q	3	5	7	9	11	13	17	19
2(q+3)	12	16	20	24	28	32	40	44
$q+1+3[2\sqrt{q}]$	13	18	23	28	30	35	42	44
$N_q(3)$	10	16	20	28	28	32	40	44

Thus, for q odd with $q \leq 19$ and $q \neq 3$ or 9, the theorem gives the best possible result. A curve achieving $N_9(3)$ is $\mathcal{U}_{2,9}$.

5. WEIERSTRASS POINTS IN CHARACTERISTIC ZERO.

First consider the canonical curve $\mathcal{C}^{2g-2}$ of genus $g \geq 3$ in $PG(g-1,\mathbb{C})$. The <u>Weierstrass points</u>, W-points for short, are the points at which the osculating hyperplane has g coincident intersections. In this case, with w the number of W-points

$$w = g(g^2-1).$$

In any case,

$$2g + 2 \leq w \leq g(g^2-1)$$

with the lower bounded achieved only for hyperelliptic curves. A curve of genus $g > 1$ is <u>hyperelliptic</u> if it has a linear series γ^1_2 (a 2-sheeted covering) on it; for example, a plane quartic with a double point. It has equation

$$y^2 = f(x)$$

with genus $g = [\frac{1}{2}(d-1)]$ where $d = \deg f$.

Consider the case g=3 of the canonical curve $\mathcal{C}^4$, a non-singular plane quartic. The W-points are the 24 inflexions. We note that in characteristic $p > 0$, there is different behaviour; for example, $\mathcal{U}_{2,q}$ has 28 undulations (points where the tangent has 4-point contact). When $g=4$, the curve $\mathcal{C}^6 = \mathcal{F}^3 \cap \mathcal{F}^2$, the intersection of a cubic and a quadric surface, has 60 stalls where the osculating plane meets the curve at four coincident points.

More generally, still with characteristic zero, if $\mathcal{C}$ has genus $g \geq 1$ and $P \in \mathcal{C}$, there exist integers $n_1, n_2, \ldots, n_g$ such that no function has pole divisor precisely $n_i P$. Also $\{n_1, n_2, \ldots, n_g\} = \{1, 2, \ldots, g\}$ for all but a finite number of points. We elaborate this idea and make it more precise in §§8-10.

6. FUNDAMENTAL DEFINITIONS IN ALGEBRAIC GEOMETRY

Let $\mathcal{C} \subset A^n(K)$ be an irreducible non-singular algebraic curve defined over K, let $I(\mathcal{C}) \subset K[X_1, \ldots, X_n]$ be the ideal of polynomials wich are zero at all points of $\mathcal{C}$, let $\Gamma(\mathcal{C}) = K = [X_1, \ldots, X_n]/I(\mathcal{C})$; and $K(\mathcal{C})$ be the quotient field of $\Gamma(\mathcal{C})$; then $K(\mathcal{C})$ is called the function field of $\mathcal{C}$. Also, for P in $\mathcal{C}$, let $O_P = \{f/g \mid f, g \in \Gamma, g(P) \neq 0\}$, the local ring of $\mathcal{C}$ at P. Then, by natural inclusions, $K \subset \Gamma(\mathcal{C}) \subset O_P(\mathcal{C}) \subset K(\mathcal{C})$. Also $O_P \setminus \{\text{units}\} = M_P = \langle t \rangle$, the maximal ideal, and for any z in O_P there exist a unique unit u and a unique non-negative integer m such that $z = ut^m$; write $m = \mathrm{ord}_P(z)$. Hence, if $G \in K[X_1, \ldots, X_n]$ and g is the image of G in $\Gamma(\mathcal{C})$ with $G(P) \neq 0$, define $\mathrm{ord}_P(G) = \mathrm{ord}_P(g)$. In particular, if $\mathcal{C}$ is a plane curve and $V(L)$ the tangent at P, then $\mathrm{ord}_P(L)$ gives the multiplicity of contact of the tangent with $\mathcal{C}$.