Appendix A

Maximum principles

In this appendix we state and prove the maximum principles used in the previous chapters. They are not classical, since the coefficients of the involved operator are unbounded. More precisely, let us consider

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i\, D_i - V, \tag{A.0.1}$$

with $q_{ij} = q_{ji}$, F_i, V continuous real-valued functions in $\mathbb{R}^N$, satisfying

$$V \geq 0, \qquad \sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2,\ \nu_0 > 0.$$

To overcome the unboundedness of the coefficients, we make the following assumption

(H) *there exists a positive function $\varphi \in C^2(\mathbb{R}^N)$, such that $\lim_{|x|\to+\infty} \varphi(x) = +\infty$ and $A\varphi - \lambda_0\,\varphi \leq 0$, for some $\lambda_0 > 0$.*

φ is called a *Liapunov function.* Clearly, assumption (H) gives growth bounds on the coefficients of A. If for instance $\varphi(x) = 1 + |x|^2$, then (H) is satisfied if there exists $\lambda_0 > 0$ such that

$$\mathrm{Tr}\, Q(x) + \langle F(x), x\rangle \leq \lambda_0(1 + |x|^2).$$

It can be assumed that $\sup_{\mathbb{R}^N}(A\varphi - \lambda_0\varphi) < +\infty$. This does not make any difference since replacing φ with $\varphi + C$ for a suitable constant C, we return exactly to (H). Moreover, when one deals with parabolic problems, it is possible to consider φ dependent also on time and to require that $\varphi \in C^2([0,T]\times\mathbb{R}^N)$, $\varphi \geq 0$, $\lim_{|x|\to+\infty}\varphi(t,x) = +\infty$ uniformly in $[0,T]$ and $(D_t - A + \lambda_0)\varphi \geq 0$. Since we are concerned both with parabolic and elliptic problems and since the coefficients of A do not depend on t, we keep assumption (H) throughout the manuscript.

We start by proving maximum principles for parabolic and elliptic problems in a regular, (possibly) unbounded open set Ω of $\mathbb{R}^N$ with Neumann boundary conditions. Such results have been used in Chapter 2. In this case it is sufficient for φ to be defined in $\overline{\Omega}$, but we have to require an additional condition concerning its normal derivatives on $\partial\Omega$. The proof is similar to [34, Proposition 2.1].

Proposition A.0.5 *Let Ω be an open set in $\mathbb{R}^N$ with C^1 boundary. Assume (H) and in addition suppose that $\dfrac{\partial\varphi}{\partial\eta} \geq 0$ on $\partial\Omega$, where η is the outward unit normal vector to $\partial\Omega$. Let $z \in C([0,T]\times$*

$\overline{\Omega}) \cap C^1(]0,T] \times \overline{\Omega}) \cap C^{1,2}(]0,T] \times \Omega)$ *be a bounded function satisfying*

$$\begin{cases} z_t(t,x) - Az(t,x) \le 0, & 0 < t \le T,\ x \in \Omega, \\ \dfrac{\partial z}{\partial \eta}(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ z(0,x) \le 0 & x \in \Omega. \end{cases}$$

Then $z \le 0$.

Proof. Set $v(t,x) = e^{-\lambda_0 t} z(t,x)$; we prove that $v \le 0$, then the statement follows. We consider the sequence

$$v_n(t,x) = v(t,x) - \frac{1}{n}\varphi(x), \qquad 0 \le t \le T,\ x \in \Omega,$$

and we observe that

$$\begin{cases} D_t v_n(t,x) - (A - \lambda_0) v_n(t,x) \le 0, & 0 < t \le T,\ x \in \Omega, \\ \dfrac{\partial v_n}{\partial \eta}(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ v_n(0,x) \le 0, & x \in \overline{\Omega}. \end{cases}$$

For every $n \in \mathbb{N}$ the function v_n attains its maximum in $[0,T] \times \overline{\Omega}$ at some point (t_n, x_n). If $t_n > 0$ and $x_n \in \Omega$ then

$$D_t v_n(t_n, v_n) \ge 0, \qquad A v_n(t_n, x_n) + V(x_n) v_n(t_n, x_n) \le 0,$$

and consequently, using the equation

$$(\lambda_0 + V(x_n)) v_n(t_n, x_n) \le (\lambda_0 + D_t - A) v_n(t_n, x_n) \le 0.$$

Since $\lambda_0 > 0$ this implies that $v_n(t_n, x_n) \le 0$.

If $t_n = 0$ we immediately have $v_n(t_n, x_n) \le 0$. Finally, it is not possible that $t_n > 0$ and $x_n \in \partial\Omega$, without any interior maximum point because of the strong maximum principle ([24, Theorem 2.14]).

Therefore we have proved that $v(t,x) \le n^{-1}\varphi(x)$ for all $0 \le t \le T$ and $x \in \overline{\Omega}$. Thus letting $n \to +\infty$ we conclude that $v \le 0$, as claimed. $\square$

A similar maximum principle holds in the elliptic case. However, we point out that the involved solutions are only of class $W^{2,p}$ and not C^2 in general. To prove such a result we need a maximum principle for operators with bounded coefficients, which is due to Bony (see [9]).

Lemma A.0.6 *Let Ω be an open subset of $\mathbb{R}^N$ and let $F : \Omega \to \mathbb{R}^N$ be a function of class $W^{1,p}$, with $p > N$. Then the image through F of a set with measure zero has still measure zero.*

Proof. Let Q_1 be a unitary cube of $\mathbb{R}^N$. By Morrey's inequality (see [10, Teorema IX.12]), if $\varphi \in W^{1,p}(Q_1)$ then

$$|\varphi(x) - \varphi(y)| \le C|x-y|^{1-\frac{N}{p}} \left(\int_{Q_1} |D\varphi|^p \right)^{\frac{1}{p}}, \qquad x, y \in Q_1, \tag{A.0.2}$$

where C is a positive constant depending on p and N. In the sequel, we keep the same notation to denote a constant which has such a dependence. If Q_α is a cube with side l_α and ψ is a function in $W^{1,p}(Q_\alpha)$, then $\varphi(x) = \psi(l_\alpha x)$ belongs to $W^{1,p}(Q_1)$ and (A.0.2) applied to φ yields

$$|\psi(l_\alpha x) - \psi(l_\alpha y)| \le C|x-y|^{1-\frac{N}{p}} \left(\int_{Q_1} l_\alpha^p |D\psi(l_\alpha z)|^p dz \right)^{\frac{1}{p}}, \qquad x, y \in Q_1.$$

By changing variables in the integral we get

$$\begin{aligned}
|\psi(l_\alpha x) - \psi(l_\alpha y)| &\le C|x-y|^{1-\frac{N}{p}} \Big(\int_{Q_\alpha} l_\alpha^{p-N} |D\psi(z)|^p dz \Big)^{\frac{1}{p}} \\
&= C\, l_\alpha^{1-\frac{N}{p}} |x-y|^{1-\frac{N}{p}} \Big(\int_{Q_\alpha} |D\psi(z)|^p dz \Big)^{\frac{1}{p}} \\
&\le C\, l_\alpha^{1-\frac{N}{p}} \Big(\int_{Q_\alpha} |D\psi(z)|^p dz \Big)^{\frac{1}{p}}, \qquad x, y \in Q_1.
\end{aligned}$$

Therefore

$$|\psi(\xi) - \psi(\eta)| \le C\, l_\alpha^{1-\frac{N}{p}} \Big(\int_{Q_\alpha} |D\psi(x)|^p dx \Big)^{\frac{1}{p}}, \qquad \xi, \eta \in Q_\alpha. \tag{A.0.3}$$

Let M be a subset of Ω with $|M| = 0$, where $|\cdot|$ denotes the Lebesgue measure. Then, for every $\varepsilon > 0$ there exists a family $\{Q_\alpha\}_\alpha$ of disjoint cubes such that $M \subseteq \cup_\alpha Q_\alpha \subseteq \Omega$ and $\sum_\alpha l_\alpha^N \le \varepsilon$, where l_α denotes the side of Q_α. By applying (A.0.3) to the scalar components $F_1, ..., F_N$ of the function F, we obtain for every α and every $x, y \in Q_\alpha$

$$\begin{aligned}
|F(x) - F(y)| &\le \sum_{i=1}^N |F_i(x) - F_i(y)| \le C\, l_\alpha^{1-\frac{N}{p}} \sum_{i=1}^N \Big(\int_{Q_\alpha} |DF_i(z)|^p dz \Big)^{\frac{1}{p}} \\
&\le C\, l_\alpha^{1-\frac{N}{p}} \Big(\int_{Q_\alpha} \Big(\sum_{i,j=1}^N |D_j F_i| \Big)^p \Big)^{\frac{1}{p}} =: \lambda_\alpha.
\end{aligned}$$

This means that $F(Q_\alpha)$ is contained in the cube $\widetilde{Q}_\alpha$ with side λ_α. It follows that

$$F(M) \subseteq F\Big(\bigcup_\alpha Q_\alpha \Big) \subseteq \bigcup_\alpha F(Q_\alpha) \subseteq \bigcup_\alpha \widetilde{Q}_\alpha$$

and consequently

$$|F(M)| \le \sum_\alpha |\widetilde{Q}_\alpha| = \sum_\alpha \lambda_\alpha^N = C^N \sum_\alpha \Big[l_\alpha^{N(1-\frac{N}{p})} \Big(\int_{Q_\alpha} \Big(\sum_{i,j=1}^N |D_j F_i| \Big)^p \Big)^{\frac{N}{p}} \Big].$$

Applying Hölder's inequality with exponents $r = p/N$ and $r' = (1 - N/p)^{-1}$, we get

$$\begin{aligned}
|F(M)| &\le C^N \Big(\sum_\alpha l_\alpha^N \Big)^{1-\frac{N}{p}} \Big(\sum_\alpha \int_{Q_\alpha} \Big(\sum_{i,j=1}^N |D_j F_i| \Big)^p \Big)^{\frac{N}{p}} \\
&\le C^N \varepsilon^{1-\frac{N}{p}} \Big(\int_\Omega \Big(\sum_{i,j=1}^N |D_j F_i| \Big)^p \Big)^{\frac{N}{p}}.
\end{aligned}$$

Since ε was arbitrary, the thesis follows. □

Proposition A.0.7 *Let Ω be a bounded open set of $\mathbb{R}^N$ with C^1 boundary and let $u \in W^{2,p}(\Omega)$, with $p > N$. Assume that u attains its maximum M at $x_0 \in \Omega$ and that $u(x) < M$, for every $x \in \overline{\Omega} \setminus \{x_0\}$. Then, for each closed neighborhood V of x_0 there exists $E \subseteq V$ with $|E| > 0$, such that for almost all $x \in E$ the Hessian matrix of u, $(D^2 u(x))$, is nonpositive, i.e. $\langle D^2 u(x)\xi, \xi \rangle \le 0$, for all $\xi \in \mathbb{R}^N$.*

PROOF. Let S be the hypersurface of $\mathbb{R}^{N+1}$ given by the equation $y = u(x)$, $x \in \Omega, y \in \mathbb{R}$. Since $p > N$, by the Sobolev embeddings the function u belongs to $C^1(\overline{\Omega})$, hence S is of class C^1. This ensures that the tangent hyperplane is well defined at each point of S. Let V be a closed neighborhood of x_0 contained in Ω and let us denote by E the set of points x in V with the property that S lies locally under the tangent hyperplane at $(x, u(x))$. We observe that E is nonempty since it contains x_0. Now, we claim that E has positive measure. Let us first show that there exists $\delta > 0$ such that if $h \in \mathbb{R}^N$ and $|h| < \delta$, then there are a point $\xi \in E$ and a real number α such that the hyperplane of equation $y = \langle h, x\rangle + \alpha$ is tangent to S at the point $(\xi, u(\xi))$. To this aim, we observe that $\inf_{\overline{\Omega}\setminus V}(M - u(x)) > 0$. Otherwise, there exists a sequence $(x_n) \subseteq \overline{\Omega} \setminus V$ such that $u(x_n)$ converges to M. By compactness, we can find $y \in \overline{\Omega} \setminus \{x_0\}$ and a subsequence (x_{n_k}) such that $x_{n_k} \to y$ and therefore, by continuity, $u(x_{n_k}) \to u(y) = M$. But this is impossible since x_0 was, by the assumption, the unique maximum point of u in $\overline{\Omega}$. Now consider $\lambda = \inf\limits_{\overline{\Omega}\setminus V}(M - u(x))\Big(\sup\limits_{\overline{\Omega}\setminus V}|x - x_0|\Big)^{-1} > 0$ and choose $0 < \delta < \lambda$. Then, for every $h \in \mathbb{R}^N$ with $|h| < \delta$ and every $x \in \overline{\Omega} \setminus V$ we have

$$\begin{aligned} u(x) - M - \langle h, x - x_0\rangle &< u(x) - M + \inf_{\overline{\Omega}\setminus V}(M - u(x))\Big(\sup_{\overline{\Omega}\setminus V}|x - x_0|\Big)^{-1}|x - x_0| \\ &\leq \inf_{\overline{\Omega}\setminus V}(M - u(x)) - (M - u(x)) \leq 0, \end{aligned}$$

hence

$$u(x) < \langle h, x\rangle + M - \langle h, x_0\rangle, \qquad \text{for all } x \in \overline{\Omega} \setminus V. \tag{A.0.4}$$

Since V is compact and $u(x) - \langle h, x\rangle$ is a continuous function in V, there exists $\xi \in V$ such that

$$\max_{x\in V}(u(x) - \langle h, x\rangle) = u(\xi) - \langle h, \xi\rangle =: \alpha.$$

In particular, $\alpha \geq u(x_0) - \langle h, x_0\rangle = M - \langle h, x_0\rangle$ and therefore from (A.0.4) it follows that

$$u(x) < \langle h, x\rangle + \alpha, \qquad \text{for all } x \in \overline{\Omega} \setminus V.$$

On the other hand, by construction,

$$u(x) \leq \langle h, x\rangle + \alpha, \qquad \text{for all } x \in V,$$

then $u(x) \leq \langle h, x\rangle + \alpha$, for every $x \in \overline{\Omega}$. Since $u(\xi) = \langle h, \xi\rangle + \alpha$, we deduce also that $Du(\xi) = h$ and therefore the hyperplane $y = \langle h, x\rangle + \alpha$ is in fact the tangent hyperplane to S at $(\xi, u(\xi))$. Since it lies over S, we have that $\xi \in E$. Now, define $F : \Omega \to \mathbb{R}^N$ as $F(x) = Du(x)$. From the previous step, if $h \in \mathbb{R}^N$ and $|h| < \delta$, then there exists $\xi \in E$ such that $h = Du(\xi) = F(\xi)$. This means that $B_\delta \subseteq F(E)$ and, as a consequence, $|F(E)| > 0$. Since F is of class $W^{1,p}(\Omega)$, from the previous lemma it follows that E has positive measure, too.

Now, the regularity of u implies that u is almost everywhere twice differentiable in the classical sense. Let $x \in E$ be such that u is twice differentiable at x in the classical sense and assume, by contradiction, that there exists $y \in \mathbb{R}^N$ such that $\sum_{i,j=1}^N D_{ij}u(x)\, y_i y_j > 0$. Without loss of generality we can suppose that $|y| = 1$. Set $f(t) = u(x + ty) - t\langle Du(x), y\rangle$, for $|t| < \varepsilon$, for some $\varepsilon > 0$. Then f is differentiable in $(-\varepsilon, \varepsilon)$ with $f'(0) = 0$ and f'' exists at $t = 0$ with $f''(0) = \sum_{i,j=1}^N D_{ij}u(x)\, y_i y_j > 0$. This implies that $t = 0$ is a strict relative minimum point for f, hence $f(t) > f(0)$ for $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$, which means $u(x + ty) > u(x) + t\langle Du(x), y\rangle$, for $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$. On the other hand, since $x \in E$, for every z sufficiently close to x we have

$$u(z) \leq u(x) + \langle Du(x), z - x\rangle.$$

Choosing $z = x + ty$ we find

$$u(x+ty) \leq u(x) + t\langle Du(x), y\rangle,$$

which is a contradiction. Thus, we have established that at each point $x \in E$ where u is twice differentiable in the classical sense, $(D^2u(x))$ is nonpositive. This concludes the proof. □

At this point, we are ready to prove the announced maximum principle for $W^{2,p}$ functions involving operators with bounded coefficients. More precisely, let

$$L = \sum_{i,j=1}^{N} \alpha_{ij} D_{ij} + \sum_{i=1}^{N} \beta_i \, D_i + \gamma.$$

Assume that all the coefficients are real-valued functions in $L^\infty(\Omega)$ and that the matrix (α_{ij}) is symmetric and nonnegative and that $\gamma \leq 0$.

Theorem A.0.8 *Let Ω be a bounded open set with C^1 boundary and let $u \in W^{2,p}(\Omega)$, with $p > N$. Assume that u attains a nonnegative maximum at $x_0 \in \Omega$. Then*

$$\liminf_{x\to x_0} \operatorname{ess} (Lu)(x) \leq 0,$$

where $\liminf\limits_{x\to x_0} \operatorname{ess} (Lu)(x) = \sup\limits_{\rho>0} \operatorname*{inf\,ess}\limits_{x\in\overline{B_\rho(x_0)}} Lu(x)$.

Proof. Let $\varepsilon > 0$ and set $v(x) = u(x) - \varepsilon|x - x_0|^2$. It is readily seen that $v \in W^{2,p}(\Omega)$ and that x_0 is a strict maximum point for v. Then, from Proposition A.0.7 for each $\rho > 0$, there exists a set $E_\rho \subset \overline{B_\rho(x_0)}$ such that $|E_\rho| > 0$ and $(D^2v(x))$ is nonpositive for almost all $x \in E_\rho$. Since (α_{ij}) is nonnegative a.e., we deduce that

$$\sum_{i,j=1}^{N} \alpha_{ij}(x) D_{ij} v(x) \leq 0, \quad \text{for almost all } x \in E_\rho.$$

On the other hand, since $v \in C^1(\Omega)$, we have that $\lim\limits_{x\to x_0} D_i v(x) = D_i v(x_0) = 0$ and hence, using the boundedness of β_i

$$\lim_{x\to x_0} \sum_{i=1}^{N} \beta_i(x) D_i v(x) = 0.$$

Finally, since $\gamma(x) \leq 0$ and $v(x_0) = u(x_0) \geq 0$ we have that $\lim_{x\to x_0} \gamma(x)v(x) = 0$, if $v(x_0) = 0$. If $v(x_0) > 0$ then, by continuity, $v(x) > 0$ for x close to x_0, hence $\gamma(x)v(x) \leq 0$. Therefore we have

$$\begin{aligned}
\liminf_{x\to x_0} \operatorname{ess} (Lv)(x) &= \sup_{\rho>0} \operatorname*{inf\,ess}_{x\in\overline{B_\rho(x_0)}} (Lv)(x) \\
&\leq \sup_{\rho>0} \operatorname*{inf\,ess}_{x\in E_\rho} \left(\sum_{i,j=1}^{N} \alpha_{ij}(x) D_{ij} v(x) + \sum_{i=1}^{N} \beta_i(x) D_i v(x) + \gamma(x)v(x) \right) \\
&\leq 0.
\end{aligned}$$

Thus we have established that $\liminf\limits_{x\to x_0} \operatorname{ess} (Lv)(x) \leq 0$. Since

$$Lv(x) = Lu(x) - 2\varepsilon \sum_{i=1}^{N} \alpha_{ii}(x) - 2\varepsilon \sum_{i=1}^{N} \beta_i(x)(x_i - x_0^i) - \varepsilon\gamma(x)|x - x_0|^2,$$

we obtain that

$$\liminf_{x\to x_0} \operatorname{ess} Lu(x) \leq 2\varepsilon \sum_{i=1}^{N} \|\alpha_{ii}\|_\infty.$$

Letting $\varepsilon \to 0$, we get the statement. □

In the sequel, we use the previous result to derive an elliptic maximum principle for the operator A defined in (A.0.1). First we state an easy corollary of Theorem A.0.8, which is more useful for our aims.

Corollary A.0.9 *Let u belong to $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $p < \infty$ and suppose that $Au \in C(\mathbb{R}^N)$. If u has a relative nonnegative maximum at the point x_0, then $Au(x_0) \le 0$.*

Proposition A.0.10 *Let Ω be an open set in $\mathbb{R}^N$ with C^2 boundary. Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B_R)$ for all $R > 0$ and $p < \infty$, such that $Au \in C_b(\overline{\Omega})$ and*

$$\lambda u(x) - Au(x) \le 0, \qquad x \in \Omega,$$

for some $\lambda > 0$. Let $x_0 \in \partial\Omega$ such that $u(x_0) > 0$ and $u(x) < u(x_0)$ for all $x \in \Omega$. Then

$$\frac{\partial u}{\partial \eta}(x_0) > 0. \tag{A.0.5}$$

Proof. We follow the proof of the classical Hopf maximum principle (see e.g. [26, Lemma 3.4]). By the regularity assumption on $\partial\Omega$, we can consider a ball $B(y, r) \subset \Omega$ such that $\overline{B}(y, r) \cap \partial\Omega = \{x_0\}$. Assume that $u > 0$ in $B(y, r)$. It is readily seen that there exists $\alpha > 0$ such that the function $z(x) = e^{-\alpha|x-y|^2} - e^{-\alpha r^2}$ satisfies $Az > 0$ in $D = B(y, r) \setminus \overline{B}(y, r/2)$. Set $w = u + \varepsilon z$, where $\varepsilon > 0$ is chosen in such a way that $w(x) < u(x_0)$ for all $x \in \partial B(y, r/2)$. Then $w(x) \le u(x_0)$ in ∂D and

$$Aw(x) = Au(x) + \varepsilon Az(x) > \lambda u(x) > 0, \qquad x \in D. \tag{A.0.6}$$

Let $\overline{x} \in \overline{D}$ the maximum point of w in $\overline{D}$. It is not possible that $\overline{x} \in D$, otherwise from Corollary A.0.9 we should have $Aw(\overline{x}) \le 0$, which is in contradiction with (A.0.6). Then $\overline{x} \in \partial D$ and necessarily $\overline{x} = x_0$. It follows that

$$\frac{\partial w}{\partial \eta}(x_0) = \frac{\partial u}{\partial \eta}(x_0) + \varepsilon \frac{\partial z}{\partial \eta}(x_0) \ge 0.$$

Since $\partial z/\partial\eta(x_0) < 0$, this implies (A.0.5). $\square$

Proposition A.0.11 *Let Ω be an open set in $\mathbb{R}^N$ with C^2 boundary. Assume (H) and in addition suppose that $\dfrac{\partial \varphi}{\partial \eta} \ge 0$ on $\partial\Omega$, where η is the outward unit normal vector to $\partial\Omega$. Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B_R)$ for all $R > 0$ and $p < \infty$, such that $Au \in C_b(\overline{\Omega})$ and*

$$\begin{cases} \lambda u(x) - Au(x) \le 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta}(x) \le 0, & x \in \partial\Omega, \end{cases} \tag{A.0.7}$$

for some $\lambda \ge \lambda_0$. Then $u \le 0$.

Proof. As in Proposition A.0.5, we introduce the sequence

$$u_n(x) = u(x) - \frac{1}{n}\varphi(x), \qquad x \in \Omega$$

and we note that

$$\begin{cases} \lambda u_n(x) - Au_n(x) \le 0, & x \in \Omega, \\ \dfrac{\partial u_n}{\partial \eta}(x) \le 0, & x \in \partial\Omega. \end{cases} \tag{A.0.8}$$

We prove that $u_n \le 0$, for all $n \in \mathbb{N}$; then the conclusion follows letting $n \to \infty$. Each u_n has a maximum point $x_n \in \overline{\Omega}$. If $x_n \in \Omega$ then $u_n(x_n) \le 0$. Indeed, if $u_n(x_n) > 0$, then from Corollary A.0.9 it follows that $Au_n(x_n) \le 0$ and, using (A.0.8), $u_n(x_n) \le 0$, which is a contradiction. Now assume that $x_n \in \partial\Omega$ and $u_n(x) < u_n(x_n)$ for all $x \in \Omega$ (otherwise there would exist an interior maximum point and we could apply the previous step). Then from Proposition A.0.10 and (A.0.8) it follows that $u_n(x_n) \le 0$ and this completes the proof. □

Next, we deal with Dirichlet parabolic problems. We skip the proof of the following proposition, since it is exactly the same as that of Proposition A.0.5.

Proposition A.0.12 *Let Ω be an open set of $\mathbb{R}^N$ and assume hypothesis (H). Let $u \in C([0,T] \times \overline{\Omega}) \cap C^{1,2}(]0,T[\times\Omega)$ be a bounded function satisfying*

$$
\text{(A.0.9)} \qquad \begin{cases} u_t(t,x) \le Au(t,x), & 0 < t \le T,\ x \in \Omega, \\ u(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ u(0,x) \le 0 & x \in \Omega, \end{cases}
$$

Then $u \le 0$.

Now we present a maximum principle for discontinuous solutions to the Dirichlet parabolic problem (A.0.9). The result is suggested in [29] and involves special domains.

Theorem A.0.13 *Assume hypothesis (H). Let Ω be an open subset of $\mathbb{R}^N$, $g_i : \overline{\Omega} \to \mathbb{R}$, $i = 1, \ldots, n$, be C^2-functions. Suppose that*

$$\Omega = \{x : g_i(x) > 0, \ \ i = 1, \ldots, n\}, \ \ |Dg_i| \ge 1 \text{ on } \Gamma_i = \partial\Omega \cap \{g_i = 0\}.$$

Define $Q = (0,T) \times \Omega$, $\partial' Q = (0,T) \times \partial\Omega \cup \{0\} \times \overline{\Omega}$ and $\partial_{tx} Q = \{0\} \times \partial\Omega$. Let $u \in C^{1,2}(Q)$, u continuous on $\overline{Q} \setminus \partial_{tx}Q$, bounded on Q. If $u_t \le Au$ in Q and $u \le 0$ in $\partial' Q \setminus \partial_{tx} Q$, then $u \le 0$ in Q.

Finally, if $u_t = Au$, $|u(t,\xi)| \le K$ for $t > 0$, $\xi \in \partial\Omega$ and $|u(0,x)| \le K$, $x \in \Omega$, then $\|u\|_\infty \le K$.

Proof. The proof is given into two steps.
Step 1. We assume in addition that Ω is bounded.

In this case the functions g_i are bounded in Ω together with their derivatives up to the second order. A long but straightforward computation shows that the functions

$$
\text{(A.0.10)} \qquad \psi_i(t,x) = \frac{1}{t^{\varepsilon\nu}} \exp\left(\lambda t - \frac{\varepsilon g_i^2(x)}{t}\right)
$$

verify, for $\varepsilon > 0$ small enough and λ large enough, $(D_t - A)\psi_i \ge 0$, $i = 1, \ldots, n$, in $(0,\infty) \times \Omega$.

Let $M = \sup\limits_Q u = \sup\limits_{\overline{Q} \setminus \partial_{tx} Q} u > 0$ (otherwise the proof is finished). Let $\gamma > 0$ and define

$$u_\gamma(t,x) = u(t,x) - M\gamma^{\varepsilon\nu} \sum_{i=1}^n \frac{1}{(t+\gamma)^{\varepsilon\nu}} \exp\left(\lambda(t+\gamma) - \frac{\varepsilon g_i^2(x)}{t+\gamma}\right),$$

where ε and λ are given in (A.0.10). Clearly $(D_t - A)u_\gamma \le 0$. Take $\eta > 0$ such that $\lambda\gamma - \frac{\varepsilon\eta}{\gamma} > 0$ and consider

$$I_\eta = \{x \in \overline{\Omega} : \ \exists i = i(x) = 1, \ldots, n \ : \ g_i^2(x) \le \eta\}.$$

For each $x \in I_\eta$, one has

$$\gamma^{\varepsilon\nu} \sum_{i=1}^n \frac{1}{\gamma^{\varepsilon\nu}} \exp\left(\lambda\gamma - \frac{\varepsilon g_i^2(x)}{\gamma}\right) \ge \exp\left(\lambda\gamma - \frac{\varepsilon\eta}{\gamma}\right) > 1.$$

By continuity, there exists $\delta > 0$ such that for any $(t, x) \in [0, \delta] \times I_\eta$,

$$\gamma^{\varepsilon\nu} \sum_{i=1}^{n} \frac{1}{(t+\gamma)^{\varepsilon\nu}} \exp\left(\lambda(t+\gamma) - \frac{\varepsilon g_i^2(x)}{t+\gamma}\right) > 1.$$

It follows that $u_\gamma \le M - M = 0$ in $([0, \delta] \times I_\eta) \setminus \partial_{tx} Q$.

Since $u(0, x) \le 0$, $x \in \Omega \setminus I_\eta$, we have $u_\gamma(0, x) < 0$, $x \in \Omega \setminus I_\eta$ as well. Because Ω is bounded, by continuity we obtain $u_\gamma(t, x) \le 0$, $(t, x) \in [0, \delta] \times \Omega \setminus I_\eta$, for some $\delta > 0$.

We have obtained that $u_\gamma \le 0$ in $([0, \delta] \times \overline{\Omega}) \setminus \partial_{tx} Q$. Applying the classical maximum principle in $[\delta, T] \times \overline{\Omega}$, we get that $u_\gamma \le 0$ in $([0, T] \times \overline{\Omega}) \setminus \partial_{tx} Q$. Letting $\gamma \to 0^+$, we infer the claim.

Step 2. We consider a possibly unbounded Ω.

Here we will use the Lyapunov function φ. Set $v = e^{-\lambda_0 t} u$ and observe that $v_t - Av + \lambda_0 v \le 0$. We prove that $v \le 0$ in Q. Fix $R > 1$ and consider

$$\Omega_R = \Omega \cap B_R = \{g_i > 0\} \cap \{R^2 - |x|^2 > 0\}, \qquad Q_R = (0, T) \times \Omega_R.$$

Note that Ω_R satisfies the same geometric assumptions of Ω if one adds to the set $\{g_1, \ldots, g_n\}$ the function $g_0(x) = R^2 - |x|^2$. Let $C_R = \inf_{\partial B_R \cap \Omega} \varphi$. Remark that $C_R \to \infty$ as $R \to \infty$. Define

$$v_R(t, x) = v(t, x) - \|v\|_\infty \frac{\varphi(x)}{C_R}, \ \ (t, x) \in Q_R.$$

It is easy to see that $(D_t - A + \lambda_0) v_R \le 0$ in Q_R. Moreover $v_R(0, x) \le 0$, $x \in \Omega_R$.

If $t \in (0, T)$, then $v_R(t, x) \le 0$ for $x \in \partial B_R \cap \Omega$, since $\frac{\varphi}{C_R} \ge 1$. Moreover $v_R(t, x) \le 0$ for $x \in \partial\Omega$, $t \in (0, T)$. This shows that $v_R \le 0$ on the parabolic boundary of Q_R.

Applying Step 1 to the operator $\tilde{A} = A - \lambda_0$ in Ω_R, we get $v_R \le 0$, in Q_R, that is

$$v(t, x) \le \|v\|_\infty \frac{\varphi(x)}{C_R}.$$

Letting $R \to \infty$, we get the claim.

The last statement easily follows considering the functions $\pm u - K$. □

Observe that the above theorem covers also the case of certain non smooth domains, whose boundaries can be described by a finite number of functions g_i as in the statement, see e.g. Example 3.6.1.

Let us show that uniformly C^2 domains are covered by Theorem A.0.13.

Corollary A.0.14 . *Theorem A.0.13 holds for uniformly C^2-domains.*

Proof. It suffices to show that there exists a C^2-function $g : \overline{\Omega} \to \mathbb{R}$ such that $g > 0$ in Ω, $|Dg| \ge 1$ in $\partial\Omega = \{g = 0\}$. Let r be the distance function from $\partial\Omega$. Then $r \in C^2(\Omega_\delta)$ for some $\delta > 0$ and $|Dr| = 1$ on $\partial\Omega$. Let moreover θ be a smooth function such that $0 \le \theta \le 1$, $\theta = 1$ in $\Omega_{\delta/2}$, $\theta = 0$ outside Ω_δ. It is easy to check that $g = \theta r + 1 - \theta$ satisfies the claim. □