+$ oscillate.

Proof. First we shall prove that inequality (2) has no eventually positive solution. Suppose that this is not true, i.e., there exists a solution $x(t)$ of (2) such that for t_0 large enough $x(t)$ is defined for $t \geq t_0$ and $x(t) > 0, t \geq t_0$.

Since $\xi_n \to \infty$ as $n \to \infty$, then there exists $\nu_0 > 0$ such that for $n \geq \nu_0$ we have $\xi_n > t_0$ and thus from (2) and (4) it follows that $x'(t) \geq 0, \Delta x(t_k) \geq 0$ for $t, t_k \in J_n$, i.e. $x(t)$ is a nondecreasing function for $t \in J_n, n \geq \nu_0$.

Let $\nu = \max(\nu_0, \nu_1)$ and $n \geq \nu$.

By the theorem on the impulsive differential inequalities ([1], Theorem 2.3), from (2) it follows that

$$x(\xi_n + \tau) \geq x(\xi_n) \prod_{\xi_n \leq t_k < \xi_n + \tau} (1 + b_k) + \int_{\xi_n}^{\xi_n+\tau} p(s) \prod_{s \leq t_k < \xi_n + \tau} (1 + b_k) x(s + \tau) ds. \quad (6)$$

Since $x(s)$ is a nondecreasing function in $(\eta_n - \tau, \eta_n)$ and $(s + \tau) \in (\eta_n - \tau, \eta_n)$ for $s \in (\xi_n, \xi_n + \tau)$, then (6) implies

$$x(\xi_n + \tau) \geq x(\xi_n) \prod_{\xi_n \leq t_k < \xi_n + \tau} (1 + b_k) + x(\xi_n + \tau + 0) \int_{\xi_n}^{\xi_n+\tau} p(s) \prod_{s \leq t_k < \xi_n + \tau} (1 + b_k) ds,$$

whence

$$x(\xi_n) \prod_{\xi_n \leq t_k < \xi_n + \tau} (1 + b_k) + x(\xi_n + \tau) \left\{ \int_{\xi_n}^{\xi_n+\tau} p(s) \prod_{s \leq t_k < \xi_n + \tau} (1 + b_k) ds - 1 \right\} \leq 0. \quad (7)$$

From (7) we conclude that for each $n \geq \nu$ the following inequality holds

$$\int_{\xi_n}^{\xi_n+\tau} p(s) \prod_{s \leq t_k < \xi_n + \tau} (1 - b_k) ds < 1,$$

which contradicts (5).

In order to prove that (3) has no eventually negative solution it suffices to note that if $x(t)$ is a solution of (3), then $-x(t)$ is a solution of (2).

From assertions 1 and 2 it follows that equation (1) has neither an eventually positive nor an eventually negative solution. Thus each solution of (1) defined in an interval of the form $[\alpha, \infty) \subset \mathbb{R}_+$ oscillates.

Remark 1. The fact that $x(t)$ is a nondecreasing function (in particular, in the interval $(\eta_n - \tau, \eta_n)$) is used in the proof of Theorem 1 just to deduce (7) as a corollary from (6). This property of $x(t)$ is ensured by condition (4).

However, if in the interval $(\eta_n - \tau, \eta_n)$ there are no moments of impulse effect, then $x(t)$ is a nondecreasing function only if $p(t) \geq 0, t \in (\eta_n - \tau, \eta_n)$, and in this case it is not necessary to impose the condition $b_k \geq 0, t_k \in J_n$, but it suffices to require that

$$1 + b_k > 0, \quad t_k \in [\xi_n, \xi_n + \tau].$$

In particular, if in each interval J_n there is just one moment of impulse effect, say t_n, and $t_n \in [\xi_n, \xi_n + \tau]$, then condition (5) takes on the form

$$(1 + b_n) \int_{\xi_n}^{t_n} p(s) ds + \int_{\xi_n}^{\xi_n+\tau} p(s) ds \geq 1. \quad (8)$$

Theorem 2. *Let conditions A1, A2 hold and a sequence of disjoint intervals* $J_n = (\xi_n, \eta_n)$ *exists with* $\eta_n - \xi_n = 2\tau$ *such that:*

1. For any $n \in N, t \in J_n$ *and* $t_k \in J_n$

$$p(t) \geq 0, \quad b_k \geq 0. \tag{9}$$

2. There exist a real number K and an integer $\nu_1 > 0$ *such that for any* $n \geq \nu_1$ *and* $t \in (\xi_n, \eta_n - \tau)$

$$\int_t^{t+\tau} p(s) \prod_{s \leq t_k \leq t+\tau} (1 + b_k) ds \geq K > e^{-1}. \tag{10}$$

3. There exist a real number $\delta > 0$ *and an integer* $\nu_2 > 0$ *such that for any* $n \geq \nu_2$ *there exists* $t_n^* \in [\xi_n, \xi_n + \tau]$ *such that*

$$A_n(t_n^*) B_n(t_n^*) \geq \delta, \tag{11}$$

where

$$A_n(t) = \int_{\xi_n}^t p(s) \prod_{s \leq t_k \leq t} (1 + b_k) ds, \quad B_n(t) = \int_t^{\xi_n + \tau} p(s) \prod_{s \leq t_k \leq \xi_n + \tau} (1 + b_k) ds.$$

4. There exists $\nu_3 > 0$ *such that for any* $n \geq \nu_3$ *the inequality*

$$\eta_n - \xi_n > (m_0 + 1)\tau, \tag{12}$$

is valid, where

$$m_0 = \min\{m \in N : \delta(eK)^m > 1\}. \tag{13}$$

Then:

1. Inequality (2) has no eventually positive solution.

2. Inequality (3) has no eventually negative solution.

3. All solutions of equation (1) defined in an interval of the form $[\alpha, +\infty) \subset \mathbb{R}_+$ *oscillate.*

Proof. Suppose that the inequality (2) has a solution $x(t)$ such that for t_0 large enough we have $x(t) > 0, t \geq t_0$.

Since $\xi_n \to \infty$ as $n \to \infty$, then there exists $\nu_0 > 0$ such that $\xi_n > t_0, n \geq \nu_0$ and then from (2) and (9) it follows that $x(t)$ is a nondecreasing function in $J_n, n \geq \nu_0$.

Let $\nu = \max(\nu_0, \nu_1, \nu_2, \nu_3)$. Then for any $n \geq \nu$ the solution $x(t)$ is a nondecreasing function in J_n and conditions (10), (11) and (12) are valid.

From (2), by the theorem on the impulsive differential inequalities ([1], Theorem 2.3) we have that

$$x(t_n^* + 0) \geq x(\xi_n) \prod_{\xi_n \leq t_k \leq t_n^*} (1 + b_k) + \int_{\xi_n}^{t_n^*} p(s) \prod_{s \leq t_k \leq t_n^*} (1 + b_k) x(s + \tau) ds$$

and since $x(s+\tau)$ is nondecreasing in (ξ_n, t_n^*) and $x(\xi_n) > 0$ then

$$x(t_n^* + 0) \geq x(\xi_n + \tau + 0) \int_{\xi_n}^{t_n^*} p(s) \prod_{s \leq t_k \leq t_n^*} (1 + b_k) ds$$

or

$$x(t_n^* + 0) \geq x(\xi_n + \tau + 0) A_n(t_n^*). \tag{14}$$

Analogously, from the inequality

$$x(\xi_n + \tau + 0) \geq x(t_n^* + 0) \prod_{t_n^* < t_k \leq \xi_n + \tau} (1 + b_k) + \int_{t_n^*}^{\xi_n + \tau} p(s) \prod_{s \leq t_k \leq \xi_n + \tau} (1 + b_k) x(s + \tau) ds$$

we obtain that

$$x(\xi_n + \tau + 0) \geq x(t_n^* + \tau + 0) B_n(t_n^*). \tag{15}$$

Then from (14) and (15) it follows that

$$\frac{x(t_n^* + \tau + 0)}{x(t_n^* + 0)} \leq \frac{1}{\delta}, \quad n \geq \nu. \tag{16}$$

From (2), by the Theorem on the impulsive differential inequalities ([1], Theorem 2.3) we have that

$$x(t + \tau) \geq x(t) \prod_{t \leq t_k < t + \tau} (1 + b_k) + \int_t^{t+\tau} p(s) \prod_{s \leq t_k < t + \tau} (1 + b_k) x(s + \tau) ds$$

for $n \geq \nu$ and $t \in (\eta_n, \eta_n - \tau)$. This implies

$$x(t) \leq x(t + \tau) \left\{ 1 - \int_t^{t+\tau} p(s) \prod_{s \leq t_k \leq t + \tau} (1 + b_k) ds \right\},$$

since $1 + b_k \geq 1$ and $x(s + \tau) \geq x(t + \tau + 0)$ for $s \in (t, t + \tau)$. Taking into account that $1 - x \leq e^{-x}, x \in \mathbb{R}$, we obtain

$$x(t) \leq x(t + \tau) \exp \left\{ - \int_t^{t+\tau} p(s) \prod_{s \leq t_k \leq t + \tau} (1 + b_k) ds \right\}.$$

Consequently, for each $n \geq \nu$ and $t \in (\xi_n, \eta_n - \tau)$

$$\frac{x(t + \tau)}{x(t)} \geq e^k \geq eK$$

Repeating the above arguments, we get to

$$\frac{x(t + \tau)}{x(t)} \geq (eK)^{m_0} \tag{17}$$

for any $n \geq \nu$ and $t \in (\xi_n, \eta_n - m_0\tau)$.

Since $\xi_n + \tau < \eta_n - m_0\tau$, then (17) is valid for any $n \geq \nu$ and $t = t_n^* \in [\xi_n, \xi_n + \tau]$, i.e.,

$$\frac{x(t_n^* + \tau + 0)}{x(t_n^* + 0)} \geq (eK)^{m_0}. \qquad \text{ä}(18)$$

Then from (16) and (18) there follows the inequality

$$\delta^{-1} \geq (eK)^{m_0}$$

which contradicts (13).

The proof of assertions 2 and 3 is carried out as in Theorem 1.

Remark 2. Condition (10) can be given in the form

$$\liminf_{t \to +\infty} \left\{ \int_t^{t+\tau} p(s) \prod_{s \leq t_k \leq t+\tau} (1 + b_k) ds \right\} > e^{-1} \tag{19}$$

for $t \in \cup_{n=1}^{\infty} (\xi_n, \eta_n - \tau)$.

Remark 3. In the case when equation (1) is without impulse effect ($b_k = 0, k \in \mathbb{N}$) condition (10) has the form

$$\int_t^{t+\tau} p(s)ds \geq K > e^{-1}, \quad t \in (\xi_n, \eta_n - \tau), \quad n \geq \nu_1$$

and it implies that condition (11) is valid with $\delta = \frac{K^2}{4}$.

In the case when $b_k \neq 0$, (10) not always implies (11).

However, under some additional assumptions about the moments t_k and the magnitudes b_k of the impulse effects, condition 2 implies condition 3 of Theorem 2. This is so, for instance, if the number i_n of the moments of impulse effect in each interval $[\xi_n, \xi_n + \tau]$ is bounded ($i_n \leq \gamma, n \geq \nu$) and there exists $\mu > 0$ such that $1 + b_k \leq \mu$ for each $n \geq \nu$ and $t_k \in [\xi_n, \xi_n + \tau]$.

Remark 4. Condition 4 of Theorem 2 is met if we suppose that

$$\lim_{n \to \infty} (\eta_n - \xi_n) = +\infty.$$

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REFERENCES

[1] BAINOV D.D., SIMEONOV P.S., *Systems with Impulse Effect. Stability, Theory and Applications*, Ellis Horwood Series in Mathematics and its Applications, Ellis Horwood, Chichester, 1989.

[2] GOPALSAMY K., ZHANG B.G., *On delay differential equations with impulses*, J. Math. Anal. Appl., 139 (1989), 110-122.

[3] GYÖRI I., LADAS G., *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[4] LADDE G.S., LAKSHMIKANTHAM V., ZHANG B.G., *Oscillation Theory of Differential Equations with Deviating Arguments*, Pure and Applied Mathematics, Vol. 110, Marcel Dekker, New York 1987.

[5] SHEVELO V.N., *Oscillations of Solutions of Differential Equations with Deviating Arguments*, Naukova Dumka, Kiev, 1978 (in Russian).

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