1 Introduction

In *The Economics of Imperfect Competition*, Joan Robinson (1969, p.70) wrote:

An increase in wealth is likely to make the demand of the individual buyer of any particular commodity less elastic. Thus an increase in demand due to an increase of wealth is likely to reduce the elasticity of the demand curve, and may reduce the elasticity so much that the slope of the curve is increased.

While the idea that higher individual income implies lower price elasticity of the *individual* demand curve is an assumption on preferences,[1] the relationship between an overall increase in income and *market* demand hinges on some assumption on how income is distributed across consumers. Indeed, increases in aggregate income rarely take place without affecting how income is distributed – and, according to most accounts, income growth over the last decades has occurred together with 'increasing inequality', or 'income polarization' (see, e.g., Gottshalk and Smeeding, 2000). In a partial equilibrium perspective, if an increase in the consumers' aggregate income is associated with changes in the elasticity of the market demand curve, this should in principle affect the behaviour of firms and market structure (Benassi *et al.*, 2002a): Joan Robinson herself argues that such a shock would affect the mark-up levels and the co-movement of prices and quantities in monopolistic markets.

Clearly, any statement about the behaviour of market demand elasticity following a change in aggregate income generally requires some assumption on the individual demand curve; however, one would like to know whether the aggregate reaction to an aggregate shock depends *only* on such assumptions at the individual level. Relying on the above quotation, one may call 'Robinson effect' the idea that the sign of the relationship between aggregate income and market price elasticity is the same as that of the relationship between individual income and the price elasticity of the individual demand curve.

This paper asks what restrictions on the shape of the income distribution are sufficient to ensure that a negative (positive) relationship between indivual income and individual price elasticity translates into a negative (positive) relationship between mean income and market demand elasticity. A natural way to model increases in mean income is via first-order stochastic-

[1]The idea that the price elasticity of demand decreases as individual income increases is arguably more reasonable than the converse. For some empirical evidence on the relevance of the elasticity-income link, see e.g., Gertler *et al.* (1987).

dominance (FOSD) shifts of the income distribution. Hence, our model provides sufficient conditions for the Robinson effect to hold when income distribution is hit by a FOSD shock – it being the case (as shown in section 3) that such a shock may not in general lower market elasticity, even though the price elasticity of the individual demand is decreasing in income.

The paper is organized as follows. In the next section a simple general framework is developed to study the relationship between income distribution and the elasticity of market demand. In Section 3 the main result of the paper is presented, which identifies sufficient conditions on the income distribution for the 'Robinson effect' to take place, when the income distribution is hit by shocks in the first-order stochastic-dominance sense. These conditions are satisfied by a wide range of commonly used distributions. Section 4 offers some concluding remarks.

2 Income distribution and demand elasticity

In this section we present a partial equilibrium framework to assess the role of income distribution and the effects of distribution changes on market demand, when income is the only source of heterogeneity.

Consumers differ only in income, and their behavior is described by a continuous standard Marshallian demand curve $q(p, y)$, where the prices of commodities other than q are held fixed throughout. Each agent is accordingly identified by his income $y \in Y = (y_m, y_M)$, where $0 < y_m < y_M \leq \infty$. The good q is normal, that is (letting subscripts denote derivatives) $q_y(p, y) > 0$ and $q_p(p, y) < 0$, for all $(p, y) \in P \times Y$, where P is a subset of non-negative reals. A natural specification might be $P = (0, p_M)$, with p_M satisfying $q(p_M, y_M) = 0$: it would be the choking price for the highest income consumers (in the limit, if $y_M = \infty$). For any $p \in P$, one clearly has $\lim_{y \to y_M} q(p, y) > \lim_{y \to y_m} q(p, y) \geq 0$.

Income is continuously distributed according to the density $f(y, \theta) > 0$, where $\theta \in \Theta$ is a real parameter of the distribution. In the next section it will measure a FOSD shock. The income distribution $F : Y \times \Theta \to [0, 1]$ is obviously defined by

$$F(y, \theta) = \int_{y_m}^{y} f(x, \theta) dx \tag{1}$$

Clearly, $F_\theta(y_M, \theta) = 0$, since by definition $F(y_M, \cdot) = 1$ for all θ. Aggregate (mean) market demand is

$$Q(p, \theta) = \int_{y_m}^{y_M} q(p, y) f(y, \theta) dy \tag{2}$$

A natural question is, what happens to market demand when the income distribution shifts, following a change in θ. Trivially,

$$Q_\theta(p,\theta) = \int_{y_m}^{y_M} q(p,y) f_\theta(y,\theta) dy$$

which, by standard results (e.g., Hirshleifer and Riley, 1992, ch.3), will be positive if θ is a FOSD shift, since q is increasing in y; while it will be positive or negative, depending on convexity or concavity of Engel curves, if θ measures a mean preserving, second order stochastic dominance shift of the distribution.

The focus of our paper, however, is what happens to market demand *elasticity* when the income distribution changes. Let $\eta(p,y)$ be the (positive) demand elasticity along the individual demand curve $q(p,y)$. It is straightforward to derive the market demand elasticity H satisfying

$$H(p,\theta) = \int_{y_m}^{y_M} \eta(p,y)\varphi(y,p,\theta) dy \tag{3}$$

where $\varphi(y,p,\theta)$ is defined by

$$\varphi(y,p,\theta) = \frac{q(p,y)f(y,\theta)}{Q(p,\theta)} \tag{4}$$

so that, quite naturally, market elasticity is a weighted average of individual elasticities. Given $p \in P$, $\varphi > 0$ is the density describing how market demand is distributed across income classes. The corresponding cumulative distribution is

$$\Phi(y,p,\theta) = \int_{y_m}^{y} \varphi(x,p,\theta) dx \tag{5}$$

such that $\Phi(y_M,\cdot,\cdot) = 1$. In particular, we note that by writing out the whole expression,

$$\Phi(y,p,\theta) = \frac{1}{Q(p,\theta)} \int_{y_m}^{y} q(p,x) f(x,\theta) dx$$

$\Phi(y,p,\theta)$ has the form of a Lorenz curve, since Q is the average value of q.

We gather in the next proposition two noteworthy, albeit quite intuitive, general properties of $\Phi(y,p,\theta)$.

Proposition 1 *(a) For given $(p,\theta) \in P \times \Theta$, $\Phi(y,p,\theta)$ dominates stochastically $F(y,\theta)$ in the first order sense, that is $\Phi(y,p,\theta) \leq F(y,\theta)$ for all $y \in Y$, with strict inequality somewhere; (b) If $\eta_y(p,y) < 0$ for all $y \in Y$, an increase in p affects $\Phi(y,p,\theta)$ as a first order stochastic dominance shock, i.e., $\Phi_p(y,p,\theta) \leq 0$ for all $y \in Y$, with strict inequality somewhere.*

Proof. (a) Using definitions (1) and (5), for given (p, θ) we have

$$F(y, \theta) - \Phi(y, p, \theta) = \int_{y_m}^{y} f(x, \theta) \left(1 - \frac{q(p, x)}{Q(p, \theta)}\right) dx \equiv Z(y)$$

say. Note that $Z(y_m) = Z(y_M) = 0$, while $Z_y = [1 - q(p, y)/Qp, \theta)]f(y, \theta)$. Since $f(y, \theta)$ is positive, Q is an average and q is monotonically increasing in y, there is only one value $\overline{y}$ of y such that $q(p, \overline{y}) = Q$, which is the only maximum of Z. There follows that $Z > 0$ for all $y \in Y$, since it is increasing (decreasing) around y_m (y_M). Hence, $\Phi(y, p, \theta) - F(y, \theta) = -Z(y, p, \theta) < 0$.

(b) By writing out the derivative of (5) with respect to p, we get

$$\Phi_p(y, p, \theta) = -\frac{Q_p(p, \theta)}{Q(p, \theta)} \Phi(y, p, \theta) - \frac{1}{p} \int_{y_m}^{y} \eta(p, y) \varphi(y, p, \theta) dy$$

after some rearrangement. Now multiply through by $p > 0$ and use (3) to obtain the following condition for $\Phi_p(y, p, \theta) < 0$:

$$K(y) \equiv \int_{y_m}^{y} (H(p, \theta) - \eta(p, y)) \varphi(y, p, \theta) dy < 0$$

where K is defined for given (p, θ). Clearly, $K(y_m) = 0$, and $K(y_M) = 0$ by (3). Since $H(p, \theta)$ is an average of $\eta(p, y)$ and $\eta_y(p, y) < 0$, the derivative $K_y = (H(p, \theta) - \eta(p, y)) \varphi(y, p, \theta)$ is increasing in y and vanishes at $y = \widetilde{y}$ such that $\eta(p, \widetilde{y}) = H(p, \theta)$, which is a minimum. This implies that $K(y) < 0$ for all y, and hence $\Phi_p < 0$. ■

These properties hold in general – in particular, as is obvious, they do not depend on θ. Property (a) implies that $\mu(\theta) = \int_{y_m}^{y_M} x f(x, \theta) dx < m(p, \theta) = \int_{y_m}^{y_M} x \varphi(x, p, \theta) dx$ for all $p \in P$: the average income weighted by the demand share of each income class on overall demand, is higher than mean income (i.e., average income weighted by the income share of each income class on overall income): this follows naturally from the commodity being normal. By property (b), following and increase in p, the implied decrease in demand is such that the degree of income heterogeneity among buyers increases – in the sense that demand is more unevenly distributed across income classes; also, the share of high income buyers on overall demand increases, which, though naturally to be expected, may be empirically not trivial, and in some circumstances significant from a welfare point of view.[2]

[2]This applies, e.g., to commodities like pharmaceuticals or health services, where the issue of price controls and availability for low income consumers may be relevant. Gertler *et al.* (1987) provide some empirical evidence in this respect.

Equation (3) makes it clear that, when working on elasticity, the crucial question is how shifts in F translate themselves into shifts in Φ: that is, how changes in income distribution affect the income distribution of market demand (or its Lorenz curve). We now turn to the case where an exogenous shock generates a FOSD shift to the income distribution.

3 First order stochastic dominance

In this section we enquire about the effects of a FOSD shock to the income distribution: hence, we interpret θ as an index of FOSD and impose that $F_\theta(y,\theta) \leq 0$ for all $y \in Y$ (with strict inequality somewhere), which implies that aggregate (average) income is increasing in θ, $\mu_\theta(\theta) > 0$. As individual demand $q(p,y)$ is increasing in income y, this also immediately implies that $Q_\theta(p,\theta) > 0$: not surprisingly, a FOSD shock increases demand at all prices.[3]

But how about elasticity? In principle, there is no reason to expect that Robinson's assumption on preferences (an increase in individual income affects negatively the price elasticity of individual demand) delivers a negative relationship between aggregate income and the price elasticity of market demand. The following example shows that an increase in mean income may leave market elasticity unaltered, even though the elasticity of individual demand is decreasing in individual income.

Let the consumer's demand for commodity q be

$$q(p,y) = \max\left\{1 - \frac{p}{y}, 0\right\}$$

such that its elasticity (whenever the consumer buys the commodity) is $\eta(p,y) = p/(y-p)$, which is positive and clearly decreasing in income.[4] Let now the latter be distributed across consumers as a standard exponential,

$$f(y,\theta) = e^{-(y-\theta)}$$

with $y_m = \theta$ and $y_M = \infty$. An increase in $\theta > 0$ amounts to a FOSD shock, which increases linearly aggregate (mean) income.[5] We show in the Appendix that in this case the aggregate demand function takes the form

$$Q(p,\theta) = G(p)e^{\theta}$$

[3] For a simple proof, see e.g. Hirshleifer and Riley (1992, ch.3).

[4] This demand function can be rationalized as deriving from a separable utility function (see, e.g., Tirole, 1989, p.144).

[5] Indeed, it is easily seen that $\mu(\theta) = 1 + \theta$, and that $F_\theta(y,\theta) = -e^{-y+\theta} < 0$.

for any $p > \theta$: there follows trivially that $H_\theta(p,\theta) = 0$: an increase in mean income has no effect on the price elasticity of market demand. The same Appendix also presents a simple general argument, to the effect that a shock being FOSD does not ensure that the sign of the individual relationship between elasticity and income carries over to the aggregate relationship between market elasticity and mean income.

3.1 Elasticity and the income distribution of demand

A preliminary step is now required to see how θ may affect the income distribution of demand. This involves considering Esteban's (1986) income share elasticity, defined as follows

$$\pi(y,\theta) = \lim_{h\to 0} \frac{d\log\left(\frac{1}{\mu}\int_y^{y+h} xf(x,\theta)dx\right)}{d\log y} = 1 + \frac{yf_y(y,\theta)}{f(y,\theta)} \tag{6}$$

The function π measures the percentage change in the share of income accruing to class y, brought about by a marginal increase y. Esteban shows that there is a one-to-one relationship between any given income density and the corresponding income share elasticity, so that the former can be characterized in terms of the latter. Given that, a natural question is what is the relationship between a FOSD shock to the distribution, and the behaviour of the corresponding income share elasticity. In this respect, the following proposition is noteworthy:

Proposition 2 *Let θ be a continuous shift to the density $f(y,\theta)$, such that $F_\theta(y_m,\theta) = 0$. If $\pi_\theta(y,\theta) > 0$ for all $y \in Y$, then θ is a* FOSD *variable, i.e. $F_\theta(y,\theta) \leq 0$ for all $y \in Y$ (strictly somewhere).*

Proof. To ease notation, let $s(y,\theta) = f_\theta(y,\theta)/f(y,\theta)$, with $f(y,\theta) > 0$ for all $y \in Y$. Simple differentiation then shows that $\pi_\theta(y,\theta) = ys_y(y,\theta)$, so that $\pi_\theta(y,\theta) > 0$ means that $s(y,\theta)$ is monotonically increasing in y for any given θ. Now notice that by definition $F_\theta(y_M,\theta) = \int_{y_m}^{y_M} s(y,\theta)f(y,\theta)dy$ $= 0$: as $f(y,\theta) > 0$ and the overall integral is nil, $s(y,\theta)$ has to take on both negative and positive values. Since $s(y,\theta)$ is increasing in y, the (negative) minimum of s occurs at $y = y_m$ and, by the same token, $s(y_M,\theta) > 0$ is a maximum for s: there is a unique value $\widehat{y}$ of y such that $s(\widehat{y},\theta) = 0$. Consider now the function $F_\theta(y,\theta) = \int_{y_m}^{y} f_\theta(y,\theta)dy$, the first derivative of which is $f_\theta(y,\theta) = s(y,\theta)f(y,\theta)$. Clearly, $\text{sign}\{f_\theta(y,\theta)\} = \text{sign}\{s(y,\theta)\}$, and $f_\theta(y,\theta)$ vanishes at $\widehat{y}$ which is the unique minimum for $F_\theta(y,\theta)$. Since $s(y,\theta)$ is negative (positive) for y near y_m (y_M) so will be $f_\theta(y,\theta)$: $F_\theta(y,\theta)$

points down (up) around y_m (y_M). As $F_\theta(y_m,\theta) = F_\theta(y_M,\theta) = 0$, $F_\theta(y,\theta)$ lies below the zero line: θ is then a FOSD parameter. ■

Under the assumption that a shock on θ does not affect the lower bound of the support of the distribution, the proof takes advantage of the fact that if θ raises the income share elasticity, the θ-elasticity of the density (equivalently, the function s) must be increasing in income, and negative for low income levels.

Proposition 2 is the key to the paper's main result, to the effect that the condition $\pi_\theta(y,\theta) > 0$ is actually sufficient for the Robinson effect to take place. To see this, notice that, given (3), the derivative of market elasticity H with respect to θ is clearly,

$$H_\theta(p,\theta) = \int_{y_m}^{y_M} \eta(p,y)\varphi_\theta(y,p,\theta)dy$$

Integrating by parts one obtains

$$H_\theta(p,\theta) = -\int_{y_m}^{y_M} \eta_y(p,y)\Phi_\theta(y,p,\theta)dy \tag{7}$$

indeed, a crucial piece of information is obviouly how individual elasticity $\eta(p,y)$ reacts to y.

Looking at (7), one may rely exclusively on Robinson's assumption that $\eta_y(\cdot,y) < 0$ to draw the conclusion that $H_\theta(\cdot,\theta) < 0$, whenever one can safely assert that $\Phi_\theta(y,\cdot,\theta) \leq 0$ for all y (with strict inequality somewhere). In other words: it is enough to know that individual elasticity is such that $\eta_y(\cdot,y) < (>) 0$ to conclude that $H_\theta(\cdot,\theta) < (>) 0$, when a FOSD shock to $F(y,\theta)$ translates into a FOSD shock to $\Phi(y,\cdot,\theta)$: monotonicity of the individual relationship is then enough to sign the aggregate relationship. However, $\Phi(p,y,\theta)$ depends on preferences *via* the individual demand curve, so that a given shock to F does not necessarily translate into a shock of the same type to Φ: in fact, we are interested on what restriction on F o*nly* are such that this occurs. As the following proposition establishes, it turns out that one such restriction is that the income share elasticity be raised by an increase in θ – which obviously raises mean income.

Proposition 3 *Assume* $F_\theta(y_m,\theta) = 0$*. If* $\pi_\theta(y,\theta) > 0$*, then* $\Phi_\theta \leq 0$ *for all* $y \in Y$*.*

Proof. The proof is straightforward, by noting that Proposition 2 can also be applied to the income distribution of demand: if the corresponding

Esteban elasticity is raised by an increase in θ, then a change in θ is a FOSD shift to $\Phi(y,p,\theta)$. Let such elasticity be denoted by $\widehat{\pi}(y,p,\theta)$: it is easily checked that

$$\widehat{\pi}(y,p,\theta)=1+\frac{y\varphi_y(y,p,\theta)}{\varphi(y,p,\theta)}=\varepsilon(y,p)+\pi(y,\theta) \tag{8}$$

where $\varepsilon(y,p)$ is the income elasticity of demand. There follows that $\pi_\theta(y,\theta)>0$ implies $\widehat{\pi}_\theta(y,\theta)>0$ and hence θ is a FOSD variable for both F and Φ, since $F_\theta(y_m,\theta)$ implies trivially $\Phi_\theta(y_m,p,\theta)=0$. ■

It should be stressed that π_θ is positive in many, well known and widely used cases, where it is associated with the densities intersecting only once following a shock on θ. Moreover, this property is a well known feature (in an obviously different context) of many contract theoretic models, where it is known as 'monotone likelihood ratio property' (e.g., Hart and Holström, 1987).[6]

By Proposition 3, if the distributive shock on θ has no effect on the income share elasticity, there is no effect on the price elasticity of market demand – incidentally, this is what happens in the previous example, since for the exponential distribution $\pi(y,\theta)=1-y$, independent of θ.[7] The economics behind this result can be put as follows. It is obvious that aggregate price elasticity is an average of individual elasticities, weighted by the individual demand share on total demand. A FOSD shock increases market demand (as agents are on average richer), but does not necessarily increase the weight of high income (low elasticity) classes *vis à vis* that of low income (high elasticity) classes: for this to happen, the increase in the density of high income classes must be such that their demand increases more than aggregate demand: that is, $\varphi_\theta(\cdot,p,\theta)>0$. This implies that for some other classes, $\varphi_\theta(\cdot,p,\theta)<0$, while for at least one value of y it will be $\varphi_\theta(\cdot,p,\theta)=0$ (since obviously $\int_{y_m}^{y_M}\varphi_\theta(y,p,\theta)dy=0$). Given this, a decrease in aggregate elasticity is clearly to be had whenever $\varphi_\theta(\cdot,p,\theta)$ is monotonically increasing in y, i.e. the shock raises the high income (and decreases the low-income)

[6] The property $\pi_\theta>0$ (θ being an appropriately defined FOSD shift variables) holds for distributions such as Pareto, lognormal, Beta, and Gamma. The implication of MLRP is apparent when recalling, from Proposition 2, that $\pi_\theta>0$ implies that $s(y,\theta)$ is increasing in y.

[7] In the example we also have, contrary to the assumptions in Proposition 3, $F_\theta(y_m,\theta)=-1\neq 0$. Referring to the proof of Proposition 2, this is implied by π being independent of θ, since the latter means that the θ-elasticity of the density is independent of income. In fact, for the exponential distribution this elasticity equals θ: hence $f_\theta(\cdot,\theta)=f(\cdot,\theta)$, and $F_\theta(y_M,\theta)=F_\theta(y_m,\theta)+\int_{y_m}^{y_M}f_\theta(\cdot,\theta)dy=0$ requires $F_\theta(y_m,\theta)=-1$.

demand share; on the other hand, Proposition 2 tells us that a necessary and sufficient condition for the θ-elasticity of any density to be monotonically increasing in y, is that the corresponding income share elasticity be raised by θ. Hence if $\widehat{\pi}_\theta$ is positive, φ_θ is indeed increasing in y.

4 Concluding remarks

The effects of income distribution on market demand are generally studied under the assumption that prices be given – the main focus being on Engel curves, consumption patterns and the size of the market (e.g., Lambert and Pfähler, 1997). However, the link with price elasticity should in principle also matter, as elasticity is crucial to the firms' choices and market structure (Benassi *et al.*, 2002b). Clearly, the crucial obstacle to this kind of analysis is that the relationship between market demand elasticity and income distribution depends heavily on preferences.

The premise of this paper is that it is anyway useful to know to what extent the link between income distribution and the price elasticity of demand is affected by specific assumptions about preferences. In this respect, our main result is that there exist restrictions on the shape of the income distribution (holding for a wide class of functional forms), such that the 'Robinson effect' operates – that is, the sign of the income-elasticity link at the aggregate level is the same as that dictated by preferences at the individual level, whenever the increase in aggregate income is due to a first-order, stochastic dominance shock to the distribution of income. For example, one practical consequence of this is that, when individual price elasticity is decreasing in income, one such shock is bound to raise the firms' market power in a traditional Cournot setting, whenever it also raises the income share elasticity at all income classes.

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Appendix

In this Appendix we (a) present in more detail the example discussed in the text; and (b) present a simple general argument to show that, under a FOSD shock, knowing the sign of the relationship between price elasticity and income along the individual demand curve says nothing on the relationship between market elasticity and average income.

(a) The example

The income distribution is a standard exponential, with density $f(y,\theta) = e^{-(y-\theta)}$, and cumulative distribution $F(y,\theta) = 1 - e^{-y+\theta}$, $y \in [\theta, \infty)$. As explained in the text (see also f.note 5), $\theta > 0$ is a FOSD parameter and mean income is $\mu(\theta) = 1 + \theta$. We notice that, contrary to our assumption in Proposition 3, $F_\theta(y_m, \theta) = -1 < 0$ and $\pi(y,\theta)$ is independent of θ. Indeed

$$\pi(y,\theta) = 1 + \frac{y f_y(y,\theta)}{f(y,\theta)} = 1 + \frac{-ye^{-(y-\theta)}}{e^{-(y-\theta)}} = 1 - y$$

As to the individual demand function, we have $q(p,y) = \max\left\{1 - \frac{p}{y}, 0\right\}$, so that aggregate demand is

$$Q(p,\theta) = \int_\theta^\infty \max\left\{1 - \frac{p}{y}, 0\right\} e^{-(y-\theta)} dy \tag{A.1}$$

Assume now that $p > \theta$. Then (A.1) becomes

$$Q(p,\theta) = \int_p^\infty \left(1 - \frac{p}{y}\right) e^{-(y-\theta)} dy$$

which gives $Q(p,\theta) = (1 - pA(p)\, e^p)e^{-p+\theta}$ where $A(p) = \int_p^\infty x^{-1}e^x dx$ is a decreasing positive function of p. Clearly, this can be written as $Q(p,\theta) = G(p)e^\theta$ (which is isoelastic in θ), with $G(p) = e^{-p} - pA(p)$.

(b) A simple argument

Assume θ is a FOSD shock to the income distribution, such that $F_\theta(y,\theta) \leq 0$ (strictly somewhere) for all $y \in Y$, which implies $Q_\theta(p,\theta) > 0$ for all $p \in P$. Upon differentiation, a necessary and sufficient condition for $H_\theta(p,\theta) < 0$ is that

$$-pQ_{p\theta}(p,\theta) < Q_\theta(p,\theta)H(p,\theta)$$

where subscripts denote (cross) partials and (obviously) p, $Q_\theta(p,\theta)$, and $H(p,\theta)$ are all positive. We now show that $\eta_y(p,y) < 0$ implies $Q_{p\theta}(p,\theta) < 0$, which means that the LHS is itself positive: some specific assumption on $F(y,\theta)$ is accordingly required beyond FOSD, to ensure that $\eta_y(p,y) < 0$ implies $H_\theta(p,\theta) < 0$.

Integration by parts yields

$$Q_{p\theta}(p,\theta) = -\int_{y_m}^{y_M} q_{py}(p,y) F_\theta(y,\theta) dy$$

Since $\eta_y(p,y) < 0$ implies trivially $-pq_{py}(p,y) < q_y(p,y)\eta(p,y)$ for all y, there follows that

$$pQ_{p\theta}(p,\theta) = -p\int_{y_m}^{y_M} q_{py}(p,y) F_\theta(y,\theta) dy < \int_{y_m}^{y_M} q_y(p,y)\eta(p,y) F_\theta(y,\theta) dy < 0$$

the last inequality deriving from $q_y(p,y)$ and $\eta(p,y)$ being both positive for all y, while $F_\theta(y,\theta) \leq 0$ (strictly somewhere) by the definition of FOSD. Since p is obviously positive, this implies $Q_{p\theta}(p,\theta) < 0$.