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DUALITY THEOREMS FOR REGULAR HOMOTOPY OF FINITE DIRECTED GRAPHS. (*)

RIASSUNTO. - *Dati uno spazio topologico normale e numerabilmente paracompatto* S *ed un grafo finito ed orientato* G *si prova che tra gli insiemi* Q(S,G) *e* Q^*(S,G) *delle classi di o-omotopia e di* o^**-omotopia esiste una biiezione naturale. Nelle stesse condizioni, se* S' *è un sottospazio chiuso di* S *e* G' *un sottografo di* G, *esiste ancora una biiezione naturale tra gli insiemi* Q(S,S';G,G') *e* Q^*(S,S';G,G') *delle classi di omotopia. Si mostra infine che in condizioni meno restrittive per lo spazio* S *le precedenti biiezioni possono non sussistere.*

INTRODUCTION

In the extension from the undirected graphs to the directed ones, we have two possible definitions of regular function. In fact, given a topological space S and a finite directed graph G, a function $f\colon S \to G$ is called *o-regular* (resp. *o^*-regular*) if for all $v,w \in G$ such that $v \neq w$ and $v \nrightarrow w$, it is $\overline{f^{-1}(v)} \cap f^{-1}(w) = \phi$ (resp. $f^{-1}(v) \cap \overline{f^{-1}(w)} = \phi$). Therefore we can deal with two different homotopies, the o-homotopy and the o^*-homotopy. Hence we examine the problem

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of seeing if, under suitable conditions for the space S, the o-homotopy and the o*-homotopy get to coincide necessarily, i.e. if there exists a natural bijection between the sets of homotopy classes $Q(S,G)$ and $Q^*(S,G)$. As we observed in [2], by the Duality Principle the o-homotopy and o*-homotopy are interchanged by replacing the graph G by the dually directed graph G^*; thus we can identify the four sets $Q(S,G)$, $Q^*(S,G)$, $Q(S,G^*)$, $Q^*(S,G^*)$ at the same time.

Briefly we show how to solve the foregoing statements. In Part one , at first, we just consider functions and homotopies that are *completely regular*, i.e. without singularities; hence we examine the sets of complete o-homotopy classes $Q_c(S,G)$ and the ones of complete o*-homotopy classes $Q^*_c(S,G)$. Then we obtain some properties which characterize the regular and completely regular functions (§ 1) and we give the definition of *pattern*, by which we construct a relation from the set of completely o-regular functions to the one of completely o*-regular functions. Consequently, we have (§ 3) the Duality Theorem for complete homotopy classes (Theorem 9): *"There exists a natural bijection between the sets of complete homotopy classes* $Q_c(S,G)$ *and* $Q^*_c(S,G)$*".*

Now we recall the results obtained in [3], Theorems 12, 12*, 16, 16*:

i) If the space S is normal (*), in every class of $Q(S,G)$ (resp. $Q^*(S,G)$) there exists a completely o-regular (resp. o*-regular) function.

ii) If $S\times I$ is normal, two completely o-regular (resp. completely o*-regular) functions, which are homotopic, are also completely homotopic.

Hence it follows (§ 4) that if S and $S\times I$ are normal spaces, there exists a natural bijection from $Q_c(S,G)$ to $Q(S,G)$ and from $Q^*_c(S,G)$ to $Q^*(S,G)$. From here and Theorem 9 the Duality Theorem follows. Now if we recall that a normal space S such that the product $S\times I$ is normal, is said a *countably paracompact normal*

(*) We distinguish between normal space and T_4-space, according to whether it is a T_2-space or not.

space (see [12], pp.168-169) we can enunciate the Duality Theorem (Theorem 11): *"If* S *is a countably paracompact normal space, then there exists a natural bijection from* Q(S,G) *to* $Q^*(S,G)$*"*.

In Part two we consider the same problem for couples of topological spaces (S,S') and of directed graphs (G,G'). That is not a trivial generalization of Part one, because new difficulties rise. In general, indeed, we cannot construct patterns of completely o-regular functions, then we must add the further condition that the completely regular functions are *balanced* in S' as regards S (§ 5), i.e. such that for all $x' \in S'$, for all $v \in G$, $x' \in \overline{f^{-1}(v)}$ implies that $x' \in \overline{f^{-1}(v) \cap S'}$. Thus we can repeat the construction of patterns (§ 6). A second difficulty rises in that the so constructed patterns are not in general balanced functions. Hence we must choose as subspace S' an *open subspace* (§ 7) and under this condition the duality for complete homotopy is solved. Unfortunately we cannot deduce the Duality Theorem since the Normalization Theorems proved in [3] for S and $S \times I$ normal spaces hold only if S' is a closed set. We eliminate this last difficulty (§ 8,9) by considering the *decreasingly filtrated set* of open subspaces including S' and the *inductive limit* of the functions balanced in any open neighbourhood of S'. Thus by proceeding as in Part one we obtain the Duality Theorem (Theorem 32): *"If* S *is a countably paracompact normal space and* S' *a closed subspace of* S, *then there exists a natural bijection from the set of o-homotopy classes* Q(S,S';G,G') *to the one of* o^**-homotopy classes* $Q^*(S,S';G,G')$ *"*.

In § 11 we generalize the Duality Theorem to the case of $(n+1)$-tuples of topological spaces and of $(n+1)$-tuples of graphs. In § 12 we obtain the Duality Theorem for *absolute and relative homotopy groups* and we prove that the natural bijections are isomorphisms. At last in § 13 we give some counterexamples and among these we remark 13.4 and 13.5 which show that under weaker conditions for the space S (quasi compact, T_0 but not T_1) the two Duality Theorems do not hold.

0) Background.

Graphs and their subsets. (See [2] § 1, [3] § 1).

Let G be a *finite directed graph*.
If v, w are two vertices of G, we use the symbol $v \to w$ (resp. $v \not\to w$) to denote that vw is (resp. is not) a directed edge of G. If $v \to w$, we call v a *predecessor* of w and w a *successor* of v.

The graph G^* with the same vertices of G and such that $(u \to v$ in $G) \Leftrightarrow (v \to u$ in $G^*)$, is called the *dually directed graph* as regards G. (If $G \equiv G^*$, i.e. if for all $v,w \in G$ we have $(v \to w) \Leftrightarrow (w \to v)$, the graph is called *undirected*).

Let X be a non-empty subset of G. A vertex of X is called a *head* (resp. a *tail*) of X in G, if it is a predecessor (resp. a successor) of all the other vertices of X. We denote by $H_G(X)$ (resp. $T_G(X)$) or, simply, by $H(X)$ (resp. $T(X)$) the set of the heads (resp. tails) of X in G. If $H(X) \neq \phi$ (resp. $T(X) \neq \phi$), X is called *headed* (resp. *tailed*); otherwise, X is called *non-headed* (resp. *non-tailed*) Finally, X is called *totally headed* (resp. *totally tailed*), if all the non-empty subsets of X are headed (resp. tailed). If X is a singleton, we agree to say that X is headed.

Regular and completely regular functions. (See [2] § 1, [3] § 2).

Let S be a *topological space*.
Given a function $f: S \to G$ from S to G, we denote by capital letter V the set of all the f-counterimages of $v \in G$, and if we want to emphasize the function f, we write $V^f = f^{-1}(v)$.

A function $f: S \to G$ is called *o-regular* (resp. *o*-regular*), if for all $v,w \in G$ such that $v \neq w$ and $v \not\to w$, it is $V \cap \overline{W} = \phi$ (resp $\overline{V} \cap W = \phi$).

Let $I = [0,1]$ be the unit interval in R^1. Two o-regular (resp. o*-regular) functions $f,g: S \to G$ are called *o-homotopic* (resp. *o*-homotopic*), if there exists an o-regular (resp. o*-regular) function $F: S \times I \to G$, such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for all $x \in S$. The o-regular (resp. o*-regular) function F is called an *o-homotopy* (resp. *o^*-homotopy*) between f and g. The o-homotopy (resp. o^*-homotopy) is an equivalence relation and we denote by $Q(S,G)$ (resp. $Q^*(S,G)$) the set of o-homotopy (resp. o^*-homotopy) classes. We note that $Q^*(S,G)$ coincides with $Q(S,G^*)$ and $Q^*(S,G^*)$ with $Q(S,G)$.

DUALITY PRINCIPLE. - *Every true proposition in which appear the concepts of headed set, tailed set, o-regularity, o^*-regularity, o-homotopy, o^*-homotopy,* Q(S,G), $Q^*(S,G)$, *remains true if the concepts of headed set and tailed set, o-regularity and o^*-regularity, o-homotopy and o^*-homotopy,* Q(S,G) *and* $Q^*(S,G)$, *are interchanged throught the statement of the proposition.*

Given an o-regular (resp. o^*-regular) function $f: S \to G$, a n-tuple $X = \{v_1,\dots, v_n\}$, $(n \geq 2)$ is called a *singularity* of f if:

i) X is non-headed (resp. non-tailed);

ii) $\overline{V_1^f} \cap \dots \cap \overline{V_n^f} \neq \phi$.

An o-regular (resp. o^*-regular) function $f: S \to G$ from S to G is called *completely o-regular* (resp. *completely o^*-regular*), or simply *c.o-regular* (resp. *c.o^*-regular*), if there are no singularities of f. (If the graph G is undirected, then all the singularities are couples and the c.regular functions are called *strongly regular* functions).

Functions between pairs. (See [2] §5,[3] §2).

Let S' be a *subspace* of S and G' a *subgraph* of G.
A function $f: S,S' \to G,G'$ is called *o-regular* (resp. *o^*-regular*) if both $f: S \to G$ and its restriction $f' = f/_{S'}: S' \to G'$ are o-regular (resp. o^*-regular) functions.

Two o-regular (resp. o^*-regular) functions $f,g: S,S' \to G,G'$ are called *o-homotopic* (resp. *o^*-homotopic*), if there exists an o-regular (resp. o^*-regular) homotopy $F: S\times I,\ S'\times I \to G,G'$, between f and g. The o-homotopy (resp. o^*-homotopy) is an equivalence relation and we denote by $Q(S,S';G,G')$ (resp. $Q\ (S,S';G,G')$) the *set of o-homotopy* (resp. *o^*-homotopy*) *classes*. We note that $Q^*(S,S';G,G')$ coincides with $Q(S,S',G^*, G'^*)$ and $Q(S,S';G,G')$ with $Q^*(S,S';G^*,G'^*)$.

A function $f: S,S' \to G,G'$ is called *c.o-regular* (resp. *c.o^*-regular*) if both $f: S \to G$ and $f': S' \to G'$ are c.o-regular (resp. c.o^*-regular) functions.

As before, the *Duality Principle* holds for functions between pairs.

Main results of [2], [3].

R_a: $X \subseteq G$ *is totally headed, iff it is totally tailed.* (See [3], Proposition 4).

If S is a *normal topological space* and S' is a *closed subspace* of S, we have:

R_b: (The first Normalization Theorem). *Let* $f: S \to G$ *(resp.* $f: S,S' \to G,G'$*) be an o-regular function. Then there exists a c.o-regular function, o-homotopic to* f. (See [3], Theorems 12, 15).

R_c: (Extension Theorem between pairs). *Let* $f: S,S' \to G,G'$ *be an o-regular function. Then there exist a closed neighbourhood* U *of* S' *and an o-regular function* $g: S,S' \to G,G'$*, which is o-homotopic to* f *and such that the function* $g: S,U \to G,G'$ *is o-regular, i.e.* $g(U) \subseteq G'$ *and the restriction* $\hat{g}: U \to G'$ *of* g *to* U *is o-regular.* (See [2], Theorem 20).

R_d: *In the construction of* R_c, *if there exist* n *vertices* $p_1,\dots,p_n \in G$ *and* m *vertices* $q_1,\dots,q_m \in G'$*, such that* $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \overline{Q_1^{f'}} \cap \dots \cap \overline{Q_m^{f'}} = \phi$*, then also it follows* $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap \overline{Q_1^{\hat{g}}} \cap \dots \cap \overline{Q_m^{\hat{g}}} = \phi$. *Similarly, from* $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap X = \phi$ *it results* $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap U = \phi$. (See [2], Corollary 21).

Moreover, if $S\times I$ is *normal*, then it results:

R_e: (The first Normalization Theorem for homotopies). *Let* $f,g: S \to G$ *(resp.* $f,g: S,S' \to G,G'$*) be two o-homotopic c.o-regular functions. Then, between the functions* f *and* g, *there also exists an o-homotopy, which is a c.o-regular function.* (See [3], Theorem 16).

By Duality Principle, the results dual to the previous ones are also true.