Chapter 3

Laminar flows in fluids with tempearture and pressure dependent viscosity

3.1 Introduction

In this chapter and in the next one we shall consider fluids whose viscosity is an analytic function of both temperature and pressure but its coefficient of thermal expansion α, its thermal conductivity k and its specific heat at constant pressure c_p are constants. While it is true that all the physical quantities do vary with pressure, the variation in the viscosity with pressure is far more dramatic than the variation of the other quantities with pressure. We shall now use the Barus' equation (2.11) to get a rough estimate of the variation in the viscosity with pressure for common organic liquids. For Naphthalemic mineral oil the piezoviscous coefficient β has been determined experimentally to be 26.5 GPa^{-1} at 20 °C, 23.4 GPa^{-1} at 40 °C, 20 GPa^{-1} at 60 °C and 16.4 GPa^{-1} at 80 °C (see [29] for details). Thus a change of pressure from 0.1 GPa to 1.0 GPa at 80 °C leads to a change in the viscosity of $2.57 \cdot 10^8$%! The density on the other hand changes according to the relation [16]

$$\rho = \rho_0 \left(1 + \frac{0.6p}{1 + 1.7p}\right),$$

and thus, the change in density is merely 16%. While such a change in density will be taken into account if one is interested in depicting the response very accurately, in most applications one can ignore the density change and model the fluid as incompressible. The other properties also undergo much more modest changes in their values than the viscosity and hence we feel that assuming α, k and c_p constants is a reasonable first approximation.

Based on this approximation we study steady unidirectional flows subject to temperature field to assess the effect of buoyancy on the flow when the viscosity depends upon the pressure. In particular we study the laminar flows in polymer melts (section 3.3) and in bitumen (section 3.4).

3.2 Laminar flows

Let $Oxyz$ be a cartesian frame of reference with unit vector fields $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, $\mathbf{k}$ pointed vertically upward. In this section we shall determine the laminar flows in a fluid whose viscosity is an analytic function of temperature and pressure whereas the coefficient of thermal expansion α, the specific heat at constant pressure c_p and the heat conductivity k are assumed to be constants. Therefore, if gravity is the only force acting on the fluid, the equations which govern the motion we have derived in section 2.4 become:

$$\begin{cases} \nabla p + \rho_0 g \mathbf{k} = 0 \\ \rho_0 \mathbf{v}_t + \rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha(T_1 - T_2)\nabla P + \mu(p,T)\Delta \mathbf{v} \\ \qquad\qquad +2\mathbf{D} \cdot \nabla \mu(p,T) + \rho_0 g \alpha (T - T_0)\mathbf{k} \\ \operatorname{div} \mathbf{v} = 0 \\ T_t + \mathbf{v} \cdot \nabla T = \kappa \Delta T \end{cases} \tag{3.1}$$

in $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$. In (3.1) ρ_0 is the density at the reference temperature $T_0 = (T_1 + T_2)/2$, $\kappa = k/(\rho_0 c_p)$ is the thermal diffusivity, g and p are, respectively, the acceleration and the pressure field due to gravity, P is the pressure due to the thermal expansion of the fluid and by T we denote the temperature field. The boundary conditions we append to system (3.1) are

$$\begin{cases} T(x,y,d/2,t) = T_2, \quad T(x,y,-d/2,t) = T_1 \\ p(x,y,0,t) = p_0 \end{cases} \tag{3.2}$$

where p_0 is the reference pressure.

Now it is convenient to non-dimensionalize (3.1) according to the scales:

$$\begin{aligned} &\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad t^* = \frac{\mu_0}{\rho_0 d^2} t, \quad \mathbf{v}^* = \frac{\rho_0 d}{\mu_0}\mathbf{v}, \\ &p^* = \frac{p - p_0}{\rho_0 g d}, \quad P^* = \frac{P}{\rho_0 g d}, \quad \mu^* = \frac{\mu}{\mu_0}, \\ &T^* = \frac{T - T_0}{T_1 - T_2}, \quad R = \frac{\alpha(T_1 - T_2)\rho_0 g d^3}{\mu_0 \kappa}, \quad Pr = \frac{\mu_0}{\rho_0 \kappa}, \end{aligned} \tag{3.3}$$

where $\mu_0 = \mu(p_0, T_0)$ is the viscosity at the reference state (p_0, T_0), R and Pr are the Rayleigh and Prandtl numbers, respectively. With this scaling

(3.1) becomes (omitting all asterisks)

$$\begin{cases} \nabla p + \mathbf{k} = 0 \\ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{R}{Pr} \nabla P + \mu(p, T) \Delta \mathbf{v} + 2\mathbf{D} \cdot \nabla \mu(p, T) + \frac{R}{Pr} T \mathbf{k} \\ \operatorname{div} \mathbf{v} = 0 \\ Pr(T_t + \mathbf{v} \cdot \nabla T) = \Delta T \end{cases} \tag{3.4}$$

in $\mathbb{R}^2 \times (-1/2, 1/2)$. Then to determine the steady flows of the type

$$\mathbf{v} = v(z)\mathbf{i}, \quad T = T(z),$$

we have to solve the following system

$$\begin{cases} p_x = p_y = P_y = 0 \\ p_z = -1 \\ -\frac{R}{Pr} P_x + \mu_z v_z + \mu v_{zz} = 0 \\ -\frac{R}{Pr} P_z + \mu_x v_z + \frac{R}{Pr} T = 0 \\ T_{zz} = 0 \end{cases} \tag{3.5}$$

with boundary conditions

$$\begin{cases} v(-1/2) = V_1, \quad v(1/2) = V_2 \\ T(-1/2) = 1/2, \quad T(1/2) = -1/2 \\ p(0) = 0. \end{cases} \tag{3.6}$$

It is easy to check that the boundary value problem (3.5)-(3.6) admits the solution

$$\begin{cases} p = T = -z \\ P = -\frac{z^2}{2} + \frac{Pr}{R} A_0 x + P_0 \\ v = V_1 + \displaystyle\int_{-1/2}^{z} \frac{A_0 \zeta + c}{\mu(\zeta)} \mathrm{d}\zeta, \end{cases} \tag{3.7}$$

where A_0 is the pressure gradient and

$$c = \left[V_2 - V_1 - A_0 \int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta \right] \left[\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right]^{-1}.$$

We have therefore a one-parameter family of laminar flows, the pressure gradient A_0 being the variable parameter, which includes two important special cases:

- for $A_0 = 0$, $V_2 = V$ and $V_1 = -V$, the *Couette flow*

$$v = -V + \frac{2V}{\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)}} \int_{-1/2}^{z} \frac{\mathrm{d}\zeta}{\mu(\zeta)}; \tag{3.8}$$

- for $A_0 \neq 0$ and $V_1 = V_2 = 0$, the *Poiseuille flow*

$$v = A_0 \left[\int_{-1/2}^{z} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta - \frac{\displaystyle\int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta}{\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)}} \int_{-1/2}^{z} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right]. \tag{3.9}$$

Observe that each laminar flow (3.14) can be thought of as a linear combination of Couette and Poiseuille flows. Finally we normalize (3.8) and (3.9) by dividing them by V, where, in the former case, V is the velocity of the upper plate, and, in the latter,

$$V = A_0 \left[\int_{-1/2}^{\bar{z}} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta - \frac{\displaystyle\int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta}{\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)}} \int_{-1/2}^{\bar{z}} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right] \tag{3.10}$$

is the velocity at the stationary surface

$$z = \bar{z} = \left[\int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta \right] \left[\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right]^{-1}. \tag{3.11}$$

3.3 Laminar flows in polymer melts

We now consider the exponential dependence of viscosity on temperature and pressure proposed by Laun for polymer melts [36],

$$\mu = \mu_0 \exp[\beta(p - p_0) - \gamma(T - T_0)], \tag{3.12}$$

where the non-negative numbers β and γ are the pressure and temperature coefficients of viscosity. Obviously, for $\beta = 0$ and $\gamma = 0$ (3.12) yields the classical case with constant viscosity. According to (3.3) and $(3.7)_1$, the dimensionless viscosity (3.12) is given by

$$\mu = \exp(\Gamma z) \tag{3.13}$$

with $\Gamma = \gamma(T_1 - T_2) - \beta\rho_0 g d$, and hence $(3.7)_3$ becomes

$$v = -\left[\frac{A_0}{\Gamma^2}(\Gamma z + 1) + \frac{k_1}{\Gamma}\right]\exp(-\Gamma z) + k_2, \tag{3.14}$$

where

$$k_1 = \frac{(V_2 - V_1)\Gamma}{2\sinh(\Gamma/2)} + \frac{A_0}{2}\coth(\Gamma/2) - \frac{A_0}{\Gamma}$$

and

$$k_2 = \frac{V_2\exp(\Gamma/2) - V_1\exp(-\Gamma/2)}{2\sinh(\Gamma/2)} + \frac{A_0}{2\Gamma\sinh(\Gamma/2)}.$$

The Couette and Poiseuille flows are, respectively, given by

$$v = \frac{V}{\sinh(\Gamma/2)}\left[\cosh(\Gamma/2) - \exp(-\Gamma z)\right]; \tag{3.15}$$

and

$$v = \frac{A_0}{2\Gamma\sinh(\Gamma/2)}\left\{1 - \left[2z\sinh(\Gamma/2) + \cosh(\Gamma/2)\right]\exp(-\Gamma z)\right\}. \tag{3.16}$$

We now remark that in the limit as $\Gamma \to 0$ (3.15) and (3.16) give the Couette and the Poiseuille flows in a fluid whose viscosity is assumed to be constant (see for example [17] page 154):

$$v = 2Vz \quad \text{(Couette flow)}$$

and

$$v = \frac{A_0}{2}\left(z^2 - \frac{1}{4}\right) \quad \text{(Poiseuille flow).}$$

Next we normalize (3.15) and (3.16) by dividing them by V, where, in the former case, V is the velocity at the top, and, in the latter,

$$V = \frac{A_0}{2\Gamma\sinh(\Gamma/2)}\left\{1 - \frac{2}{\Gamma}\sinh(\Gamma/2)\exp\left[\frac{\Gamma\cosh(\Gamma/2)}{2\sinh(\Gamma/2)} - 1\right]\right\}$$

is the velocity at the stationary surface

$$z = \frac{1}{\Gamma} - \frac{\cosh(\Gamma/2)}{2\sinh(\Gamma/2)}.$$

Normalized velocity profiles of Couette and Poiseuille flows are plotted for different values of the non-dimensional parameter Γ in Figures 3.1-3.5. We observe that the normalized velocity profiles of Couette flow are convex for negative values of Γ, that is when the dependence of viscosity on pressure is stronger than that on temperature. Moreover, for such values of Γ,

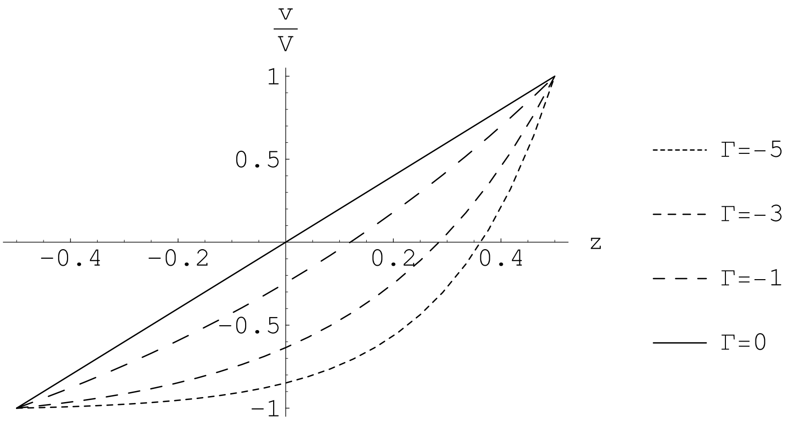

Figure 3.1: Normalized velocity profiles of Couette flow for different non-positive values of the parameter Γ.

viscosity is a decreasing function of the height z so that the fluid layer having velocity oriented as the velocity of the upper plate (i.e. as $\mathbf{i}$) is thinner than that with velocity oriented as the velocity of the lower one (i.e. as $-\mathbf{i}$). On the contrary, for positive values of Γ, that is when the dependence of viscosity on temperature is stronger than that on pressure, the normalized velocity profiles of Couette flow are concave and the fluid layer with velocity oriented as $\mathbf{i}$ is thicker than that having velocity oriented as $-\mathbf{i}$. In Figure 3.3 we show how the thickness d_+ of the fluid layer with velocity oriented as the velocity of the upper plate depends on the parameter Γ.

In Poiseuille flow, instead, for negative values of Γ the velocity profiles attain their maximum at $z_{\max} \in]0, 1/2[$, and as Γ decreases $z_{\max}$ approaches $z = 1/2$ where both pressure due to gravity and viscosity are minimum (see $(3.7)_1$ and (3.13)). For positive values of Γ the velocity profiles attain their maximum at $z_{\max} \in]-1/2, 0[$, and, as shown in Figure 3.6, as Γ increases $z_{\max}$ approaches $z = -1/2$ at which temperature is maximum whereas viscosity is minimum.

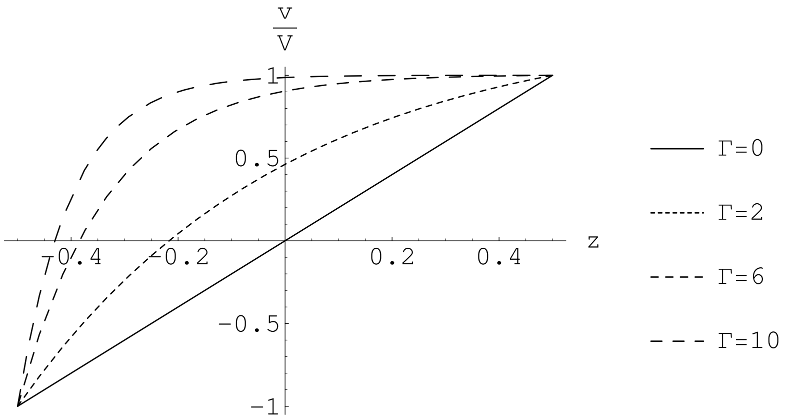

Figure 3.2: Normalized velocity profiles of Couette flow for different non-negative values of the parameter Γ.

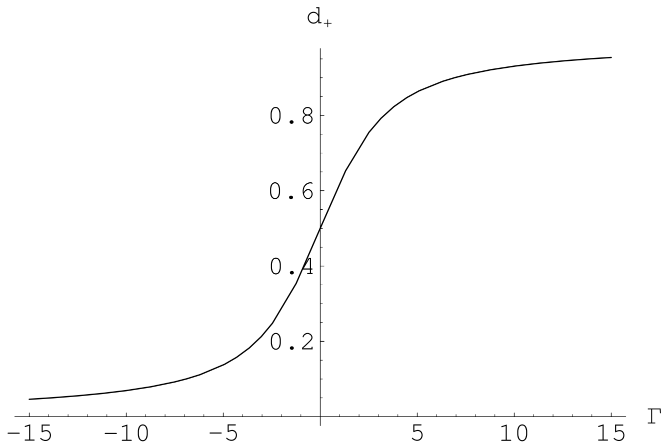

Figure 3.3: Thickness d_+ as function of Γ. For negative values of Γ, $d_+ < 1/2$, decreases as Γ decreases and in the limit as $\Gamma \to -\infty$ tends to zero. If $\Gamma = 0$, in particular in the classical case $\beta = 0$ and $\gamma = 0$, $d_+ = 1/2$. For positive value of Γ, $d_+ > 1/2$, increases as Γ increases and in the limit as $\Gamma \to +\infty$ tends to 1.

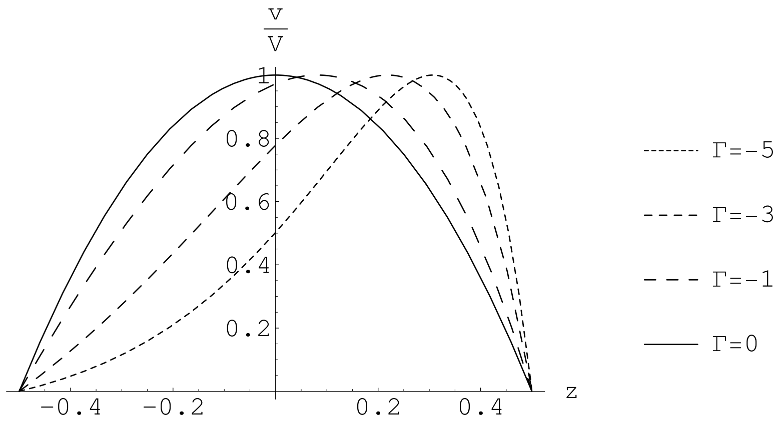

Figure 3.4: Normalized velocity profiles of Poiseuille flow for different non-positive values of the parameter Γ.

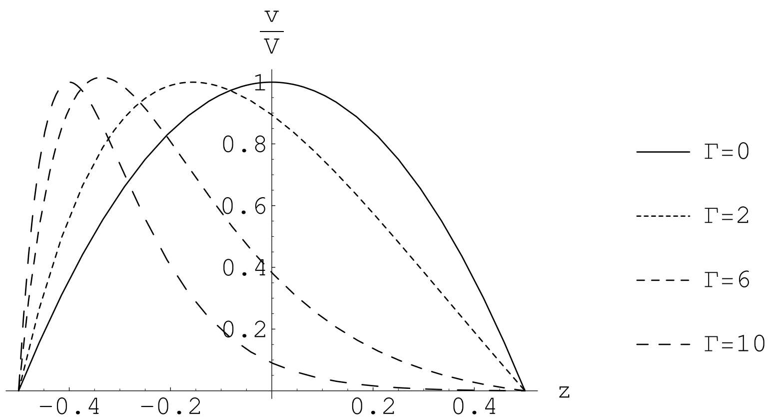

Figure 3.5: Normalized velocity profiles of Poiseuille flow for different non-negative values of the parameter Γ.

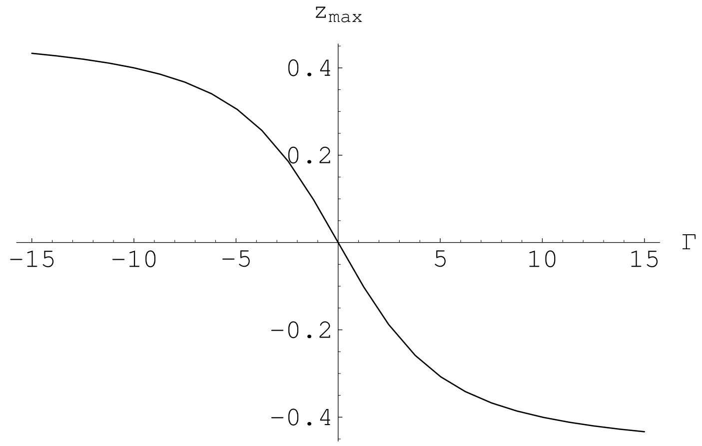

Figure 3.6: The point $z_{\max}$ as function of Γ. For negative values of Γ, $z_{\max} \in]0, 1/2[$, increases as Γ decreases and in the limit as $\Gamma \to -\infty$ tends to $1/2$. If $\Gamma = 0$, in particular in the classical case $\beta = 0$ and $\gamma = 0$, $z_{\max} = 0$. For positive values of Γ, $z_{\max} \in] - 1/2, 0[$, decreases as Γ increases and in the limit as $\Gamma \to +\infty$ tends to $-1/2$.

3.4 Couette and Poiseuille flows of bitumen

Bitumen is a hydrocarbon mixture usually produced by vacuum distillation of petroleum crude oils. The chemical composition of bitumen is very complex and thus bitumen can be separated into four fractions: saturates, aromatics, resins and alphaltenes [47]. If the proportion of these fractions vary, the resulting physical properties and microstructure of bitumen may be quite different.

Asphalt is a composite mixture of bitumen with mineral aggregates, widely used for road paving applications. The mechanical properties of asphalt are related to the rheological characteristics of bitumen, because it forms the continuous matrix and is the only deformable component. In addition, the workability (easiness of mixing, laying and compacting operations) of hot rolled asphalt depends on bitumen viscosity, among other factors [96]. Thus, bitumen is a Newtonian fluid when handled and mixed with mineral aggregates at high temperatures.

Compaction is probably the most crucial stage in the construction of road pavements because improving compaction can result in a significant improvement in road resistance to cracking and deformation. Asphalt compaction is a consequence of the static pressure that the deadweight of the roller exerts on the road surface. It is apparent that the performance of asphalt compaction will depend on bitumen viscosity. Both temperature and pressure exert an important influence on bitumen viscosity and, consequently, on its workability and road performance. The FMT model, proposed by Tschoegl, Knauss and Emri [91], describes the evolution of bitumen Newtonian viscosity, in the range between 60 °C and 160 °C and at any given differential pressure in the range 0-400 bars, fairly well. The FMT model is given as:

$$\log\left(\frac{\mu}{\mu_0}\right) = -\frac{c_1^{00}[T - T_0 - \theta(p)]}{c_2(p) + [T - T_0 - \theta(p)]}, \tag{3.17}$$

being

$$\theta(p) = c_3(p)\ln\left(\frac{1 + c_4 p}{1 + c_4 p_0}\right) - c_5(p)\ln\left(\frac{1 + c_6 p}{1 + c_6 p_0}\right), \tag{3.18}$$

$$c_1^{00} = \frac{B}{2.303 f_0}, \tag{3.19}$$

$$c_2(p) = \frac{f_0}{\alpha_f(p)}, \tag{3.20}$$

$$c_3(p) = \frac{1}{k_e \alpha_f(p)}, \tag{3.21}$$

$$c_4 = \frac{k_e}{K_e^*}, \tag{3.22}$$

$$c_5(p) = \frac{1}{k_\phi \alpha_f(p)}, \tag{3.23}$$

$$c_6 = \frac{k_\phi}{K_\phi^*}, \tag{3.24}$$

$$\alpha_f = \alpha_f^*\left(1 - \frac{mp}{K_e^* + k_e p}\right) - m\alpha_\phi^* p\left(\frac{1}{K_e^* + k_e p} - \frac{1}{K_\phi^* + k_\phi p}\right), \tag{3.25}$$

where, μ_0 is the viscosity at the reference temperature and atmospheric pressure; f_0 is the fractional free-volume at the reference temperature; B is a constant that normally is taken to be 1; $\alpha_f(p)$ is the expansivity of the free-volume, considered pressure dependent and temperature independent; α_f^* is the expansivity of the free-volume at zero differential pressure and temperature of reference, α_ϕ^* is the expansivity of the occupied volume at zero differential pressure and temperature of reference; K_e^* and K_ϕ^* are the bulk moduli of the entire and occupied volume at zero differential pressure and temperature of reference; k_e, k_ϕ and m are proportionality constants, which are indipendent of temperature and pressure; the superscript 00 indicates temperature and pressure of reference. The values of all the FMT model parameters for bitumen are shown in Table 3.1 (see also [46, 47]).

Then by non-dimensionalizing (3.17)-(3.25) by means of (3.3) and by inserting the resulting dimensionless viscosity into (3.8)-(3.11) we can plot

ρ_0	$991\ \mathrm{kg}\cdot\mathrm{m}^{-3}$	$(T_1 - T_2)/d$	$3 \cdot 10^{-2}\ \mathrm{K{\cdot}m^{-1}}$
μ_0	$228.3\ \mathrm{Pa{\cdot}s}$	B	1
f_0	0.069	k_e	3.256
K_e^*	$1.531 \cdot 10^4$ bar	k_ϕ	0.322
K_ϕ^*	$2.279 \cdot 10^4$ bar	α_f^*	$6.335 \cdot 10^{-4}\ \mathrm{K}^{-1}$
α_ϕ^*	$9.631 \cdot 10^{-4}\ \mathrm{K}^{-1}$	m	3.508

Table 3.1: Values of the different parameters of the FMT model for bitumen (60/70 penetration grade) at the reference temperature $T_0 = 60$ °C and the reference pressure $p_0 = 1$ bar.

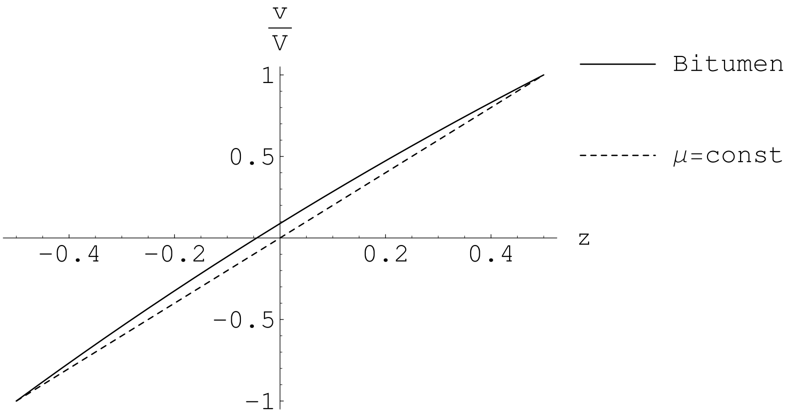

Figure 3.7: Normalized velocity profile of Couette flow in bitumen compared with the Couette flow in a fluid with constant viscosity.

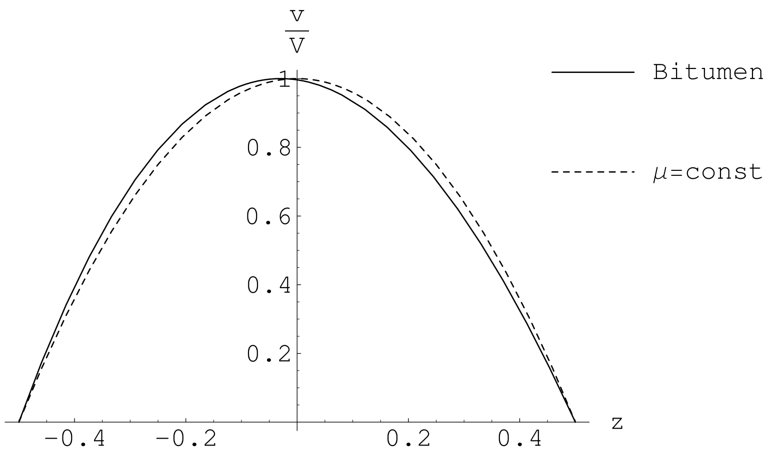

Figure 3.8: Normalized velocity profile of Poiseuille flow in bitumen compared with the Poiseuille flow in a fluid with constant viscosity.

the nomalized velocity profiles as shown in Fugures 3.7 and 3.8, respectively. These velocity profiles differ not so much from the classical case ($\mu = \text{const}$) in spite of the intricate model for bitumen viscosity given by (3.17)-(3.25). Moreover, since the normalized velocity profile in Couette flow is concave, and since the normalized velocity profile in Poiseuille flow attains its maximum approximately at $z = -0.0295$, we may conclude that the dependence of bitumen viscosity on temperature is stronger than that on pressure (see the discussion at the end of section 3.3).

Remark 3.1. In this chapter the approximate equations derived in section 2.4 are used to find the laminar flows in polymer melts and in bitumen. We think that these equations will have relevance to geophysical flows (wherein the viscosity changes with the depth of the fluid) as the approximation established in section 2.4 is valid when the dimensionless quantity $\alpha(T_1 - T_2)$ is small and does not need the fluid layer being sufficiently thin.