CHAPTER 2: THE AREA EXCESS AND DE GIORGI'S LEMMA

In this section we shall be primarily concerned with some key ideas underlying the proof of Theorem 1.9. Techniques and concepts relevant to that proof will be introduced in a rather "natural" way, by working out an explicit example in Regularity Theory.

2.1. As we showed in the preceding chapter, in the case when $\alpha(t)$ - the function controlling the deviation from minimality - is of the following type:

$$\alpha(t) = ct^{2\alpha} \quad , \quad 0 < \alpha < 1$$

then we have an "optimal regularity result", in the sense that [9]

$$\partial E \in C^{1,\alpha} \Rightarrow Dev(E,x,t) \leq ct^{2\alpha}$$

(2.1)

$$Dev(E,x,t) \leq ct^{2\alpha} \Rightarrow \partial^* E \in C^{1,\alpha}$$

See 1.12 and 1.14 (v). The appearance of the *reduced* boundary $\partial^* E$ in the last implication is unavoidable, on the account of the existence of minimal cones with singularities. In the special case when ∂E is already known to be of class C^1, we have then clearly a perfectly symmetric situation:

(2.2) if $\partial E \in C^1$, then $\partial E \in C^{1,\alpha} \Longleftrightarrow Dev(E,x,t) \leq ct^{2\alpha}$

It seems convenient to give the simple (relative to that of Theor. 1.9) proof of this fact, one reason being that while doing this we will quickly meet a certain regularity parameter, which will play a basic role in the subsequent sections.

To begin with, we introduce a new class of function spaces, including both the Morrey spaces $L^{p,\lambda}(\Omega)$ (see (1.14)) and the Hölder spaces

$C^{o,\alpha}(\Omega)$.

2.2. Definition of Campanato spaces.

Given: Ω open and bounded in $\mathbb{R}^n$, $p \geq 1$, $\lambda \geq 0$;

we say that

$$u \in \mathcal{L}^{p,\lambda}(\Omega) \text{ iff } u \in L^p(\Omega) \text{ and } \sup_{\substack{x\in\Omega \\ o<t<\operatorname{diam}\Omega}} \left(t^{-\lambda} \int_{\Omega\cap B_{x,t}} |u-u_{x,t}|^p \, dy\right) < +\infty$$

where $u_{x,t}$ (also denoted $\{u\}_{x,t}$ is the average of u on $B_{x,t}$

$$u_{x,t} = \{u\}_{x,t} = |B_{x,t}|^{-1} \int_{B_{x,t}} u(y)dy.$$

A basic fact about Campanato spaces is that $\mathcal{L}^{p,\lambda}$ is isomorphic to $C^{o,(\lambda-n)/p}$, *provided* $\lambda \in (n,n+p]$ *and* $\partial\Omega$ satisfies a suitable regularity condition (e.g. $\partial\Omega$ is locally lipschitz). See [20], Chapter 4, Theor. 1.6.

2.3. For convenience of the reader, we now recall an elementary property of averages:

if $A \subset\subset \mathbb{R}^n$, $u \in L^2(A)$, and $u_A = |A|^{-1} \int_A u dx$, then

$$(2.3) \qquad \int_A |u-u_A|^2 dx = \int_A (|u|^2 - |u_A|^2) dx \leq \int_A |u-\lambda|^2 dx \qquad \forall \lambda \in \mathbb{R}$$

along with some simple facts about harmonic functions:

if $B = B_{x,R} \subset \mathbb{R}^n$, $u \in C^1(\bar{B})$, and v is the harmonic function associated with u on B, i.e. satisfying [10]

$$(2.4)\qquad \begin{cases} \Delta v = v_{x_i x_i} = 0 & \text{in } B \\ v = u & \text{on } \partial B \end{cases}$$

then

$$(2.5)\qquad \int_B \langle Du, Dv\rangle dy = \int_B |Dv|^2\, dy \le \int_B |Du|^2\, dy$$

$$(2.6)\qquad \int_B |Du-Dv|^2 dy = \int_B (|Du|^2 - |Dv|^2)\, dy$$

$$(2.7)\qquad \{Du\}_{x,R} = \{Dv\}_{x,r} \qquad \forall r \in (0,R]$$

$$(2.8)\qquad r^{-(n+2)} \int_{B_{x,r}} |Dv - \{Dv\}_{x,r}|^2\, dy \quad \text{is a non-decreasing}$$

function of $r \in (0,R)$.

Assertions (2.5) to (2.7) are easy consequences of the Gauss-Green Theorem. As for (2.8), observe that any weak solution w of a homogeneous elliptic partial differential equation with constant coefficients:

$$a_{ij}\, w_{x_i x_j} = 0$$

satisfies

$$\int_{B_s} |w - \{w\}_s|^2 \le c_1 (s/t)^{n+2} \int_{B_t} |w - \{w\}_t|^2$$

for a suitable constant c_1 (depending on the ellipticity constant and on n), and for every s,t : $0 < s < t$; see [20], Chapter 4, Lemma 2.2.

The fact that $c_1 = 1$ when w is harmonic requires additional care: its proof may be based upon a classical result about the uniform approximation of harmonic functions by means of homogeneous harmonic polynomials (as in [8]; see e.g. [27], 2.5.2, prop. 1).

Finally, we list two elementary algebraic inequalities

(2.9) $$a^2-b^2 \leq 2(1+b^2)^{\frac{1}{2}} \cdot \left[(1+a^2)^{\frac{1}{2}} - (1+b^2)^{\frac{1}{2}}\right] + (a^2-b^2)^2/4$$

(2.10) $$a^2-b^2 \leq 2(1+a^2)^{\frac{1}{2}} \cdot \left[(1+a^2)^{\frac{1}{2}} - (1+b^2)^{\frac{1}{2}}\right]$$

both valid $\forall a,b \in \mathbb{R}$ (the proof is a straightforward calculation), together with the following result (see [17], Lemma 2.2):

2.4. A useful Lemma.

For any choice of the constants a,α,β with $a>0$, $\alpha>\beta>0$, it is possible to find two new constants $\varepsilon=\varepsilon(a,\alpha,\beta) > 0$ and $c=c(a,\alpha,\beta)>0$ such that, whenever $\omega : (0,T) \to (0,+\infty)$ is a non-decreasing function, satisfying

(2.11) $$\omega(s) \leq a\left[(s/t)^{\alpha} + \varepsilon\right]\cdot \omega(t) + bt^{\beta}$$

for some $T > 0$ and some $b \geq 0$, and for every s,t : $0<s<t<T$, then it holds:

(2.12) $$\omega(s) \leq c\left[(s/t)^{\beta}\,\omega(t) + bs^{\beta}\right]$$

still for every s,t : $0 < s < t < T$.

The proof of Lemma 2.4 goes as follows: fix $\gamma \in (\beta,\alpha)$ and $\tau \in (0,1)$

so that $2a\tau^{\alpha} < \tau^{\gamma}$, and the define

$$\varepsilon = \tau^{\alpha} \quad , \; c^{-1} = \tau^{\beta}(\tau^{\beta}-\tau^{\gamma}).$$

Given s,t : $0 < s < t < T$, consider $t' = t$, $s' = \tau t$, and apply (2.11) to obtain

$$\omega(\tau t) \leq a(\tau^{\alpha}+\varepsilon)\ \omega(t) + bt^{\beta} \leq \tau^{\gamma}\omega(t) + bt^{\beta}$$

in view of our initial assumptions. By induction:

$$\omega(\tau^{k+1}t) \leq \tau^{(k+1)\gamma}\cdot\omega(t) + bt^{\beta}\ \tau^{k\beta}\cdot \sum_{j=0}^{k} \tau^{j(\gamma-\beta)} \qquad \forall\ k \geq 0$$

whence

(2.13) $$\omega(\tau^{k}t) \leq \tau^{k\beta}(\tau^{\beta}-\tau^{\gamma})^{-1}\cdot(\omega(t)+bt^{\beta}) \qquad \forall\ k \geq 0.$$

Since $0 < s < t$, there will exist a unique $k \geq 0$ s.t. $\tau^{k+1}t \leq s < \tau^{k}t$, so that $\tau^{k} \leq \tau^{-1}\cdot(s/t)$. In conclusion, we get

$$\omega(s) \leq \omega(\tau^{k}t) \leq c\ \left[(s/t)^{\beta}\omega(t)+bs^{\beta}\right]$$

by (2.13), the monotonicity of ω , and the choice of c.

2.5. At this point, we dispose of all the ingredients needed for the proof of (2.2). Notice that the validity of the implication $\Longrightarrow$ in (2.2) has already been shown in Example 1.11 (v), hence we concentrate on the reverse implication.

To be specific, let us consider a function u of class C^1 in some (n-1)-ball $B'_{2T} = \{x' \in \mathbb{R}^{n-1} : |x'| < 2T\}$, and let us assume that

(2.14) $$p \equiv \sup\ \{|Du(x')| : x' \in B'_{2T}\} < 1 \ .$$

We fix $x' \in B'_T$ and $s,t : 0 < s < t < T$, and denote by Q_r the cylinder

$$Q_r = \{y = (y', y_n) \in \mathbb{R}^n : |y'-x'| < r,\ |y_n - u(x')| < r\},$$

by E the epigraph of u over B'_{2T}, and by v the harmonic function associated with u on $B'_{x',t}$ (see (2.4)).

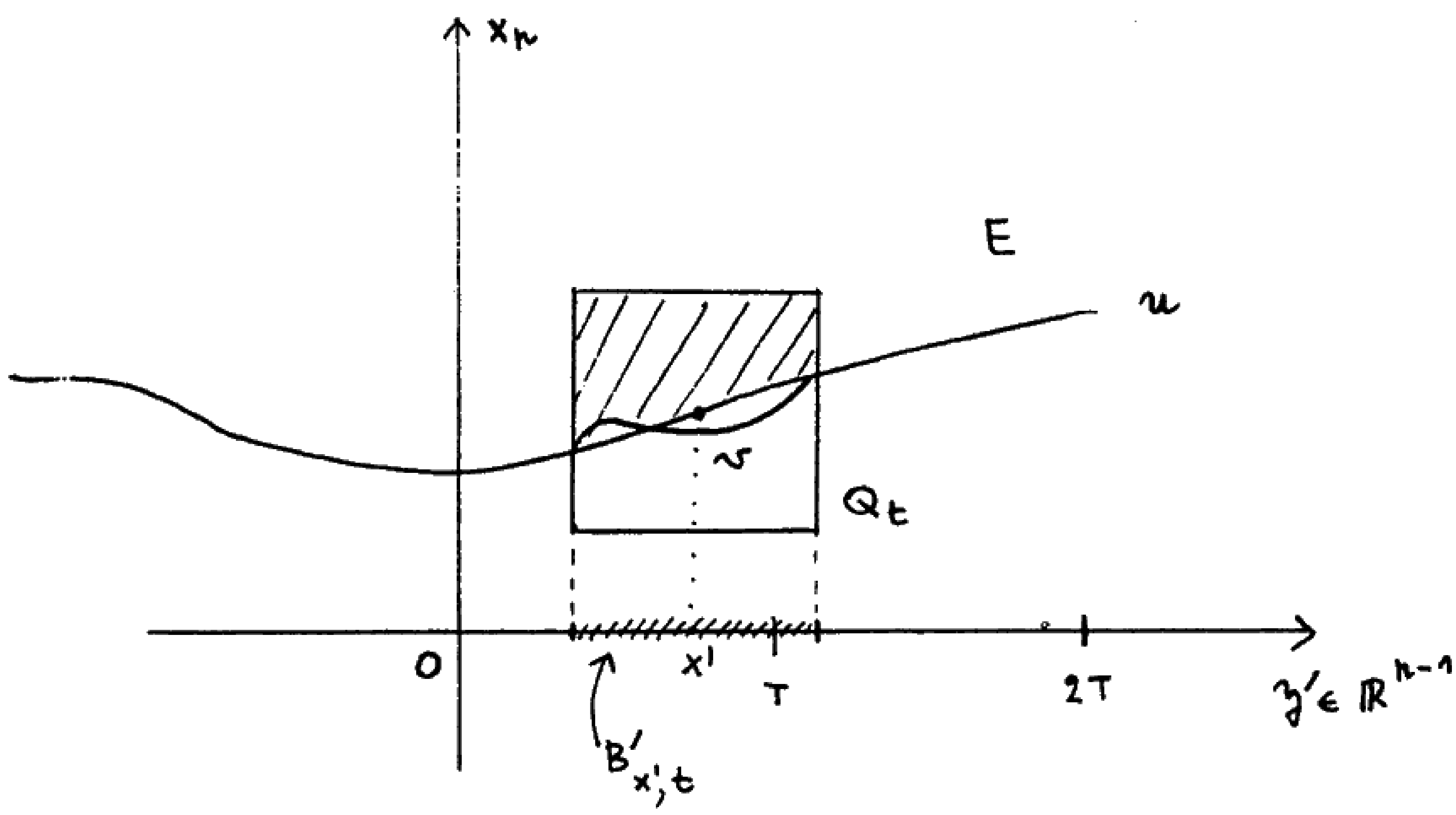

FIGURE 12.

By using successively (2.3),(2.6),(2.7),(2.8),(2.9),(2.10),(2.3), (2.5) and (2.7), we find [11]

$$\int_s |Du-\{Du\}_s|^2 \le \int_s |Du-\{Du\}_t|^2 \le 2\int_s |Du-Dv|^2 + 2\int_s |Dv-\{Du\}_t|^2$$

$$\le 2\int_t (|Du|^2 - |Dv|^2 + 2\int_s |Dv-\{Dv\}_s|^2$$

$$\leq 2 \int_t (|Du|^2 - |\{Du\}_t|^2) + 2 \int_t (|\{Du\}_t|^2 - |Dv|^2) +$$

$$+ 2(s/t)^{n+1} \cdot \int_t |Dv - \{Dv\}_t|^2$$

$$\leq 4(1+|\{Du\}_t|^2)^{\frac{1}{2}} \int_t [(1+|Du|^2)^{\frac{1}{2}} - (1+|\{Du\}_t|^2)^{\frac{1}{2}}] +$$

$$+ \frac{1}{2} \int_t (|Du|^2 - |\{Du\}_t|^2)^2 + 2(s/t)^{n-1} \cdot \int_t (|Dv|^2 - |\{Dv\}_t|^2) +$$

$$+ 4(1+|\{Du\}_t|^2)^{\frac{1}{2}} \cdot \int_t [(1+|\{Du\}_t|^2)^{\frac{1}{2}} - (1+|Dv|^2)^{\frac{1}{2}}]$$

$$\leq 4(1+|\{Du\}_t|^2)^{\frac{1}{2}} \cdot \int_t [(1+|Du|^2)^{\frac{1}{2}} - (1+|Dv|^2)^{\frac{1}{2}}] +$$

$$+ 2p^2 \int_t |Du - \{Du\}_t|^2 + 2(s/t)^{n+1} \int_t (|Du|^2 - |\{Du\}_t|^2)$$

since:

$$(2.15) \quad (|Du|^2 - |\{Du\}_t|^2)^2 = \langle Du + \{Du\}_t, Du - \{Du\}_t \rangle^2 \leq 4p^2 |Du - \{Du\}_t|^2$$

by Cauchy-Schwarz inequality and (2.14).

In conclusion, we have in view of (2.14), (1.9), and (2.3):

$$(2.16) \quad \int_{B'_{x',s}} |Du - \{Du\}_s|^2 \leq 4 (1+p^2)^{\frac{1}{2}} \psi(E, Q_t) + 2[(s/t)^{n+1} + p^2] \int_{B'_{x',t}} |Du - \{Du\}_t|^2$$

Now, if

$$(2.17) \qquad \psi(E, Q_t) \leq ct^{n-1+2\alpha} \qquad , \qquad 0 < \alpha < 1$$

then, setting

(2.18) $$\omega(r) = \int_{B'_{x',r}} |Du-\{Du\}_r|^2$$

we get from (2.14),(2.16),(2.17):

(2.19) $$\omega(s) \leq 4\cdot\bar{2}^{\frac{1}{2}}\, ct^{n-1+2\alpha} + 2\left[(s/t)^{n+1}+p^2\right]\cdot\omega(t) \quad \forall s,t:0<s<t\leq T$$

and thus also

(2.20) $$\omega(s) \leq \text{const.}s^{n-1+2\alpha} \qquad \forall s\in(0,T)$$

by virtue of Lemma 2.4, *provided* p *is sufficiently small.*

Consequelty, if

(i) (2.17) holds uniformly, for every cylinder Q_t with center at points $(x',u(x'))$ and radius t, such that $|x'| < T$ and $t\in(0,T)$;

(ii) p is sufficiently small, depending on α (see (2.19) and Lemma 2.4);

then

(2.21) $$\int_{B'_{x',t}} |Du-\{Du\}_t|^2 \leq \text{const.}t^{n-1+2\alpha} \qquad \forall x':|x'|< T,\ \forall t\in(0,T).$$

In view of the isomorphism between Campanato and Hölder spaces (particularly [12], between $\mathscr{L}^{2,n-1+2\alpha}$ and $C^{0,\alpha}$, see 2.2), we get in conclusion that $u \in C^{1,\alpha}(B'_{T/2})$.

2.6. Conditions (i) and (ii) above are clearly satisfied in the case under consideration. Indeed, whenever $E \subset \mathbb{R}^n$ has, in some open set Ω, a locally smooth (of class C^1) boundary ∂E, which in addition is almost minimal in Ω (Def. 1.5), with $\alpha(t) = ct^{2\alpha}$ and $0<\alpha<1$, then we can always arrange things so that (i) and (ii) above - with p defined by (2.14), and with u giving a local parametrization of (a piece of) $\partial E \cap \Omega$, see 1.6 and Fig. 3 - are satisfied. The preceding discussion then shows that ∂E if of class $C^{1,\alpha}$ in Ω, thus concluding the proof of (2.2).

The key role of the quantity $\int_{B'_r} |Du-\{Du\}_r|^2$ as a regularity parameter has also been stressed by the preceding discussion, see (2.18) - (2.21). Now, as the calculations above show, we have [13]

$$2(1+p^2)^{\frac{1}{2}} \int_{B'_r} [(1+|Du|^2)^{\frac{1}{2}}-(1+|\{Du\}_r|^2)^{\frac{1}{2}}] \leq \int_{B'_r} (|Du|^2-|\{Du\}_r|^2) \leq$$

$$\leq 2(1-p^2)^{-1}(1+p^2)^{\frac{1}{2}} \int_{B'_r} [(1+|Du|^2)^{\frac{1}{2}}-(1+|\{Du\}_r|^2)^{\frac{1}{2}}] \tag{2.22}$$

whenever $p < 1$. The integral in the left-hand side of (2.22) can be rewritten in terms of E (recall that $E = \mathrm{epi}(u)$, with $u \in C^1$ and $p<1$), because of the following relations (see [19], 3.4 and 4.10):

$$D_i\phi_E(B'_r \times \mathbb{R}) = D_i\phi_E(Q_r) = \int_{B'_r} D_i u(y')dy' \qquad i = 1,\ldots,n-1$$

(2.23)

$$D_n\phi_E(B'_r \times \mathbb{R}) = D_n\phi_E(Q_r) = H_{n-1}(B'_r)$$

which imply that

$$|D\phi_E(Q_r)| = \int_{B'_r} (1+|\{Du\}_r|^2)^{\frac{1}{2}} dy'$$

while clearly

$$|D\phi_E|(Q_r) = \int_{B'_r} (1+|Du|^2)^{\frac{1}{2}} dy'$$

It is then apparent that the quantity

$$|D\phi_E|(Q_r) - |D\phi_E(Q_r)|$$

also represents a fundamental regularity parameter. This justifies the following definition.

2.7. Definition of the Excess.

For every $A \subset\subset \mathbb{R}^n$ and every Caccioppoli set $E \subset \mathbb{R}^n$ we put

(2.24) $$\omega(E,A) = |D\phi_E|(A) - |D\phi_E(A)| .$$

The quantity $t^{1-n} \cdot \omega(E,B_{x,t})$ is usually known as the "area excess of E in $B_{x,t}$", denoted by Exc(E,x,t). Compare with (1.9), and the definition following (1.10).

Just as ψ was an "index of minimality", so is ω an "index of flatness": for, it is clear that if ∂E is *flat* near one of its points (so that we can assume that $\partial E \cap B_T = \{x \in B_T : x_n = 0\}$), then (see (2.23) and Fig. 13):

$$\omega(E,B_T) = |D\phi_E|(B_T) - D_n\phi_E(B_T) = 0$$

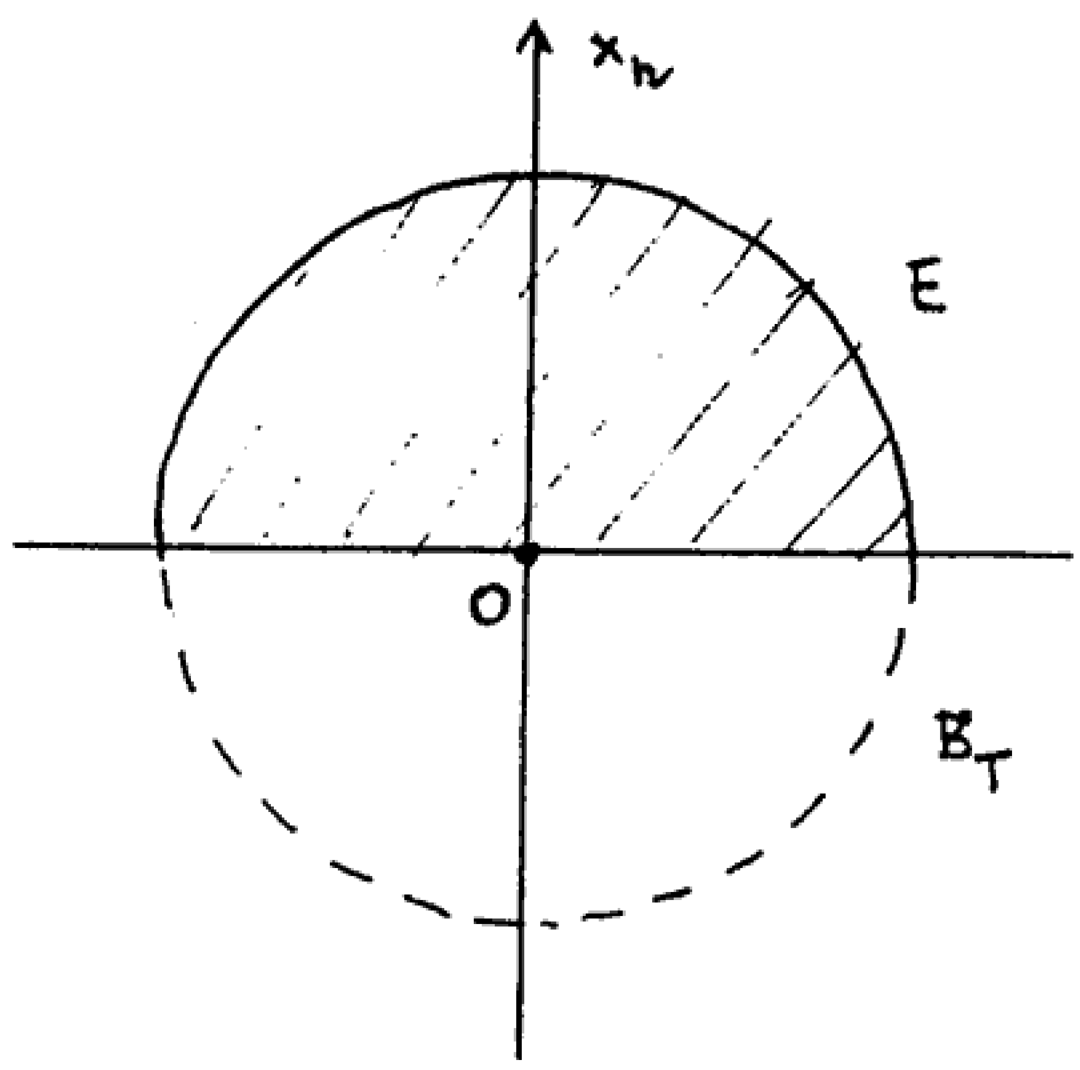

FIGURE 13.

Reciprocally, if $0 \in \partial E$ and $\omega(E,B_T) = 0$, then, on choosing the reference system so that

$$D_i\phi_E(B_T) = 0 \quad \text{when} \quad i=1,\dots,n-1; \quad D_n\phi_E(B_T) \geq 0$$

we get (see (1.29)):

$$0 < \int_{\partial^* E \cap B_T} dH_{n-1} = |D\phi_E|(B_T) = D_n\phi_E(B_T) = \int_{\partial^* E \cap B_T} \nu_E^{(n)}\, dH_{n-1}$$

Consequently

$$\nu_E^{(n)} = 1 \qquad H_{n-1}\text{-a.e. on} \quad \partial^* E \cap B_T$$

which implies, in view of known results (see e.g. [19], Theor. 4.8), that

$$\partial E \cap B_T = \{x \in B_T : x_n = 0\}.$$

Here we have a few illustrative examples:

(i) for the cone $E = \{|x_1| < |x_2|\} \subset \mathbb{R}^2$ (see 1.7) one has

$$\begin{cases} Dev(E,0,t) = 2(2-\sqrt{2}) \\ Exc(E,0,t) = 4 \end{cases}$$

(ii) for Simons'cone $C = \{x_1^2+\ldots+x_4^2 < x_5^2+\ldots+x_8^2\} \subset \mathbb{R}^8$ (see 1.4) one has instead

$$\begin{cases} Dev(C,0,t) = 0 \\ Exc(C,0,t) = \text{const.}>0 \end{cases}$$

(iii) for the epigraph $E=\{x_2 > |x_1|^{1+\alpha}\} \subset \mathbb{R}^2$, with $0\leq\alpha\leq 1$ (see 1.14 (v)) one has finally[4]

$$Dev(E,0,t) = Exc(E,0,t) \sim c_\alpha t^{2\alpha}.$$

The following proposition shows that some of the features exhibited by the preceding examples are of a general nature:

2.8. *Proposition.*

For every Caccioppoli set $E \subset \mathbb{R}^n$ we have

$$0 \leq Dev(E,x,t) \leq Exc(E,x,t) \leq t^{1-n}\, |D\phi_E|(B_{x,t}) \quad \forall x\in\mathbb{R}^n, \forall t>0. \tag{2.25}$$

Furthermore:

$$Exc(E,x,t) = o(1) \qquad \forall x \in \partial^* E. \tag{2.26}$$

Proof. Let $B = B_{x,t}$ be an arbitrary n-ball, and $F : F \Delta E \subset\subset B$. Then

$$\phi_E(B) = \int_{\partial E} \phi_E(y)(y-x)t^{-1} \, dH_{n-1}(y) = \int_{\partial B} \phi_F(y)(y-x)t^{-1} dH_{n-1}(y) = D\phi_F(B)$$

whence

$$|D\phi_E|(B) - |D\phi_E\ (B)| = |D\phi_E|(B) - |D\phi_F(B)| \geq |D\phi_E|(B) - |D\phi_F|(B)$$

and (2.25) follows at once.

Now, recall that $x \in \partial^*E$ iff

$$(\nu_1) \qquad |D\phi_E|(B_{x,t}) > 0 \qquad \forall t > 0$$

$$(2.27) \quad (\nu_2) \qquad \lim_{t \to o^+} \frac{D\phi_E(B_{x,t})}{|D\phi_E|(B_{x,t})} \triangleq \nu_E(x) \quad \text{exists, and}$$

$$(\nu_3) \qquad |\nu_E(x)| = 1$$

when this is the case, one has moreover (see (3.5)):

$$(2.28) \qquad |D\phi_E|(B_{x,t}) \sim \omega_{n-1}\ t^{n-1}$$

Conclusion (2.26) is then clear, since

$$(2.29) \qquad \mathrm{Exc}(E,x,t) = t^{1-n}\ |D\phi_E|(B_{x,t}) \left[1 - \frac{|D\phi_E\ (B_{x,t})|}{|D\phi_E|(B_{x,t})} \right]$$

2.9. We have just seen that $x \in \partial^*E$ implies $\mathrm{Exc}(E,x,t) \to 0$ as

$t \to 0$. When is the converse true? (i.e., under what additional assumptions does the infinitesimal character of the excess at a given boundary point imply the existence of the "normal" ν_E at that point?). This is a crucial point of our program. We begin our analysis by considering a simple counterexample.

Let $E = \{x_3 > r^{\frac{1}{2}}\} \subset \mathbb{R}^3$, with $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ (Fig. 14)

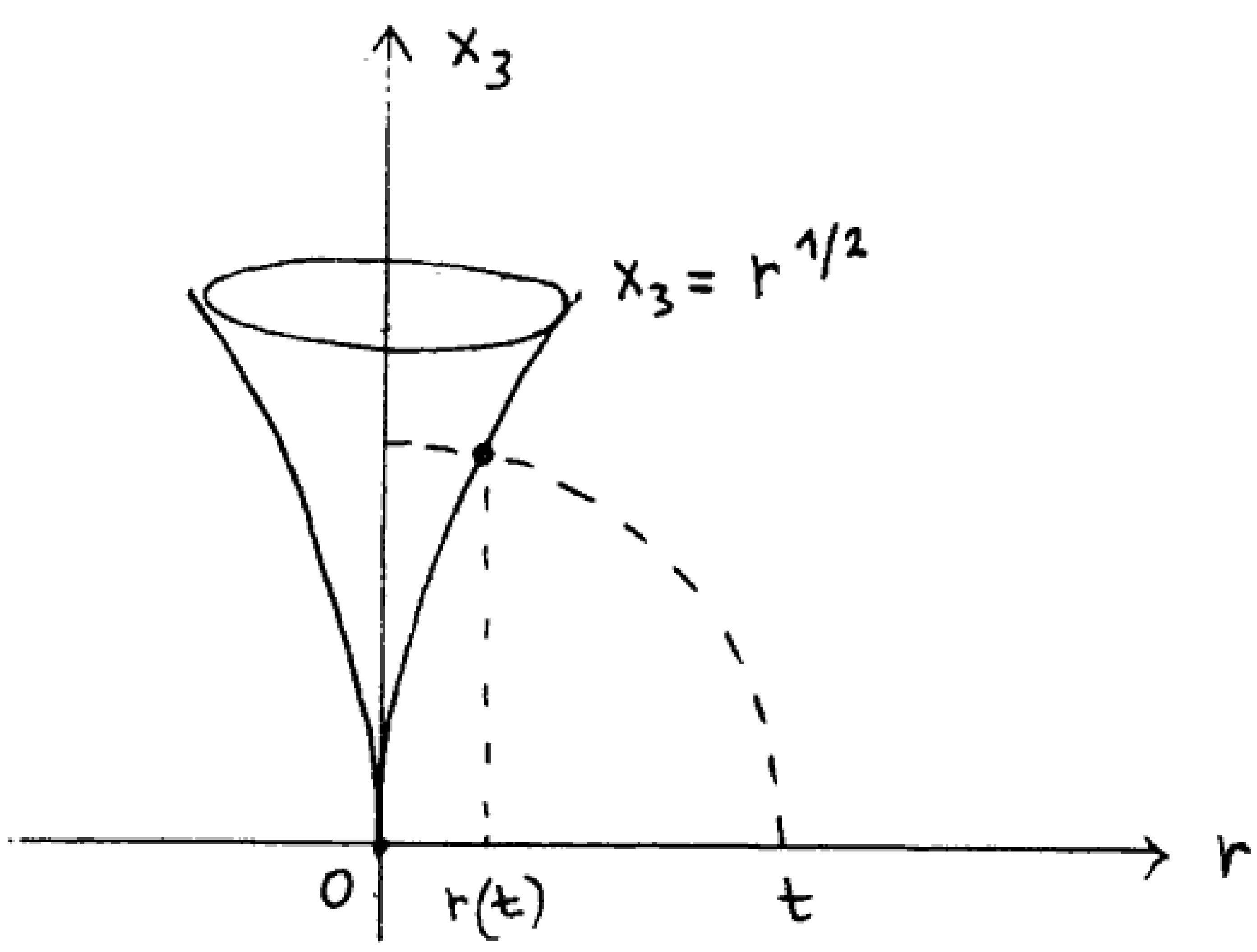

FIGURE 14.

Then $0 \in \partial E$, and

$$\begin{cases} |D\phi_E|(B_t) = 2\pi \int_0^{r(t)} r(1+1/4r)^{\frac{1}{2}}dr \\ D_1\phi_E(B_t) = D_2\phi_E(B_t) = 0 , \quad D_3\phi_E(B_t) = \pi r^2(t) \end{cases}$$

with $r(t) = \frac{1}{2}\left[(1+4t^2)^{\frac{1}{2}}-1\right]$. We immediately check that

$$(2.30) \quad \mathrm{Exc}(E,0,t) = t^{-2}\left[2\pi \int_0^{r(t)} r(1+1/4r)^{\frac{1}{2}}dr - \pi r^2(t)\right] \to 0 \quad \text{as} \quad t \to 0,$$

and that there exists

$$(2.31) \qquad \nu_E(0) = \lim_{t\to o^+} \frac{D\phi_E(B_t)}{|D\phi_E|(B_t)} = 0 \in \mathbb{R}^3$$

However, $|\nu_E(o)| = 0$ implies that $0 \notin \partial^*E$ (recall (2.27)). It follows from (2.29), (2.30) and (2.31) that

$$\lim_{t\to o} t^{-2} \; |D\phi_E|(B_t) = 0 \; ,$$

a fact that could also be checked directly.

2.10. Now, let us suppose that $x \in \partial E$ and that (contrary to what happens in the preceding example) there holds

$$(2.32) \qquad |D\phi_E|(B_{x,t}) \geq c_1 t^{n-1} \qquad \forall t \in (0,T)$$

with $c_1 > 0$. We anticipate (see prop. 3.4) that every set with almost minimal boundary does satisfy (2.32).

It follows from (2.29), (2.32) that *if* $\mathrm{Exc}(E,x,t) = o(1)$ *and* $\nu_E(x)$ *exists, then it has unit length,* and consequently $x \in \partial^*E$. In order to be sure of the existence of $\nu_E(x)$, we employ the following inequality

$$(2.33) \qquad \left| \frac{D\phi_E(G_1)}{|D\phi_E|(G_1)} - \frac{D\phi_E(G_2)}{|D\phi_E|(G_2)} \right| \leq 2 \left[\frac{\omega(E,G_2)}{|D\phi_E|(G_1)} \right]^{1/2}$$

which holds $\forall G_1 \subset G_2 \subset \mathbb{R}^n$ with $|D\phi_E|(G_1) > 0$ (see [27],2.5.4 (1)).

From (2.32) and (2.33) we deduce

$$(2.34)\qquad \left| \frac{D\phi_E(B_{x,s})}{|D\phi_E|(B_{x,s})} - \frac{D\phi_E(B_{x,t})}{|D\phi_E|(B_{x,t})} \right| \le 2\, c_1^{-1/2}\, (t/s)^{(n-1)/2}\ \mathrm{Exc}(E,x,t)^{\frac{1}{2}}$$

for every $s,t : 0<s<t<T$.

Now consider the abstract situation in which a given function

$$\nu : (0,T) \to \dot{\bar{B}}_1 \subset \mathbb{R}^n \quad \text{satisfies}$$

$$(2.35)\qquad |\nu(s) - \nu(t)| \le (t/s)^{(n-1)/2} \cdot g(t) \qquad \forall s,t: 0<s<t<T,$$

with $g(t) = o(1)$. Observe that (2.34) is a special case of (2.35). A simple calculation shows that the function

$$\nu(t) = (\text{sen lg lg}(e/t), \cos \text{lg lg}(e/t)), \quad 0<t<1$$

satisfies (2.35) with $T = 1$, $n=2$, and $g(t) = \sqrt{2}/\lg(e/t) = o(1)$. Nevertheless, $\alpha(t)$ has no limit as $t \to o$.

Condition (2.35) implies the existence of that limit, *provided* we have a reasonable "quantitative" hypothesis[14], regarding the convergence of $g(t)$ to 0. This is the case for instance when $g(t) \le ct^{\alpha}$, $\alpha > o$; indeed, given $t \in (0,T)$ and $\tau \in (0,1)$, for every $h,k \ge 1$ we find, on the account of (2.35):

$$|\nu(\tau^{h+k}t) - \nu(\tau^{h}t)| \le \sum_{i=o}^{k=1} |\nu(\tau^{h+i+1}t) - \nu(\tau^{h+i}t)|$$

$$(2.36)\qquad \le \tau^{(1-n)/2} \sum_{i=o}^{k-1} g(\tau^{h+i}t)$$

$$\leq c t^{\alpha} \tau^{(1-n)/2} \tau^{h\alpha} \sum_{i=0}^{\infty} \tau^{\alpha i}$$

$$\leq \text{const}\,(c,\tau,n,\alpha)\cdot t^{\alpha}\tau^{h\alpha}$$

which shows that $\{\nu(\tau^h t)\}_h$ is a Cauchy sequence in $\mathbb{R}^n$, for every $t \in (0,T)$ and every $\tau \in (0,1)$. Put

(2.37) $$\nu_o = \lim_{h\to+\infty} \nu(2^{-(h+1)}T)$$

and observe that $\forall t \in (0,T/2)$ there exists (and is unique) an integer $h = h(t) \geq 1$ such that

(2.38) $$2^{-(h+1)}T \leq t < 2^{-h}T$$

with in addition

(2.39) $$\lim_{t\to o^+} h(t) = +\infty .$$

Consequently

$$|\nu(t)-\nu_o| \leq |\nu(2^{-(h+1)}T)-\nu_o| + |\nu(2^{-(h+1)}T) - \nu(t)|$$

$$\leq |\quad \text{"} \quad \text{"} \quad | + c\cdot 2^{(n-1)/2}\cdot t^{\alpha}$$

by virtue of (2.35) and (2.38). From (2.37), (2.39) we then find

$$\nu_o = \lim_{t\to o} \nu(t)$$

thus proving our assertion.

2.11. We deduce from the foregoing considerations that when the set E, the point $x \in \partial E$, and the radius $T > 0$ are such that:

$$|D\phi_E|(B_{x,t}) \geq c_1 t^{n-1} \qquad \forall t \in (0,T), \text{ with } c_1 > 0, \text{ and}$$

$$Exc(E,x,t) \to 0 \quad \text{as} \quad t \to 0, \text{ in a certain "controlled may"}$$

(e.g., as $t^{2\alpha}$), then $x \in \partial^* E$.

It is not difficult to show that the first condition is satisfied, whenever ∂E is almost minimal. The point is that almost minimality implies the second condition as well, *at least when the excess, corresponding to the initial radius* T, *is conveniently small.* [15]

This fundamental result was originally proved by E.De Giorgi for minimal boundaries,in the form of the following lemma.

2.12. *Lemma* (De Giorgi [8,9])

For every $n \geq 2$ there exists a constant $\sigma = \sigma(n) > 0$ such that whenever the set $E \subset \mathbb{R}^n$, the point $x \in \partial E$, and the radius $t > 0$ satisfy

$$\begin{cases} \psi(E,B_{x,2t}) = 0 \\ Exc(E,x,2t) \leq \sigma \end{cases}$$

then:

$$Exc(E,x,t) \leq \sigma/2.$$

The iterative character of this result is apparent: a repeated application of the lemma yields the right estimation of the excess, which in addition turns out to be uniform in a neighbourhood of the given point. One derives from this the regularity of the set of boundary points, where the initial value of the excess is bounded by σ

A lemma of this sort is at the root of the various Regularity Theorems which extended De Giorgi's work: see [28,23,19,27], where the proof of such a result is obtained "by contradiction", as it was the case for the proof of Lemma 2.12 in De Giorgi's paper [8].

Moreover, a similar result is among the main tools in the Regularity Theory for almost minimal currents (and varifolds): see [4,5], where the proof is still obtained "by contradiction", and [34], where a more direct proof is developed.

It will be our aim in the next chapter to give a direct proof of a variation of Lemma 2.12, which will prove particularly useful for the demonstration of Theorem 1.9.