global solution (i.e. for $t \in [0,\infty)$)

6. Connexion with the mollified problem.

As we said in the introduction, the mollified version of the problem (1) has a unique strict global solution $u_\varepsilon(t)$ and if $u_\circ \in X_\circ^+ \cap X_\infty$ we have

$$u_\varepsilon(t) = Z_\circ(t)u_\circ + \int_0^t Z_\circ(t-s)\, F_\varepsilon(u_\varepsilon(s))ds \qquad \text{for } t \geq 0$$

If $[0,\bar{t}]$ is the existence interval for the solution of the problem (13), we have for $t \in [0,\bar{t}]$:

(17) $$||u_\varepsilon(t) - u(t)|| \leq \int_0^t ||F_\varepsilon(u_\varepsilon(s)) - F(u(s))||ds.$$

The aim of this section is to prove the following

THEOREM (3). If $u_\circ \in X_\circ^+ \cap X_\infty$, $u(t)$ is the mild solution of the problem (1) in the interval $[0,\bar{t}]$ and $u_\varepsilon(t)$ is the strict global solution of the mollified version of the problem (1), then we have

(18) $$\lim_{\varepsilon\to 0+} ||u_\varepsilon(t)-u(t)|| = 0 \qquad \text{uniformly in } t \in [0,\bar{t}] .$$

PROOF

If $f,g \in X_\circ \cap X_\infty$ then

(19) $$||F_\varepsilon(f) - F(g)|| \leq ||F_\varepsilon(f) - F_\varepsilon(g)|| + ||F_\varepsilon(g)-F(g)|| \leq 2\delta(||f|| + ||g||)||f-g|| + ||F_\varepsilon(g)-F(g)||$$

where $\delta = (v_2-v_1)\, ||k_\varepsilon||_\infty$ (see [1]).

Since we proved that the norm of the solution is invariable both in [1] and in this paper (see (16)), we have

$$||u_\varepsilon(t)|| = ||u_\circ|| = ||u(t)|| \qquad \text{for } t \in [0,\bar{t}]$$

and then, from (17) and (19)

$$||u_\varepsilon(t) - u(t)|| \leq \int_0^t ||F_\varepsilon(u(s)) - F(u(s))||ds + 4\delta||u_\circ||\int_0^t||u_\varepsilon(s)-u(s)||ds.$$

If we suppose that we have proved that

(20) $$\lim_{\varepsilon\to 0+} \int_0^t ||F_\varepsilon(u(s))-F(u(s))||ds = 0 \quad \text{uniformly in } t\in[0,\bar{t}],$$

and $\eta > 0$ is given, then a suitable $\delta > 0$ can be found such that

$$||u_\varepsilon(t) - u(t)|| \leq \eta + 4\delta\, ||u_\circ|| \int_0^t ||u_\varepsilon(s)-u(s)||ds$$

for each $\varepsilon \in (0,\delta)$ and for $t \in [0,\bar{t}]$. Hence

$$||u_\varepsilon(t) - u(t)|| \leq \eta\, e^{4\delta||u_\circ||\bar{t}} \qquad \text{for } t \in [0,\bar{t}]$$

by Gronwall's Lemma. So the theorem is proved as soon as we have proved (20). Define, for brevity

$$f(\varepsilon,s) = ||F_\varepsilon(u(s)) - F(u(s))||$$

and note that $f(\varepsilon,\cdot)$ is continuous because $F_\varepsilon(\cdot)$, $F(\cdot)$ and $u(\cdot)$ are continuous. By Lebesgue's bounded convergence theorem to prove (20) it is sufficient to prove

(21) $$\lim_{\varepsilon\to 0+} f(\varepsilon,s) = 0 \qquad \text{for } s \in[0,\bar{t}]$$

(22) $$f(\varepsilon,s) \leq g(s) \qquad \text{for } s \in[0,\bar{t}],\ \varepsilon > 0$$

where $g(s)$ is a summable function independent of ε. We will prove (21) and (22) with the help of some lemmas. But first note that

(23) $$F_\varepsilon(g) - F(g) = q[(K_\varepsilon-I)J_1gJ_2g - g(K_\varepsilon-I)J_3J_1g]$$

Lemma (10)

(a) $K_\varepsilon \in B(X_\circ)$, $||K_\varepsilon|| \leq 1$

(b) $\lim_{\varepsilon\to 0+} ||K_\varepsilon f - f|| = 0$ for $f \in X_\circ$

PROOF

(a) $$||K_\varepsilon f|| \le \int_{-\infty}^{+\infty} dv \int_{V_1}^{V_2} dw \int_{-\infty}^{+\infty} dx \int_{x}^{+\infty} k_\varepsilon(x'-x)|f(x',v,w)|dx' =$$

$$= \int_{-\infty}^{+\infty} dv \int_{V_1}^{V_2} dw \int_{-\infty}^{+\infty} dx'|f(x',v,w)| \int_{-\infty}^{x'} k_\varepsilon(x'-x)dx = ||f||$$

because $$\int_{-\infty}^{x'} k_\varepsilon(x'-x)dx = \int_{0}^{+\infty} k_\varepsilon(y)dy = 1$$

(b) $$||K_\varepsilon f-f|| = || \int_{x}^{+\infty} dx' \; k_\varepsilon(x'-x)[f(x',v,w)-f(x,v,w)]|| =$$

$$= || \int_{0}^{\varepsilon} dy \; k_\varepsilon(y)[f(x+y,v,w)-f(x,v,w)]|| \le \int_{0}^{\varepsilon} dy \; k_\varepsilon(y)||f(x+y,y,w)-f(x,v,w)||.$$

Since $f \in X_o$ we have $\lim_{y\to o} ||f(x+y,v,w)-f(x,v,w)|| = 0$ and because $y \to 0$ as $\varepsilon \to o+$ the thesis follows. ∎

<u>COROLLARY</u> (1). If $g \in X_o \cap X_\infty$ then:

(a) $||F_\varepsilon(g) - F(g)|| \le 2d \; ||g|| \; ||g||_\infty$

(b) $\lim_{\varepsilon\to o+} ||F_\varepsilon(g) - F(g)|| = 0$

PROOF

(a) By (a) of Lemma (10) we have $||k_\varepsilon - I|| \le 2$ and by (23)

$$|| F_\varepsilon(g) - F(g)|| \le 2(||J_1 g J_2 g|| + ||g||_\infty ||J_3 J_1 g||).$$

Now the assertion follows by Lemma (4).

(b) Define $g_1 = J_1 g J_2 g$, $g_2 = J_3 J_1 g$ then note that

$$||F_\varepsilon (g) - F(g)|| \le ||K_\varepsilon g_1 - g_1|| + ||g||_\infty ||K_\varepsilon g_2 - g_2||$$

by (23) and finally the assertion follows by (b) of Lemma (10). ∎

Now, if $u_\circ \in X_\circ^+ \cap X_\infty$ then $u(s) \in X_\circ \cap X_\infty$ for $s \in [0,\bar{t}]$ and (21) follows by (b) of Corollary (1). By the results of § 5, the e exists $M > 0$ such that $||u(s)||_\infty \leq M$ for $s \in [0,\bar{t}]$ and (22) follows by (a) of Corollary (1)

∎

APPENDIX

Let X be a real Banach space and in this space consider the semi-linear problem

(A1) $$\frac{du}{dt} = A u + F(u) \qquad u(0) = u_\circ \in D(A)$$

where A is linear and generates the semigroup $Z(t)$ while F is non linear but at least continuous. The integral version of (A1) is the equation

(A2) $$u(t) = Z(t)u_\circ + \int_0^t Z(t-s)F(u(s))ds.$$

A solution of (A2) is called a mild solution of (A1) but as we know it is not in general a strict solution of (A1). On the other hand a solution of (A1) is also a solution of (A2).

Under suitable conditions such as the Fréchet-differentiablity of F and the continuity of the derivative a solution of (A2) with $u_\circ \in D(A)$ is also a solution of (A1) (see [6]).

Let c be a closed cone of X, i.e. c is a closed subset of X that satisfies the condition

$x,y \in c$, $\alpha \geq 0 \Longrightarrow x + y \in c$, $\alpha x \in c$.

If we set

$Y = C([0,\bar{t}]; X)$, $C = \{u \in Y; u(t) \in c$ for $t \in [0,\bar{t}]\}$ s a closed subset of X contained in $D(F)$

$s' = s \cap c$

$S = \{u \in Y;\ u(t) \in s$ for $t \in [0,\bar{t}]\}$

$S' = S \cap C$

$$(P\,u)(t) = Z(t)u_\circ + \int_0^t Z(t-s)F(u(s))ds$$

we have the following

PROPOSITION. If

(a) $u_\circ \in s'$

(b) $\mathcal{Z}(t)\ u \in c$ for $u \in c$

(c) $F(u) \in c$ for $u \in s'$

(d) $P : S \to S$ and is strictly contractive then the unique solution u of $u = P\,u$ belongs to S'.

PROOF

The hypothesis ensure that $P : S' \to S'$ and so $u \in S'$ when (c) is not satisfied the follwing is useful.

THEOREM. With the same hypothesis (a), (b) and (d) of the preceding proposition, if

(c') $a > 0$ exists such that $F_1(u) \in c$ for $u \in s'$ where $F_1 = F + a\,I$

(d') If we set

$$(P_1 v)(t) = T(t)u_\circ + \int_0^t T(t-s)F_1(v(s))ds$$

where $T(t) = e^{-at}\ Z(t)$, P_1 maps S into itself, then we have the same conclusion of the preceding proposition, for a suitable $\bar{t}$.

In order to prove the theorem define the linear operator

$(\mathcal{Z} u)(t) = a \int_0^t Z(t-s)u(s)ds$ for $u \in Y$ and prove the following.

LEMMA. $\mathcal{Z} \in B(Y)$ and for a suitable $\bar{t}$ the operator $I + \mathcal{Z}$ is invertible. The operators P and P_1 are connected by the following equality

(A3) $$P_1 = (I + \mathcal{Z})^{-1}(P + \mathcal{Z})$$

PROOF.

$$(-a) \int_0^t Z(t-t')(P_1 u)(t')dt' = (-a) \int_0^t e^{a(t-t')}T(t-t')\cdot$$

$$\cdot\left[T(t')u_\circ + \int_0^{t'} T(t'-s)\, F_1(u(s))ds\right]dt' =$$

$$= (-a) \int_0^t e^{a(t-t')} \left[T(t)u_\circ + \int_0^{t'} T(t-s)F_1(u(s))ds\right]dt' =$$

$$= \left[1 - e^{at}\right] T(t)u_\circ + \int_0^t \left[1 - e^{a(t-s)}\right] T(t-s)F_1(u(s))ds =$$

$$= P_1 u \; - \left[Z(t)u_\circ + \int_0^t Z(t-s)(F(u(s)) + a\, u(s))ds\right]$$

So

$$(P_1 u)(t) = Z(t)u_\circ + \int_0^t Z(t-s)\{F(u(s)) + a\left[u(s) - P_1(u(s))\right]\}ds$$

and then

$$(P_1 u)(t) = (Pu)(t) + a \int_0^t Z(t-s)\left[u(s) - P_1(u(s))\right]ds$$

i.e.

(A4) $P_1 u = P\, u + \mathcal{Z} \cdot (u - P_1 u)$.

By last equality we have:

(A5) $\qquad (I + \mathcal{Z})\, P_1 u = (P + \mathcal{Z})u$.

Note that if $||Z(t)|| \leq M\, e^{bt}$ it follows that

$$||\mathcal{Z}|| \leq \begin{cases} a\, M\, \bar{t} & \text{if } b = 0 \\ a\, M\, \dfrac{e^{b\bar{t}} - 1}{b} & \text{if } b \neq 0 \end{cases}$$

. It is clear that this quantity is less than 1 for a suitable $\bar{t}$ and thus $I + \mathcal{Z}$ is invertible . So the assertion is true.

∎

REMARK (A1). (A4) and (A5) are valid in $[0,\bar{t}]$ for every $\bar{t}$, while (A3) is valid just for a suitable $\bar{t}$ (such that $||\mathcal{Z}|| < 1$).

COROLLARY (a) If u is a solution of $u = P_1 u$ in $[0,\bar{t}']$ then u is also solution of $u = Pu$ in the same interval. (b) If P is contractive then so is P_1 but in general not for the same $\bar{t}$. The converse is also true. (c) If u is a solution of $u = Pu$ in $[0,\bar{t}]$ then it is also solution of $u = P_1 u$ but, in general, in a smaller interval.

PROOF

(a) follows by (A4) and by remark (A1)

(b) By the lemma it follows that

$$||P_1|| \leq \frac{||P|| + ||\mathfrak{Z}||}{1 - ||\mathfrak{Z}||}$$

where $||\cdot||$ is the usual seminorm defined for Lipschitz operators

(i.e. $||P|| = \sup \{\frac{||P(u) - P(v)||}{||u-v||};\ u,v \in D(P)\}$).

We have $||P_1|| < 1$ if $||P|| + 2\,||\mathfrak{Z}|| < 1$ and this is true for a suitable $\bar{t}$. The inverse follows by (A4); in fact

$$||P|| \leq ||I + \mathfrak{Z}||\cdot||P_1|| + ||\mathfrak{Z}|| \quad .$$

(c) follows by (A3) and by remark (A1)

∎

PROOF OF THEOREM.

By hypothesis and by the Corollary it follows that

(b') $T(t)\, u \in c$ if $u \in c$

(c') $F_1(u) \in c$ if $u \in s'$

(d') P_1 maps S into itself and is strictly contractive, so by the Proposition it follows that a unique solution u of $u = Pu$ exists for a suitable $\bar{t}$ and it belongs to S'.

∎

Remark (A2)

The preceding results are especially useful in the cases where the mild solution is not in general the strict solution. Otherwise the preceding result is trivial because the problems

$$\frac{du}{dt} = A\,u + F(u) \quad ; \qquad u(0) = u_o \qquad \text{and}$$

$$\frac{dv}{dt} = (A-aI)v + F_1(v) \qquad v(o) = u_o$$

coincide.

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