Introduction

In the last two centuries hydrodynamic stability has been recognized as one of the central problems of fluid mechanics. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. It has many applications in engineering, in meteorology, in oceanography, in astrophysics and in geophysics. Today, the mathematical, computational and experimental aspects of the stability of laminar fluid motions have generated a huge and continuous flow of scientific researches. We refer to the book by Drazin and Reid [17] and references therein for the stability of hydrodynamic laminar flows and to the treatises by Chandrasekhar [11] and Straughan [87] for the problems related to the onset of convection in a viscous fluid in both hydrodynamic and hydromagnetic contexts.

In physical sciences one is first of all interested in linear stability of the models one applies to desribe a real world phenomenon. This is because in many situations linear stability analysis is mathematical feasible and gives the opportunity to catch the essential informations, but in some situations it is necessary to perform a nonlinear stability analysis as the it gives sufficient conditions for stability whereas the normal modes analysis furnishes only sufficient conditions for instability. The main method to study the nonlinear stability in hydrodynamics is certainly the Liapunov direct method. This method is introduced and explained in all the details in the book by Flavin and Rionero [20]. The core of this methodology is the choice of a suitable physically meaningful Liapunov functional such as the energy associated with the perturbations to a basic solution of the PDEs adopted to model a real world phenomenon. Although the subsequent analysis can be highly intricate, the Liapunov direct method is quite powerful and applicable in many situations. The aim of this thesis is to apply the Liapunov direct method to some implicit constitutive theories for fluids, in particular to fluids with temperature and pressure dependent viscosity.

In his celebrated paper on the constitutive response of fluids Stokes [85] recognized that the viscosity of a fluid could depend on the pressure although the experiments of Du Buat on the motion of water in canals and pipes at normal pressures showed that the total retardation of the velocity due to friction is not increased by increasing the pressure. This, however, does not seem to be the case at high pressures, even for incompressible liquids such as water: it is obvious that the viscosity of the water near the surface of the Pacific Ocean would be far less than viscosity near the bottom! In fact, it has long been known that the viscosity of typical liquids begins to increase sustantially with pressure when pressures on the order of 1000 atm are reached. (References to much of the experimental studies concerning the pressure dependence of viscosity can be found in [65].)

To have an idea of how viscosity changes with pressure we may refer to some of the most important experimental works on the subject. To take into account these experimental evidences Barus [5] proposed the following exponential dependence of viscosity μ on pressure p in isothermal processes:

$$\mu(p) = \mu_0 \exp[\beta(p - p_0)],$$

where μ_0 is the viscosity at the reference pressure p_0 and β is the piezo-viscous coefficient whose value varies with temperature. Later, Andrade [2] suggested a relationship among viscosity, pressure, density ρ and temperature T of the type:

$$\mu(\rho, p, T) = A\rho^{1/2} \exp\left[(p + \rho r^2)\frac{s}{T}\right]$$

where r, s and A are constants. More recently, Laun [36] modelled the viscosity of polymer melts as

$$\mu(p, T) = \mu_0 \exp[\beta(p - p_0) - \gamma(T - T_0)] \tag{1}$$

where μ_0 is the viscosity at reference pressure p_0 and temperature T_0 and the non-negative constants β and γ are, respectively, the pressure and temperature coefficient of viscosity. There have been numerous other experiments by Bair and co-workers that shows that the dependence of the viscosity on the pressure is exponential (see recent experiments of Bair and Kottke [4]). Mention must be made of the works of Martín-Alfonso and co-workers [46, 47] wherein an intricate relationship among the temperature, viscosity and pressure is provided for bitumen. In this context, it ought to be pointed out that the pressure dependence of the properties of bitumen were recognized very early. For instance, Saal and Koens [79] not only allowed for viscosity to depend on pressure (the normal stress), they even allowed it to depend on the shear stresses. Thus, they had a truly implicit constitutive model relating the stress and kinematical quantities (see also Saal and Labout [80] and Murali-Krishnan and Rajagopal [54]).

The theoretical results regarding the pressure-dependent viscosity fluids are, to our knowledge, still few and most of them are devoted to the determination of particular classes of flows (see, for instance, [30, 64, 70, 93]). The only results on the qualitative analysis of the equations governing the motion in a fluid with a pressure dependent viscosity, as far as we know, are due to Rajagopal and various co-workers [7, 44, 45] who have proved the existence of weak solutions for spatially periodic three-dimensional flows that are global in time for a large class of physically meaningful viscosity-pressure relationships. To our knowledge, nothing has been done in literature about the stability of flows in fluids with pressure dependent viscosity. In particular we found no result concerning the stability analysis of the Bénard problem for fluids with temperature and pressure dependent viscosity although it could be of practical interest in geophysics and in polymer melt processing. On the contrary, thermal convection for fluids with constant or temperature-dependent viscosity has been largely studied by many authors (see, for instance, [8, 9, 10, 11, 15, 17, 60, 86, 87, 92] and references therein).

A brief outline of the contents is now given. In Chapter 1 we introduce the basic concepts and notions on linear and nonlinear stability and illustrate the fundamental features of the Liapunov direct method. In particular, we introduce the energy method (which can be thought to be a particular case of the Liapunov direct method) we use to analyze the nonlinear stability of the Rayleigh-Bénard convection in a fluid with temperature and pressure dependent viscosity (Chapter 4), and of the laminar flows in an electrically conducting fluid saturating a porous medium (Chapter 5).

In Chapter 2 we deduce the governing equations of fluid mechanics by appealing to the implicit constitutive theories for fluids formulated by Rajagopal in [65]. We follow this innovative approach as we are taking into account the dependence on pressure of the fluid viscosity. Indeed the standard procedure in classical mechanics is to split the Cauchy stress tensor into the sum of two terms: the constraint stress and the so-called 'extra' stress. The former is assumed to not depend on the state variables (in the case of the classical fluid the velocity gradient) and, according to the Constraint Principle of Truesdell and Noll [90], does no work; the latter is constitutively prescribed but is assumed to not depend on the constrained part. Therefore, in the context of classical mechanics the fluid viscosity cannot depend on pressure and our choice of following Rajagopal [65] is then motivated by this lack in the standard theory of fluid dynamics.

When the dependence of viscosity on pressure is taken into account, the Oberbeck-Boussinesq equations, i.e. the approximate equations of motion of a heat-conducting viscous fluid under the action of gravity, must be slightly modified as one needs to distinguish between the pressure due to gravity and the pressure due to the thermal expansion of the fluid, only the former contributes to variations in viscosity at a first approximation. The first original step in this thesis is then to provide a rigorous mathematical justification for the Oberbeck-Boussinesq approximation when all the material parameters of the fluid (the viscosity μ, the thermal conductivity k, the specific heat at constant pressure c_p and the coefficient of volumetric thermal expansion α) are analytic functions of pressure and temperature, and to derive approximate equations governing the motion of a heat-conducting viscous fluid (see also [67]). The approximate governing equations we obtain reduce to the classical Oberbeck-Boussinesq equations if the material parameters of the fluid are not dependent on pressure.

While it is true that all the physical quantities do vary with pressure, the variation in the viscosity with pressure is far more dramatic than the variation of the other quantities with pressure. For instance, while viscosity might change by a factor of ten to the power of eight or more (see [4]), the density will vary by merely a few percent (see [16], [62] for details). The other properties also undergo much more modest changes in their values than the viscosity and hence we feel that assuming α, k and c_p constants is a reasonable first approximation. Based on this approximation, we both determine the laminar flows (Chapter 3) and study the onset of convection (Chapter 4) in fluids whose viscosity depends analytically on both the temperature and pressure.

In Chapter 3, by taking into account the viscosity model proposed by Laun for polymer melts, we observe how the pressure-temperature dependent viscosity influences the shape of the velocity profiles in parallel shear flows. Moreover we see that the velocity profiles in Poiseuille and Couette flows in bitumen differ not so much from the classical case in which viscosity is assumed to be a constant in spite of the intricate bitumen viscosity model proposed by Martin-Alfonso and co-workers [46, 47]. Indeed here we consider what happens in a laboratory while in geophysical field applications the effect of the pressure dependence may be more dramatic.

As concerns the onset of convection in a pressure-temperature dependent viscosity fluid, in Chapter 4 we report the results in [68] in which we prove that the principle of exchange of stabilities holds and hence instability sets in as stationary convection. Then, by following a standard procedure, we show how to find the critical Rayleigh number, the linear stability/instability threshold in Bénard problem, by appealing to a variational analysis. The nonlinear energy stability analysis yields that the thresholds for the linear theory and energy analysis coincide provided the initial disturbance to the temperature field meets a specific restriction. We may then state that the basic results of the classical theory (validity of the principle of exchange of stabilities and coincidence of the linear instability and energy stability thresholds) are extended to fluids whose viscosity depends analytically on both temperature and pressure. Moreover we present approximations to the critical Rayleigh number both for rigid and stress-free boundary conditions when the viscosity depends on pressure and temperature as in (1). These approximations are obtained by employing the Galerkin-type method developed by Chandrasekhar in [11] whose convergence is discussed in Section 5.5 in relation to the characteristic-value problem raised by the stability analysis of the laminar flows studied in Chapter 5.

Another problem involving the stability of the shear parallel flows we discuss in this thesis is the stability of the laminar motions in an electrically conducting fluid saturating a porous medium. It is known that in the geothermal region the sub-surface ground water posseses a general upward convective drift due to buoyancy induced by the high underground temperature. Since the rising ground water is cooled as it approaches the surface, where heat is removed by evaporation, radiation and movement in the surface streams, an unstable state may be induced and complicated convective motions appear in the layers near the surface. In those circumstances it is of practical interest to consider the effect of the geomagnetic field on such flows and see whether the magnetic field inhibits this instability. In [78] Rudraiah and Mariyappa studied the stability of steady hydromagnetic flows in a porous medium by assuming the fluid with a finite electrical conductivity, valid the Oberbeck-Boussinesq approximation and neglecting the effects of its viscosity with respect to the friction that manifests itself at the pores. In Chapter 5, instead, we include the frictional forces in the fluid by considering the unsteady Brinkman model for flows of a viscous fluid in a porous medium, determine the laminar flows, show how both the magnetic field and the porous matrix influence the shape of the velocity profiles and find sufficient conditions for linear and nonlinear energy stability (see also [75]).