Chapter 4

On the domain of some ordinary differential operators in spaces of continuous functions

The present chapter is devoted to the study of the following second order ordinary differential operator

$$Au = au'' + bu'$$

in spaces of continuous functions. In particular, we are interested in a precise description of the domain on which A generates a semigroup. In Chapter 1 we have computed explicitly the domain of the generator in the framework of L^p spaces, for $1 < p < \infty$, in higher dimensions. In Chapters 2 and 3 we have studied parabolic problems with Neumann or Dirichlet boundary conditions in an open set Ω of $\mathbb{R}^N$ and, by means of gradient estimates, we have obtained some information on the domains of the generators of the semigroups yielding the classical solutions to the above problems. But we did not come to a complete description of such domains. Also in the literature, one can find more results concerning L^p spaces, with $1 < p < \infty$ (see [11], [12], [37], [41]), rather than spaces of continuous functions. In [41] a complete description of the domain is given in $C_0(\mathbb{R}^N)$ when the operator contains a potential term which balances the growth of the drift coefficient. We refer to [34] for the case of Hölder spaces.

In this chapter we limit ourselves to the special case $N = 1$ and we deal with $C_b(\mathbb{R})$ and with $C(\overline{\mathbb{R}})$, the space of continuous functions having finite limits at $\pm\infty$. Here a detailed theory has been developed in the fifties by W. Feller who gave an explicit description of all the boundary conditions under which A generates a semigroup of positive contractions. An introduction to Feller's theory which is sufficient for our purposes can be found in [21, Subsection VI.4.c].

We consider A with its maximal domain in $C_b(\mathbb{R})$

$$D_{\max}(A) := \{u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R}) \mid Au \in C_b(\mathbb{R})\}$$

and we assume that

($\mathbf{H}_0$) $\lambda - A$ is injective on $D_{\max}(A)$ for some $\lambda > 0$.

This is equivalent to saying that $(A, D_{\max}(A))$ generates a semigroup of positive contractions in $C_b(\mathbb{R})$, which is not however strongly continuous (see Proposition 5.2.3).

If (H_0) holds, then $\lambda - A$ is injective on $D_{\max}(A)$ for all $\lambda > 0$. Moreover it turns out that $\lambda - A$ is injective on $D_{\max}(A)$ if and only if it is injective on $D_{\mathrm{m}}(A)$, where

$$D_{\mathrm{m}}(A) := \{u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid Au \in C(\overline{\mathbb{R}})\}$$

is the maximal domain in $C(\overline{\mathbb{R}})$, see Proposition 4.1.1 below. Then, from [21, Theorem VI.4.15], it follows that $(A, D_{\mathrm{m}}(A))$ generates a strongly continuous semigroup of positive contractions in $C(\overline{\mathbb{R}})$.

We point out that (H_0) is equivalent to requiring that $\pm\infty$ are inaccessible boundary points according to Feller's terminology, which means that, if $W(x) := \exp\left(-\int_0^x \frac{b(t)}{a(t)}\, dt\right)$, the function

$$R(x) := W(x) \int_0^x \frac{1}{a(t)W(t)}\, dt$$

is not summable either in $(-\infty, 0)$ or in $(0, +\infty)$. In many cases verifying these integral conditions is not by any means an easy task. A sufficient condition, which has the advantage to be easy to handle, is the existence of a positive function $V \in C^2(\mathbb{R})$ such that $\lim_{|x|\to\infty} V(x) = \infty$ and $AV \le \lambda V$ for some $\lambda > 0$, see again Proposition 4.1.1.

Our main results show that, under suitable conditions,

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$$

and, if a is bounded,

$$D_{\mathrm{m}}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid bu' \in C(\overline{\mathbb{R}})\}.$$

In this way, requiring that $Au \in C_b(\mathbb{R})$ (resp. $C(\overline{\mathbb{R}})$) is the same to requiring that the two terms au'' and bu' separately belong to $C_b(\mathbb{R})$ (resp. $C(\overline{\mathbb{R}})$).

Let us state our main assumptions:

$(\mathbf{H_1})$ $a \in C(\mathbb{R})$ and $a \ge \delta$ for some $\delta > 0$.

$(\mathbf{H_2})$ $b \in C^1(\mathbb{R})$ and there exist constants $c_1 \in \mathbb{R}$ and $c_2 < 1$ such that

$$a(x)b'(x) \le c_1 + c_2 b^2(x), \quad x \in \mathbb{R}\,.$$

We shall keep hypothesis (H_1) and (H_2) throughout Sections 4.1 and 4.2 together with (H_0), but we shall need stronger assumptions in Subsection 4.2.2. In fact, to describe the domain in $C(\overline{\mathbb{R}})$ we assume that $a \in C_b(\mathbb{R})$ and that b satisfies $|b'| \le c(1 + |b|)$.

4.1 Preliminary results

In this section we collect some preliminary results which will be useful for the sequel. We start by studying the injectivity of the operator $\lambda - A$ on $D_{\max}(A)$ and $D_{\mathrm{m}}(A)$, i.e. the uniqueness of the solution in $D_{\mathrm{m}}(A)$ and $D_{\max}(A)$ of the elliptic equation $\lambda u - Au = f$.

Proposition 4.1.1 *The following assertions are equivalent:*

(i) *(H_0) holds.*

(ii) *$\lambda - A$ is injective on $D_{\max}(A)$ for all $\lambda > 0$, hence $(A, D_{\max}(A))$ generates a semigroup of positive contractions in $C_b(\mathbb{R})$.*

(iii) *$\lambda - A$ is injective on $D_{\mathrm{m}}(A)$ for all $\lambda > 0$, hence $(A, D_{\mathrm{m}}(A))$ generates a strongly continuous semigroup of positive contractions in $C(\overline{\mathbb{R}})$.*

Moreover, if there exists a positive function $V \in C^2(\mathbb{R})$ such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and $AV \le \lambda V$ for some $\lambda > 0$, then the above conditions are satisfied.

PROOF. For $(i) \Leftrightarrow (ii)$ see [38, Proposition 3.5]. Implication $(ii) \Rightarrow (iii)$ is obvious, see also [21, Theorem VI.4.15].

Now we prove that (iii) implies (ii). Let $u \in D_{\max}(A)$ be such that $\lambda u - Au = 0$. From [21, Theorem VI.4.14] it follows that there exist two linearly independent solutions v_1 and v_2 of $(\lambda - A)v = 0$ such that v_1 (resp. v_2) is bounded (resp. unbounded) at $+\infty$ and unbounded (resp. bounded) at $-\infty$. Then $u = k_1 v_1 + k_2 v_2$, for some constants $k_1, k_2 \in \mathbb{R}$. Since u is bounded, $k_1 = k_2 = 0$, which means $u = 0$.

Finally if there exists a function V as above then (ii) holds as a consequence of Proposition 5.2.3. □

Now we prove some estimates which will be the main tool for the description of $D_{\max}(A)$.

Proposition 4.1.2 *Assume that $a > 0$ and that (H$_2$) holds. Let $M > 0$ and v be a function in $C^1([-M, M])$ such that $v(-M) = v(M) = 0$. Then*

$$\|bv\|_{[-M,M]} \leq \frac{1}{1-c_2}\, \|av' + bv\|_{[-M,M]} + \sqrt{\frac{c_1^+}{1-c_2}}\, \|v\|_{[-M,M]}\ , \tag{4.1.1}$$

where $c_1^+ = \max\{c_1, 0\}$.

PROOF. Set $f = av' + bv$. Let $x_0 \in [-M, M]$ be a maximum point of the function bv. We may suppose that $x_0 \in]-M, M[$ and $b(x_0) \neq 0$, otherwise $b(x_0)v(x_0) = 0$ and estimate (4.1.1) is trivially satisfied. Moreover, without loss of generality we assume that $\|bv\|_{[-M,M]} = b(x_0)v(x_0)$. Then $(bv)'(x_0) = 0$ and from hypothesis (H$_2$) it follows that

$$a(x_0)v'(x_0) = -a(x_0)b'(x_0)\frac{v(x_0)}{b(x_0)} \geq -c_1\frac{v(x_0)}{b(x_0)} - c_2 b(x_0)v(x_0)$$

and consequently

$$\|f\|_{[-M,M]} \geq f(x_0) = a(x_0)v'(x_0) + b(x_0)v(x_0) \geq (1-c_2)b(x_0)v(x_0) - c_1\frac{v(x_0)}{b(x_0)}\ .$$

Multiplying by $b(x_0)v(x_0) = \|bv\|_{[-M,M]}$ both sides of the previous inequality we get

$$\|bv\|_{[-M,M]}\|f\|_{[-M,M]} \geq (1-c_2)\|bv\|^2_{[-M,M]} - c_1 v^2(x_0) \geq (1-c_2)\|bv\|^2_{[-M,M]} - c_1^+\|v\|^2_{[-M,M]}\ .$$

If $x := \|bv\|_{[-M,M]}$, we have $x^2 \leq \alpha x + \beta$ with $\alpha = \frac{1}{1-c_2}\|f\|_{[-M,M]}$, $\beta = \frac{c_1^+}{1-c_2}\|v\|^2_{[-M,M]}$. It follows that $x \leq \alpha + \sqrt{\beta}$, that is

$$\|bv\|_{[-M,M]} \leq \frac{1}{1-c_2}\|f\|_{[-M,M]} + \sqrt{\frac{c_1^+}{1-c_2}}\, \|v\|_{[-M,M]}\ ,$$

which is the statement. □

Remark 4.1.3 Assume (H$_1$) and (H$_2$). If $u \in C^2([-M, M])$ is such that $u'(-M) = u'(M) = 0$ then Proposition 4.1.2 implies

$$\|bu'\|_{[-M,M]} \leq \frac{1}{1-c_2}\, \|Au\|_{[-M,M]} + \sqrt{\frac{c_1^+}{1-c_2}}\, \|u'\|_{[-M,M]}\ .$$

Now, if $\varepsilon > 0$ is sufficiently small, there exists a constant C_ε, independent of M, such that

$$\|u'\|_{[-M,M]} \leq \varepsilon\|u''\|_{[-M,M]} + C_\varepsilon\|u\|_{[-M,M]}\ .$$

Moreover we have that

$$\|u''\|_{[-M,M]} \le \frac{1}{\delta}\|au''\|_{[-M,M]} \le \frac{1}{\delta}\left(\|bu'\|_{[-M,M]} + \|Au\|_{[-M,M]}\right) .$$

Taking into account these estimates and choosing ε small enough we get

$$\|bu'\|_{[-M,M]} \le C\left(\|Au\|_{[-M,M]} + \|u\|_{[-M,M]}\right) \tag{4.1.2}$$

where C depends only on c_1, c_2 and δ.

Estimate (4.1.2) still holds for every function $u \in C^2(\mathbb{R})$ with compact support; indeed, it is sufficient to consider an interval containing the support of u. The next step is to show that if a is bounded then this estimate extends to every function $u \in C_b^2(\mathbb{R})$. This will be used in Subsection 4.2.2.

Proposition 4.1.4 *If $a \in C_b(\mathbb{R})$, $a \ge \delta > 0$ and (H_2) holds, then for every $u \in C_b^2(\mathbb{R})$ we have*

(i) $\|bu'\|_\infty \le C(\|Au\|_\infty + \|u\|_\infty)$;

(ii) $\|u''\|_\infty \le C\left(\|Au\|_\infty + \|u\|_\infty\right)$,

where $C = C(c_1, c_2, \delta)$.

Proof. Let $u \in C_b^2(\mathbb{R})$. We prove that

$$\|bu'\|_\infty \le \frac{1}{1-c_2}\|Au\|_\infty + \sqrt{\frac{c_1^+}{1-c_2}}\,\|u'\|_\infty. \tag{4.1.3}$$

Let $v = u'$ and $\eta \in C_c^\infty(\mathbb{R})$ be such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $[-1,1]$ and $\eta \equiv 0$ in $\mathbb{R} \setminus [-2,2]$. Set $\eta_n(x) = \eta(x/n)$. Then $a(v\,\eta_n)' + b(v\,\eta_n) = (av' + bv)\eta_n + a\,v\,\eta_n'$ and applying (4.1.1) to $v\,\eta_n \in C_c^1(\mathbb{R})$ we have

$$\|b\,v\,\eta_n\|_\infty \le \frac{1}{1-c_2}\,\|av' + bv\|_\infty + \frac{\|a\|_\infty}{1-c_2}\|v\,\eta_n'\|_\infty + \sqrt{\frac{c_1^+}{1-c_2}}\,\|v\|_\infty .$$

Letting $n \to \infty$ it follows that

$$\|bv\|_\infty \le \frac{1}{1-c_2}\,\|av' + bv\|_\infty + \sqrt{\frac{c_1^+}{1-c_2}}\,\|v\|_\infty ,$$

which is just estimate (4.1.3). Now, (i) follows from (4.1.3) as in Remark 4.1.3.

Estimate (ii) easily follows from (i). □

4.2 Characterization of the domain

4.2.1 The case of $C_b(\mathbb{R})$

In this subsection we show that $D_{\max}(A)$ is given by

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}.$$

The crucial point is to prove that $\lambda - A$ is surjective from the right-hand side above onto $C_b(\mathbb{R})$. This is done through an approximation procedure by considering the solutions of the equation $\lambda u - Au = f$ in bounded intervals with Neumann boundary conditions and applying the estimates of Section 4.1.

Proposition 4.2.1 *Assume that (H_0), (H_1) and (H_2) hold. Then*

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}.$$

Proof. Set $D(A) := \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$. Let $\lambda > 0$ and $f \in C_b(\mathbb{R})$ be fixed. For every $n \in \mathbb{N}$ consider the problem

$$\begin{cases} \lambda u - Au = f & \text{in } [-n,n] \\ u'(-n) = u'(n) = 0 \end{cases}$$

It is well known that there exists a unique solution $u_n \in C^2([-n,n])$ which satisfies the following estimate

$$\|u_n\|_{[-n,n]} \leq \frac{1}{\lambda}\|f\|_\infty \tag{4.2.1}$$

(see e.g. [21, Theorem VI.4.16]). The equality $\lambda u_n - Au_n = f$ implies that

$$\|Au_n\|_{[-n,n]} \leq 2\|f\|_\infty. \tag{4.2.2}$$

Taking into account estimate (4.1.2) we have

$$\|bu_n'\|_{[-n,n]} \leq C\left(\|Au_n\|_{[-n,n]} + \|u_n\|_{[-n,n]}\right) \leq \overline{C}\,\|f\|_\infty, \tag{4.2.3}$$

where $\overline{C} = \overline{C}(c_1, c_2, \delta, \lambda)$. Moreover

$$\delta\|u_n''\|_{[-n,n]} \leq \|au_n''\|_{[-n,n]} \leq \|Au_n\|_{[-n,n]} + \|bu_n'\|_{[-n,n]} \leq \overline{C}_1\,\|f\|_\infty \tag{4.2.4}$$

and, by interpolation

$$\|u_n'\|_{[-n,n]} \leq C_2(\|Au_n\|_{[-n,n]} + \|u_n\|_{[-n,n]}) \leq \overline{C}_2\,\|f\|_\infty \tag{4.2.5}$$

with $\overline{C}_1$ and $\overline{C}_2$ depending only on $c_1, c_2, \delta, \lambda$. Now fix $k \in \mathbb{N}$ and consider $n \geq k$. Then the previous estimates imply that $\|u_n\|_{C^2([-k,k])}$ is bounded by a constant independent of n and k. It follows that the sequences (u_n), (u_n') are bounded and equicontinuous, then by Ascoli-Arzelà Theorem there exists a subsequence of (u_n) which converges in $C^1([-k,k])$. Using a diagonal procedure we can construct a subsequence, still denoted by (u_n), and a function $u \in C^1(\mathbb{R})$ such that u_n converges to u together with the first derivatives uniformly on every compact subset of $\mathbb{R}$. It follows that bu_n' converges to bu' uniformly on compact sets and, using the equation $\lambda u_n - Au_n = f$, it turns out that au_n'' and consequently u_n'' converge, too. Therefore $u \in C^2(\mathbb{R})$ and $\lambda u - Au = f$. Writing estimates (4.2.3), (4.2.4) and (4.2.5) for the function u_n in $[-k,k]$ with $n \geq k$ and letting first $n \to \infty$ and then $k \to \infty$ we obtain that $u \in C_b^2(\mathbb{R})$ with $au'', bu' \in C_b(\mathbb{R})$, i.e. $u \in D(A)$.

This shows that $\lambda - A : D(A) \to C_b(\mathbb{R})$ is surjective. Since $D(A) \subset D_{\max}(A)$ and $\lambda - A : D_{\max}(A) \to C_b(\mathbb{R})$ is bijective we deduce that $D(A) = D_{\max}(A)$, as claimed. □

4.2.2 The case of $C(\overline{\mathbb{R}})$

As in the previous subsection we show that the domain $D_{\mathrm{m}}(A)$ on which A generates a strongly continuous semigroup in $C(\overline{\mathbb{R}})$ is given by

$$D_{\mathrm{m}}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid \ bu' \in C(\overline{\mathbb{R}})\}.$$

To this aim we require that

($\mathbf{H_0'}$) there exist positive constants d_1, d_2 such that

$$b(x)x \leq d_1(1+x^2)\log(1+x^2) + d_2, \quad x \in \mathbb{R}.$$

($\mathbf{H_1'}$) $a \in C_b(\mathbb{R})$ and $a \geq \delta$ for some $\delta > 0$.

($\mathbf{H_2'}$) $b \in C^1(\mathbb{R})$ and $|b'(x)| \leq c(1+|b(x)|)$, for some constant $c > 0$ and for all $x \in \mathbb{R}$.

Since a is bounded one easily verify that the function $V(x) = 1+\log(1+x^2)$ satisfies the hypothesis of Proposition 4.1.1. Hence $(A, D_{\mathrm{m}}(A))$ generates a semigroup in $C(\overline{\mathbb{R}})$. Clearly (H_1') and (H_2') imply (H_1) and (H_2), thus we may use the results of Subsection 4.2.1.

Proposition 4.2.2 *Assume that (H_0'), (H_1') and (H_2') hold. Then*

$$D_{\mathrm{m}}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid \; bu' \in C(\overline{\mathbb{R}})\}.$$

Proof. Set $D := \{u \in C^2(\overline{\mathbb{R}}) \mid bu' \in C(\overline{\mathbb{R}})\}$. Since $\lambda - A : D_{\mathrm{m}}(A) \to C(\overline{\mathbb{R}})$ is bijective and $D \subset D_{\mathrm{m}}(A)$, it is sufficient to prove that $\lambda - A : D \to C(\overline{\mathbb{R}})$ is surjective.

Step 1: We assume first that $a \equiv 1$. Let $\lambda > 0$ and $f \in C(\overline{\mathbb{R}})$ be fixed. From Proposition 4.2.1 we know that there exists $u \in D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid bu' \in C_b(\mathbb{R})\}$ such that $\lambda u - Au = f$. On the other hand, since $(A, D_{\mathrm{m}}(A))$ generates a strongly continuous semigroup of contractions, there is $w \in D_{\mathrm{m}}(A)$ which solves the same equation. By uniqueness $u = w$. This means that $u \in C_b^2(\mathbb{R}) \cap C(\overline{\mathbb{R}})$ with $Au \in C(\overline{\mathbb{R}})$, $bu' \in C_b(\mathbb{R})$ and $\lambda u - Au = f$. It remains to prove that $u', u'', bu' \in C(\overline{\mathbb{R}})$. Since u' is uniformly continuous and u admits finite limits at $\pm\infty$ we deduce that $\lim_{|x|\to\infty} u'(x) = 0$. In order to use the same argument for u'' we first assume $f \in C(\overline{\mathbb{R}}) \cap C_b^1(\mathbb{R})$. Then we may differentiate the equation

$$\lambda u - u'' - bu' = f \tag{4.2.6}$$

obtaining

$$\lambda v - v'' - bv' = f' + b'v \; ,$$

where $v = u'$. (H_2') implies that $g := f' + b'v \in C_b(\mathbb{R})$. Therefore $v \in D_{\max}(A)$ and Proposition 4.2.1 implies that $v \in C_b^2(\mathbb{R})$. This means that $u \in C_b^3(\mathbb{R})$ and as before it implies that $u'' \in C(\overline{\mathbb{R}})$, with $\lim_{|x|\to\infty} u''(x) = 0$.

Now take $f \in C(\overline{\mathbb{R}})$. Set $f_\varepsilon := \Phi_\varepsilon * f \in C(\overline{\mathbb{R}}) \cap C_b^1(\mathbb{R})$ for $\varepsilon > 0$, where (Φ_ε) is a family of standard mollifiers. From the previous computations, for every $\varepsilon > 0$ the solution u_ε of the equation $\lambda u_\varepsilon - Au_\varepsilon = f_\varepsilon$ belongs to D. Let $u \in D_{\max}(A)$ be the solution of $\lambda u - Au = f$ and consider the difference $u - u_\varepsilon$. Then $u - u_\varepsilon \in C_b^2(\mathbb{R})$ with $A(u - u_\varepsilon) \in C_b(\mathbb{R})$ and $\lambda(u - u_\varepsilon) - A(u - u_\varepsilon) = f - f_\varepsilon$. Moreover

$$\|u - u_\varepsilon\|_\infty \leq \frac{1}{\lambda}\|f_\varepsilon - f\|_\infty \quad ,$$

thus from the equation we get

$$\|Au - Au_\varepsilon\|_\infty \leq 2\|f_\varepsilon - f\|_\infty$$

and from Proposition 4.1.4(ii) it follows that

$$\|u'' - u_\varepsilon''\|_\infty \leq C(\|Au - Au_\varepsilon\|_\infty + \|u - u_\varepsilon\|_\infty) \; .$$

Since f_ε converges uniformly to f as $\varepsilon \to 0$, we obtain that u_ε'' converges uniformly to u'' as $\varepsilon \to 0$. Since each u_ε'' tends to 0 as $|x| \to \infty$, we conclude that $\lim_{|x|\to\infty} u'' = 0$. Therefore $u \in C^2(\overline{\mathbb{R}})$ and $bu' \in C(\overline{\mathbb{R}})$, i.e. $u \in D$.

Step 2: Now we consider a generic function a satisfying (H_1'). We endow the domain D with the canonical norm

$$\|u\|_D = \|u\|_{C^2(\overline{\mathbb{R}})} + \|bu'\|_\infty ,$$

and we apply the method of continuity to the operators

$$A_t := (ta + 1 - t)\frac{d^2}{dx^2} + b\frac{d}{dx}, \qquad t \in [0,1] .$$

Let $u \in D \subset D_{\max}(A)$. We observe that the constants c_1, c_2 in (H_2) and δ in (H_1') are independent of $t \in [0,1]$, so, applying Proposition 4.2.1 with A_t instead of A and letting $n \to \infty$ in estimates (4.2.1), (4.2.3), (4.2.4) and (4.2.5), we obtain for $\lambda > 0$

$$\|u\|_D \le C\|(\lambda - A_t)u\|_\infty ,$$

where the constant C is independent of $t \in [0,1]$.

Since $\lambda - A_0 : D \to C(\overline{\mathbb{R}})$ is bijective from step 1, we conclude that $\lambda - A_1 = \lambda - A$ is bijective, too. $\square$

4.2.3 Examples

Assume for simplicity that $a \equiv 1$. If b is given by $b(x) = -|x|^r x$, with $r \ge 0$, then it is readily seen that the function $V(x) = 1 + x^2$ satisfies $AV \le \lambda V$ for $\lambda > 0$ sufficiently large. Then Proposition 4.1.1 holds and A endowed with its maximal domain is a generator both in $C_b(\mathbb{R})$ and in $C(\overline{\mathbb{R}})$. The corresponding semigroup is differentiable for $r > 0$, but never analytic in $C_b(\mathbb{R})$ (see [40, Propositions 4.4 and 3.5]). Since (H_1') and (H_2') are satisfied, Propositions 4.2.1 and 4.2.2 hold.

Condition (H_2) is satisfied by all polynomials and functions like e^P with P a polynomial. But if b oscillates too fast then (H_2) is not true and $D_{\max}(A)$ is not contained in general in $C_b^1(\mathbb{R})$ as shown in Example 2.4.7.

As far as hypothesis (H_2') is concerned, we remark that it holds for example for e^x but not for e^{x^2}. In this last situation we do not know whether Proposition 4.2.2 still holds.