Chapter 3

Estimates of the derivatives of solution of parabolic problems in $L^1(\Omega)$

As a consequence of Theorem 2.5.2 and Proposition 1.2.7 we have that $(A_1, D(A_1))$ is sectorial in $L^1(\Omega)$, then it generates a bounded analytic semigroup $T(t)$ and $T(t)u_0$ is the solution of

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}$$

for each $u_0 \in L^1(\Omega)$. Moreover there exist $c_i = c_i(\Omega, \mu, M_1)$, $i = 0, 1$ such that

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_0, \quad t > 0 \tag{3.1}$$

and

$$t\|A_1 T(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_1, \quad t > 0. \tag{3.2}$$

Moreover since $D(A_1)$ is dense in $L^1(\Omega)$ by construction, $T(t)$ is strongly continuous in $L^1(\Omega)$. Hence

$$\lim_{t\to 0^+} \|T(t)u_0 - u_0\|_{L^1(\Omega)} = 0 \quad \text{for all } u_0 \in L^1(\Omega) \tag{3.3}$$

Notice that for every $u \in L^1(\Omega)$ and for every $t > 0$, $T(t)u \in W^{2,1}(\Omega)$.

3.0.1 Estimates of first order derivatives

Now, using the gradient estimate (2.115) of the resolvent operator $R(\lambda, A_1)$, we estabilish the following further property of the semigroup $T(t)$.

Proposition 3.0.4. *Let Ω, $\mathcal{A}$ and $\mathcal{B}$ be as in Section 2.5 and let $T(t)$ be the semigroup generated by $(A_1, D(A_1))$. Then, there exists c_2 depending on Ω, μ, M_1 such that for $t > 0$,*

$$t^{1/2}\|DT(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_2. \tag{3.4}$$

Proof. Let θ_1' be as in Theorem 2.5.3 and suppose $\omega_1' = 0$ (otherwise we consider $A_1 - \omega_1'$). Let consider the curve

$$\Gamma = \{\lambda \in \mathbf{C}; |\arg\lambda| = \theta_1', |\lambda| \ge 1\} \cup \{\lambda \in \mathbf{C} : |\arg\lambda| \le \theta_1', |\lambda| = 1\}$$

oriented counterclockwise. We know that for $t > 0$

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda} R(\lambda, A_1)d\lambda.$$

Setting $\lambda' = \lambda t$ we get

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda'} R(\lambda'/t, A_1)t^{-1}d\lambda'$$

and

$$D_iT(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda'} t^{-1} D_i R(\lambda'/t, A_1)d\lambda' \qquad i = 1, \dots, n$$

therefore by (2.115)

$$\|D_iT(t)\|_{\mathcal{L}(L^1(\Omega))} \le t^{-1/2}\int_\Gamma e^{\mathrm{Re}\,\lambda'}|\lambda'|^{-1/2}d|\lambda'| \le ct^{-1/2} \qquad i = 1, \dots, n$$

and the result is proved. □

Remark 3.0.5. [Neumann boundary conditions] We have stated Theorem 2.5.2 in the form we most frequently use, but the estimates hold under more general assumptions. In particular, all non tangential boundary conditions are allowed. We denote by c_ν a constant which can be used in the inequalities (3.1)–(3.4), when Neumann boundary conditions are associated with a general uniformly elliptic operator.

Remark 3.0.6. [Assumptions on the coefficients b_i] The result of generation in L^1 and estimates (3.1), (3.2) can be achieved under weaker assumptions on coefficients b_i. Assume $\mathcal{A}, \mathcal{B}$ as in (2.106), (2.110) with coefficients satisfying (2.108), (2.107). Then we know that $(A_1, D(A_1))$ generates an analytic semigroup in $L^1(\Omega)$.
We consider a first order perturbing operator $\mathcal{C} = \sum_{i=1}^n(\tilde{b}_i - b_i)D_i$ with $\tilde{b}_i \in L^\infty(\Omega)$ $b_i \ne \tilde{b}_i$. Let C_1 be the realization of $\mathcal{C}$ in $L^1(\Omega)$ with domain $D(C_1) = W^{1,1}(\Omega)$. The operator C_1 is A_1- bounded and more precisely for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\|C_1u\|_{L^1(\Omega)} \le \varepsilon\|A_1u\|_{L^1(\Omega)} + c(\varepsilon)\|u\|_{L^1(\Omega)}$$

holds for every $u \in D(A_1)$. Indeed let $u \in D(A_1)$, (suppose $\omega_1 = 0$, otherwise consider $A_1 - \omega_1$) then $u = R(\lambda, A_1)f$ for every $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda > 0$ and $f \in L^1(\Omega)$. Moreover, by (1.7) we can write

$$u = \int_0^\infty e^{-\lambda s}T(s)f ds, \quad \mathrm{Re}\,\lambda > 0.$$

Thus, in particular for $\lambda > 0$

$$\|Du\|_{L^1(\Omega)} \le c\|f\|_{L^1(\Omega)} \int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}} ds = \frac{c}{\sqrt{\lambda}}\|f\|_{L^1(\Omega)} \le c(\sqrt{\lambda}\|u\|_{L^1(\Omega)} + \frac{1}{\sqrt{\lambda}}\|A_1 u\|_{L^1(\Omega)})$$

This implies that $D(A_1) \hookrightarrow W^{1,1}(\Omega)$; moreover, minimizing over $\lambda > 0$, we get

$$\|Du\|_{L^1(\Omega)} \le c\|u\|^{1/2}_{L^1(\Omega)}\|A_1 u\|^{1/2}_{L^1(\Omega)} \le \varepsilon\|A_1 u\|_{L^1(\Omega)} + \frac{c}{\varepsilon}\|u\|_{L^1(\Omega)} \tag{3.5}$$

and by Theorem 1.2.10 we conclude. We point out that the first inequality in (3.5) asserts that $W^{1,1}(\Omega) \in J_{1/2}(L^1(\Omega), D(A_1))$.

3.1 Estimates of second order derivatives

In order to proceed, we also need a precise L^1-estimate of the second (spatial) derivatives of $T(t)u_0$, for $u_0 \in D(A_1)$. This is proved in Proposition 3.1.3 below. The argument used here is similar to the one used in [18, Theorem 2.4], where Ω is bounded and different boundary conditions are imposed. The scheme is the following: we estimate the second order derivatives in Proposition 3.1.1, and then, using this result, we characterize the interpolation space $D_{\mathcal{A}}(\alpha, 1) = (L^1(\Omega), D(\mathcal{A}))_{\alpha,1}$ as a fractional Sobolev space and use this to improve estimate (3.6) using the $W^{1,1}$ norm of u instead of the L^1 norm. We start with the following result.

Proposition 3.1.1. *Let $\Omega, \mathcal{A}, \mathcal{B}$ be as in Section 2.5. Assume, in addition, $c \in W^{1,\infty}(\Omega)$; then, there exists c_3 depending on n, μ, Ω, M_1, $\|c\|_{W^{1,\infty}(\Omega)}$, c_0 c_1, c_2, c_ν such that for every $t \in (0,1)$ and $u \in L^1(\Omega)$ we have*

$$t\|D^2T(t)u\|_{L^1(\Omega)} \le c_3\|u\|_{L^1(\Omega)}. \tag{3.6}$$

Proof. We set for $\sigma > 0$ $u_\sigma = T(\sigma)u$ and

$$M_2 = \max\{\|A\|_{2,\infty}, \|B\|_{2,\infty}, \|c\|_{1,\infty}\}. \tag{3.7}$$

By the regularity of the boundary $\partial\Omega$ we can consider a partition of unity $\{(\eta_h, U_h)\}_{h\in\mathbf{N}}$ such that $\operatorname{supp}\eta_h \subset U_h$, $\sum_{h=0}^\infty \eta_h(x) = 1$ for every $x \in \overline{\Omega}$ and $0 \le \eta_h \le 1$ for every $h \in \mathbf{N}$, $\overline{U}_0 \subset \Omega$, U_h for $h \ge 1$ is a ball such that $\{U_h\}_{h\ge1}$ is a covering of $\partial\Omega$ and $\{U_h\}_{h\in\mathbf{N}}$ is a covering of Ω with bounded overlapping, that is there is $\kappa > 0$ such that

$$\sum_{h\in\mathbf{N}} \chi_{U_h}(x) \le \kappa, \quad \forall x \in \overline{\Omega}. \tag{3.8}$$

Moreover we choose η_h in such a way $\langle A(x)D\eta_h(x), \nu(x)\rangle = 0$ for every $x \in \partial\Omega$ and set $\overline{M} := \sup_{h\in\mathbf{N}} \|\eta_h\|_{2,\infty}$, which is finite by the uniform C^2 regularity of $\partial\Omega$. We can also consider coordinate functions $\psi_h : V_h \to B(0,1)$ such that $\psi_h(V_h \cap \Omega) = B^+(0,1) = \{y = (y', y_n) \in B(0,1) : y_n > 0\}$, $\psi_h(V_h \cap \partial\Omega) = \{y = (y', y_n) \in B(0,1) : y_n = 0\}$,

$d(\psi_h)_x(a(x)\nu(x)) = -e_n$ for every $x \in \partial\Omega$ where $d(\psi_h)_x$ denotes the differential of ψ_h at x. Finally we suppose that there is a constant M_ψ such that

$$\sup_{h\geq 1}\left\{\|D^2\psi_h\|_{2,\infty}, \|D^2\psi_h^{-1}\|_{2,\infty}\right\} \leq M_\psi.$$

Notice also that we may assume that for all $h \geq 1$ the inclusion $U_h \subset\subset V_h$ holds, and that we can choose a C^2 domain E such that $\psi_h(U_h \cap \Omega) \subset E \subset B^+(0,1)$. Notice that $u_\sigma \in W^{1,1}(\Omega)$ and denote by $u(t) = T(t)u_\sigma$ the solution of the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_\sigma & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}$$

We want to estimate the L^1–norm of $tD^2u(t)$ by the L^1-norm of u; we shall use estimates (3.1)–(3.4). The functions $v_h(t) = u(t)\eta_h$ solve, for every $h \in \mathbf{N}$, the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{A}_h u(t) & \text{in } (0,\infty)\times\Omega \\ w(0) = \eta_h u_\sigma & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{3.9}$$

where

$$\mathcal{A}_h u(t) = -2\langle AD\eta_h, Du(t)\rangle - u(t)\,\mathrm{div}(AD\eta_h) - u(t)\,\langle B, D\eta_h\rangle\,. \tag{3.10}$$

Notice that the derivative $D_k v_h(t)$ satisfies the equation $\partial_t(D_k v_h(t)) - \mathcal{A}(D_k v_h(t)) = \mathcal{A}_h^k u(t)$, where

$$\begin{aligned} \mathcal{A}_h^k u(t) =& \mathrm{div}\,((D_kA)D(u(t)\eta_h)) + \langle (D_kB), D(u(t)\eta_h)\rangle + (D_kc)u(t)\eta_h + D_k(\mathcal{A}_h u(t)) \\ =& \mathrm{div}\,((D_kA)D(u(t)\eta_h)) + \langle (D_kB), D(u(t)\eta_h)\rangle + (D_kc)u(t)\eta_h \\ &+ D_k[-2\langle AD\eta_h, Du(t)\rangle - u(t)\,\mathrm{div}(AD\eta_h) - u(t)\,\langle B, D\eta_h\rangle] \end{aligned} \tag{3.11}$$

For $D_k v_h(t)$ we consider the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{A}_h^k u(t) & \text{in } (0,\infty)\times\Omega \\ w(0) = D_k(\eta_h u_\sigma) & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{3.12}$$

whose solution is $v_{hk}(t) = T(t)D_k(\eta_h u_\sigma) + \int_0^t T(t-s)\mathcal{A}_h^k u(s)ds$. Now we consider $h = 0$, i.e., we draw our attention to the inner part. Since $v_0(t) = \eta_0 u(t) = 0$ in $\Omega \setminus U_0$, it turns out that $D_k v_0(t)$ is the solution of (3.12) with $h = 0$. Then

$$D_k v_0(t) = T(t)D_k(\eta_0 u_\sigma) + \int_0^t T(t-s)\mathcal{A}_0^k u(s)ds, \tag{3.13}$$

where $\mathcal{A}_0^k$ is the operator defined in (3.11). Then, differentiating, we obtain

$$D^2_{lk} v_0(t) = D_l[T(t)D_k(\eta_0 u_\sigma)] + \int_0^t D_l[T(t-s)\mathcal{A}_0^k v(s)]ds.$$

by which, using (3.4),

$$\begin{aligned}\|D^2_{lk}v_0(t)\|_{L^1(\Omega)} &\le \|D_lT(t)D_k(\eta_0u_\sigma)\|_{L^1(\Omega)} + \int_0^t \|D_lT(t-s)\mathcal{A}_0^k u(s)\|_{L^1(\Omega)}ds\\ &\le \frac{c_2}{\sqrt{t}}\|D_k(\eta_0u_\sigma)\|_{L^1(\Omega)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)}ds\\ &\le \frac{c_2^2}{\sqrt{t}}\|\eta_0\|_{W^{1,\infty}}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)}ds\end{aligned}$$

Finally, estimating $\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)}$ by (3.11) we get $\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} \le c\,\|u(s)\|_{W^{2,1}(\Omega)}$ where $c = c(\overline{M}, M_2)$. Summing on l and k, using (A.1) and again (3.1), we get

$$\|D^2 v_0(t)\|_{L^1(\Omega)} \le c\Big(\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds\Big)$$

where $c = c(\overline{M}, M_2, c_2, n)$. We now consider $h \ge 1$, i.e., we consider a ball intersecting $\partial\Omega$.
Using the transformation $\hat f(y) := f(\psi_h^{-1}(y))$ for a generic f defined in $\Omega \cap V_h$, and since v_h is the solution of (3.9), we get that for every $h \ge 1$ the function $\hat v_h(t,y) = \eta_h(\psi_h^{-1}(y))u(t,\psi_h^{-1}(y))$ is the solution of the following initial-boundary value problem with homogeneous Neumann boundary conditions

$$\begin{cases}\partial_t w - \hat{\mathcal{A}}w = \hat{\mathcal{A}}_h\hat v & \text{in } (0,+\infty)\times E\\ w(0) = \hat\eta_h\hat u_\sigma & \text{in } E\\ \dfrac{\partial w}{\partial\nu} = 0 & \text{in } (0,+\infty)\times\partial E\end{cases} \tag{3.14}$$

where $\hat{\mathcal{A}}$ is the operator defined on $B(0,1)$ as follows

$$\hat{\mathcal{A}}w := \mathrm{div}(\hat A\,Dw) + \langle\hat B, Dw\rangle + \hat c w$$

whose coefficients (here we omit the index h to simplify the notations and by analogy with (3.9)) are given by

$$\begin{aligned}\hat A(y) :=& \big(D\psi_h\big)(\psi_h^{-1}(y))\cdot A(\psi_h^{-1}(y))\cdot\big(D\psi_h\big)^t(\psi_h^{-1}(y))\\ (\hat B(y))_l :=& \mathrm{Tr}\Big[\big(D\psi_h\big)(\psi_h^{-1}(y))\cdot A(\psi_h^{-1}(y))\cdot H^l(\psi_h^{-1}(y))\cdot\big(D\psi_h^{-1}\big)^t(y)\Big]\\ &+\mathrm{Tr}\Big[\big(D\psi_h\big)(\psi_h^{-1}(y))\cdot G^j(y)\Big]\big(D\psi_h\big)^t_{jl}(\psi_h^{-1}(y)) - \frac{\partial}{\partial y_j}\Big[\hat a_{jl}(y)\Big]\\ &+\Big[\big(D\psi_h\big)(\psi_h^{-1}(y))\cdot B(\psi_h^{-1}(y))\Big]_l\\ \hat c(y) :=& c(\psi_h^{-1}(y))\end{aligned}$$

where $H^l_{ki} = D^2_{ki}(\psi_h)_l$ and $G^j_{ki} = D_k a_{ij}(\psi_h^{-1}(y))$ and (see (3.10))

$$\hat{\mathcal{A}}_h\hat u(t) = -2\big\langle A(\psi_h^{-1}(y))(D\psi_h)^t D\hat\eta_h, (D\psi_h)^t D\hat u(t)\big\rangle - \hat u(t)\big[\,\mathrm{div}(\hat A D\hat\eta_h) + \langle\hat A, D\hat\eta_h\rangle\big]\,.$$

Now, as done before for $h = 0$, differentiating the equation (now $D_k = \frac{\partial}{\partial y_k}$) we obtain that $D_k\hat v_h$ solves $\partial_t(D_k\hat v_h(t)) - \hat{\mathcal{A}}(D_k\hat v_h(t)) = \hat{\mathcal{A}}_h^k\hat u(t)$, where $\hat{\mathcal{A}}_h^k\hat v$ can be obtained by taking the corresponding term in (3.11). Associated with this operator, we can consider the problem

$$\begin{cases} \partial_t w - \hat{\mathcal{A}} w = \hat{\mathcal{A}}_h^k \hat{u}(t) & \text{in } (0,\infty)\times E \\ w(0) = D_k\big(\hat{\eta}_h \hat{u}_\sigma\big) & \text{in } E \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{in } (0,\infty)\times \partial E. \end{cases}$$

The function $D_k \hat{v}_h$ satisfies the equation and the initial condition. Notice that if $k \neq n$ also the boundary condition is satisfied since $\hat{v}_h = 0$ in a neighborhood of $\partial E \cap \{y \in \mathbf{R}^n \,|\, y_n > 0\}$, in the other part of ∂E the operator D_k is a tangential derivative and $\frac{\partial \hat{v}_h}{\partial y_n}$ is constant for $y_n = 0$. Denote by S the semigroup which gives the solution of this problem and notice that the estimates (3.1)–(3.4) hold for $S(t)$, see Remark 3.0.5. Then

$$D_k \hat{v}_h(t) = S(t) D_k \hat{v}_h(0) + \int_0^t S(t-s)\hat{\mathcal{A}}_h^k \hat{u}(s)ds. \tag{3.15}$$

Differentiating (3.15) with respect to D_j for any j, we have then proved that the following holds

$$D^2_{kj}\hat{v}_h(t) = D_j S(t) D_k \hat{v}_h(0) + \int_0^t D_j S(t-s)\hat{\mathcal{A}}_h^k \hat{u}(s)ds. \tag{3.16}$$

Thus, as for $v_0(t)$, we have for $(k,j) \neq (n,n)$

$$\begin{aligned} \|D^2_{kj}\hat{v}_h(t)\|_{L^1(E)} &\leq \frac{c_2}{\sqrt{t}}\|D_k(\hat{\eta}_h \hat{u}_\sigma)\|_{L^1(E)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\hat{\mathcal{A}}_h^k \hat{u}(s)\|_{L^1(E)}ds \\ &\leq \frac{c}{\sqrt{t\sigma}}\|\hat{u}\|_{L^1(E)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\hat{\mathcal{A}}_h^k \hat{u}(s)\|_{L^1(E)}ds. \end{aligned}$$

We now estimate $D^2_{nn}\hat{v}_h(t)$. Since

$$\begin{aligned} \hat{a}_{nn} D^2_{nn}\hat{v}_h(t) \;&=\; \hat{\mathcal{A}}\hat{v}_h(t) - \sum_{(i,j)\neq(n,n)} \hat{a}_{ij} D^2_{ij}\hat{v}_h(t) - \sum_{i,j=1}^n (D_i \hat{a}_{ij}) D_j \hat{v}_h(t) \\ &\quad - \sum_{i=1}^n \hat{b}_i D_i \hat{v}_h(t) - \hat{c}\hat{v}_h(t) \end{aligned}$$

and since $\hat{a}$ is uniformly elliptic with ellipticity constant proportional to μ, we can find a constant c (depending only on $n, M_2, \mu, \partial\Omega$) such that

$$\begin{aligned} \|D^2_{nn}\hat{v}_h(t)\|_{L^1(E)} &= \Big\| \frac{1}{\hat{a}_{nn}}\Big(\hat{\mathcal{A}}\hat{v}_h(t) - \sum_{(i,j)\neq(n,n)} \hat{a}_{ij} D^2_{ij}\hat{v}_h(t) + \\ &\quad - \sum_{i,j=1}^n (D_i \hat{a}_{ij}) D_j \hat{v}_h(t) - \sum_{i=1}^n \hat{b}_i D_i \hat{v}_h(t) - \hat{c}\hat{v}_h(t)\Big)\Big\|_{L^1(E)} \\ &\leq c\Big[\sum_{(i,j)\neq(n,n)} \|D^2_{ij}\hat{v}_h(t)\|_{L^1(E)} + \|\hat{\mathcal{A}}\hat{v}_h(t)\|_{L^1(E)} + \|D\hat{v}_h(t)\|_{L^1(E)} + \|\hat{v}_h(t)\|_{L^1(E)}\Big]. \end{aligned}$$

Summing up, we may argue in the same way as for $h=0$, and get

$$\begin{aligned}\|D^2\hat{v}_h(t)\|_{L^1(E)} &\le c'\Big[\frac{1}{\sqrt{t}}\|u_\sigma\circ\psi_h^{-1}\|_{W^{1,1}(E)}+\int_0^t\frac{1}{\sqrt{t-s}}\|D^2\hat{u}(s)\|_{L^1(E)}ds\\ &\quad+\|\hat{\mathcal{A}}\hat{v}_h(t)\|_{L^1(E)}\Big]\\ &\le c'\Big[\frac{1}{\sqrt{t\sigma}}\|u\circ\psi_h^{-1}\|_{L^1(E)}+\int_0^t\frac{1}{\sqrt{t-s}}\|D^2\hat{u}(s)\|_{L^1(E)}ds+\|\hat{\mathcal{A}}\hat{v}_h(t)\|_{L^1(E)}\Big]\end{aligned}$$

where $c'=c(\overline{M},M_2,M_\psi,n,c_2,c_\nu)$. Coming back to $\Omega\cap U_h$ we obtain

$$\begin{aligned}\|D^2v_h(t)\|_{L^1(\Omega\cap U_h)} &\le c''\Big[\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega\cap U_h)}+\int_0^t\frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega\cap U_h)}ds\\ &\quad+\|\mathcal{A}v_h(t)\|_{L^1(\Omega\cap U_h)}\Big] \qquad (3.17)\\ &\le c''\Big[\frac{1}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega\cap U_h)}+\int_0^t\frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega\cap U_h)}ds+\|\mathcal{A}v_h(t)\|_{L^1(\Omega\cap U_h)}\Big]\end{aligned}$$

where c'' depends on $\overline{M},M_2,M_\psi,n,c_2,c_\nu$. Now, using (3.1), (3.2) and (3.8), we have

$$\begin{aligned}\|D^2u(t)\|_{L^1(\Omega)} &= \|D^2\big(\sum_{h=0}^\infty v_h(t)\big)\|_{L^1(\Omega)}=\|\sum_{h=0}^\infty D^2v_h(t)\|_{L^1(\Omega)} \qquad (3.18)\\ &\le \kappa c''\Big[\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega)}+\int_0^t\frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds+\|\mathcal{A}u(t)\|_{L^1(\Omega)}\Big]\\ &\le c'''\Big[\frac{1}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega)}+\int_0^t\frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds+\frac{1}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega)}\Big],\end{aligned}$$

where c''' depends on κ,c'',c_0,c_1. Now using Gronwall's generalized inequality (see Lemma 1.5.7), we get

$$\|D^2u(t)\|_{L^1(\Omega)}\le\frac{c}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega)}. \qquad (3.19)$$

Then, by taking $\sigma=t$, we get $\|D^2u(t)\|_{L^1(\Omega)}\le c_3t^{-1}\|u\|_{L^1(\Omega)}$ for every $t\in(0,1)$. □

3.1.1 Characterization of interpolation spaces between $D(A_1)$ and $L^1(\Omega)$

We can use Proposition 3.1.1 to characterize some interpolation spaces between $D(A_1)$ and $L^1(\Omega)$.

Theorem 3.1.2. *Let A_1 be as in Proposition 3.1.1; then for every $\alpha\in(0,1/2)$ we have*

$$(L^1(\Omega),D(A_1))_{\alpha,1}=W^{2\alpha,1}(\Omega)$$

where $W^{2\alpha,1}$ denotes the Sobolev space of fractional order (see Section A.2.1 for details).

Proof. It is sufficient to prove that

$$(L^1(\Omega),D(A_1))_{\alpha,1}=(L^1(\Omega),W^{2,1}(\Omega)\cap W^{1,1}_{A,\nu}(\Omega))_{\alpha,1}$$

in fact using Theorem A.2.7 we complete the proof.
First of all, let us observe that $W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega) \hookrightarrow D(A_1)$. Therefore, using Definition A.2.2, we obtain

$$(L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\alpha,1} \hookrightarrow (L^1(\Omega), D(A_1))_{\alpha,1}.$$

Conversely, let $u_0 \in (L^1(\Omega), D(A_1))_{\alpha,1}$ and set for $t \in [0,1]$

$$u_0 = u_0 - T(t)u_0 + T(t)u_0 = -\int_0^t A_1 T(s)u_0 ds + T(t)u_0 = v_1 + v_2.$$

We have

$$\|v_1\|_{L^1(\Omega)} \le \int_0^t \|A_1 T(s)u_0\|_{L^1(\Omega)} ds$$

and since $v_2 \in W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega)$, using (A.1), (3.1) and Proposition 3.1.1, we have

$$\begin{aligned}\|v_2\|_{W^{2,1}(\Omega)} &= \|T(t)u_0\|_{L^1(\Omega)} + \sum_{i,j=1}^n \|D_{ij}[T(t)u_0 - T(1)u_0 + T(1)u_0]\|_{L^1(\Omega)} \\ &\le c_0\|u_0\|_{L^1(\Omega)} + \sum_{i,j=1}^n \|D_{ij}\int_t^1 T(s/2)A_1T(s/2)u_0 ds\|_{L^1(\Omega)} + c_3\|u_0\|_{L^1(\Omega)} \\ &\le c\left\{\|u_0\|_{L^1(\Omega)} + \int_t^1 s^{-1}\|A_1T(s/2)u_0\|_{L^1(\Omega)} ds\right\}\end{aligned}$$

Therefore for $t \in [0,1]$, setting $K(t,u_0) := K(t,u_0,L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))$ we obtain

$$\begin{aligned}K(t,u_0) &= \inf_{u_0=u_0^1+u_0^2}(\|u_0^1\|_{L^1(\Omega)} + t\|u_0^2\|_{W^{2,1}(\Omega)}) \\ &\le \|v_1\|_{L^1(\Omega)} + t\|v_2\|_{W^{2,1}(\Omega)} \\ &\le c\Big(\int_0^t \|A_1T(s)u_0\|_{L^1(\Omega)} ds + t\|u_0\|_{L^1(\Omega)} \\ &+ t\int_t^1 s^{-1}\|A_1T(s/2)u_0\|_{L^1(\Omega)} ds\Big)\end{aligned}$$

On the other hand, choosing $u_0^1 = u_0$ and $u_0^2 = 0$ we get

$$K(t,u_0) \le \|u_0\|_{L^1(\Omega)}.$$

Therefore

$$\begin{aligned}K(t,u_0) \le c\Big(\min(1,t)\|u_0\|_{L^1(\Omega)} &+ \int_0^t \|A_1T(s)u_0\|_{L^1(\Omega)} ds \\ &+ t\int_t^1 s^{-1}\|A_1T(s/2)u_0\|_{L^1(\Omega)} ds\Big).\end{aligned}$$

Therefore for each $\alpha \in (0,1)$ we get

$$\begin{aligned}\int_0^\infty t^{-(1+\alpha)} K(t,u_0)dt \le c\Big\{&\|u_0\|_{L^1(\Omega)} \int_0^\infty t^{-(1+\alpha)} \min(1,t)dt \\ &+ \int_0^\infty (t^{-(1+\alpha)} \int_0^t \|A_1 T(s)u_0\|_{L^1(\Omega)} ds)dt \\ &+ \int_0^\infty (t^{-\alpha} \int_t^\infty s^{-1}\|A_1T(s)u_0\|_{L^1(\Omega)} ds)dt\Big\}\end{aligned}$$

so that using Hardy inequalities stated in Theorem 1.5.6, we get

$$\int_0^\infty t^{-(1+\alpha)} K(t,u_0)dt \le c\Big\{\|u_0\|_{L^1(\Omega)} + \int_0^\infty s^{-\alpha}\|A_1T(s)u_0\|_{L^1(\Omega)} ds\Big\}$$

and hence from Theorem 1.3.2 we get

$$(L^1(\Omega), D(A_1))_{\alpha,1} \hookrightarrow (L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\alpha,1}$$

so, the result is proved. □

Using Theorem 3.1.2 we can improve the estimate of Proposition 3.1.1, under additional assumption on the initial datum; in fact, we have the following.

Proposition 3.1.3. *Let $\Omega, \mathcal{A}, \mathcal{B}$ be as in Section 2.5. Assume, in addition, $c \in W^{1,\infty}(\Omega)$; then, there exist $\delta \in (1/2, 1)$ and c_4 depending on n, μ, Ω, M_2, c_0, c_1, c_2, c_3 c_ν such that for every $t \in (0,1)$ and $u \in D(A_1)$ we have*

$$t^\delta \|D^2T(t)u\|_{L^1(\Omega)} \le c_4\|u\|_{W^{1,1}(\Omega)}. \tag{3.20}$$

Proof. We can repeat the proof of Proposition 3.1.1 until the first inequality in (3.18), with $\sigma > 0$, so that we have

$$\begin{aligned}\|D^2u(t)\|_{L^1(\Omega)} \le &\kappa c''\Big[\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds \\ &+ \|\mathcal{A}u(t)\|_{L^1(\Omega)}\Big]\end{aligned} \tag{3.21}$$

Using (1.10), we get that for any $\alpha, \beta \in (0,1)$ there is C such that

$$t^{1-\alpha+\beta}\|\mathcal{A}T(t)u\|_{D_{\mathcal{A}}(\beta,1)} \le C\|u\|_{D_{\mathcal{A}}(\alpha,1)}.$$

By definition of interpolation, $D_{\mathcal{A}}(\beta,1)$ is continuously embedded in $L^1(\Omega)$ for any $\beta \in (0,1)$. Using the fact that $D_{\mathcal{A}}(\alpha,1)$ is the fractional Sobolev space $W^{2\alpha,1}(\Omega)$ for $\alpha < 1/2$ and that $W^{1,1}(\Omega)$ embeds in $W^{2\alpha,1}(\Omega)$ for such α, we obtain, with constants C that may change from a line to the other,

$$\begin{aligned}\|\mathcal{A}T(t)u\|_{L^1(\Omega)} \le &C\|\mathcal{A}T(t)u\|_{D_{\mathcal{A}}(\beta,1)} \le \frac{C}{t^{1-\alpha+\beta}}\|u\|_{D_{\mathcal{A}}(\alpha,1)} \\ =&\frac{C}{t^{1-\alpha+\beta}}\|u\|_{W^{2\alpha,1}(\Omega)} \le \frac{C}{t^{1-\alpha+\beta}}\|u\|_{W^{1,1}(\Omega)}\end{aligned}$$

We choose then $\alpha \in (0, 1/2)$ and $\beta \in (0, 1)$ is such a way that $\delta = 1 - \alpha + \beta \in (1/2, 1)$, and (3.21) becomes

$$\|D^2 u(t)\|_{L^1(\Omega)} \leq \frac{C}{t^\delta} \|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{C'}{\sqrt{t-s}} \|D^2 u(s)\|_{L^1(\Omega)} ds.$$

Therefore applying the Gronwall's lemma and passing to the limit as $\sigma \to 0$ we get (3.20). $\square$