# Chapter 5

# $BV$ functions and parabolic problems: the second characterization

In this chapter we present a second characterization of $BV$ functions obtained using in a different way the semigroup $T(t)$ generated by the $L^1$ realization of

$$\mathcal{A} = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j) + \sum_{i=1}^{n} b_i(x)D_i + c(x) \tag{5.1}$$

with coefficients

$$a_{ij} \in W^{2,\infty}(\Omega) \quad b_i,\, c \in L^\infty(\Omega) \tag{5.2}$$

satisfying (2.107) and with homogeneous boundary condition given by $\mathcal{B}$ in (2.5); in that case, it is possible to associate a positive function $p(t,x,y) \in C_b^1((0,\infty) \times \Omega \times \Omega)$ to the semigroup $T(t)$ (see [45, Sections 5.3, 5.4] for more details) generated by $(A_1, D(A_1))$ and the following representation holds

$$(T(t)u_0)(x) = \int_\Omega p(t,x,y)u_0(y)\,dy. \tag{5.3}$$

This function $p(t,x,y)$ is called the kernel of $T(t)$ and this formula is a keystone for proving some interesting relations between $BV$ functions and solutions of parabolic initial boundary value problems; more precisely, in the spirit of [33], we give a complete characterization of sets of finite perimeter and then, using it in connection with the coarea formula, we prove that

$$|Du|_A(\Omega) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_\Omega \int_\Omega p(t,x,y)|u(x) - u(y)|\,dydx, \tag{5.4}$$

where $|Du|_A$ denotes the $A$-weighted total variation of $u$. This characterization is analogous to some results in [8], [14] and [27], [33], where general kernels depending on $|x-y|$ are considered.

## 5.1 The heat kernel in $\mathbf{R}^n$

In [27], Ledoux investigated in a different perspective some connections between the heat semigroup $(W(t))_{t\geq 0}$ on $L^2(\mathbf{R}^n)$ and the isoperimetric inequality.
We recall that the classical isoperimetric inequality in $\mathbf{R}^n$ states that *among all subset $E \subset \mathbf{R}^n$ with fixed volume and smooth boundary, Euclidean balls minimize the surface measure of the boundary.* In [27] Ledoux observed that the $L^2-$ inequality for the Gauss-Weiesrstrass semigroup in $\mathbf{R}^n$

$$\|W(t)\chi_E\|_{L^2(\mathbf{R}^n)} \leq \|W(t)\chi_B\|_{L^2(\mathbf{R}^n)} \qquad t \geq 0 \tag{5.5}$$

for sets $E$ with smooth boundary and with $|E| = |B|$ can be used to prove the isoperimetric inequality. In order to reach this, he provided an estimate for the $L^2$ norm of $W(t)\chi_E$ in terms of the perimeter of $E$ in $\mathbf{R}^n$. We refer to [27, Proposition 1.1] for the proof.

**Proposition 5.1.1** (Ledoux). *For every subset $E$ of finite measure in $\mathbf{R}^n$ and smooth boundary $\partial E$ and for every $t \geq 0$, the inequality*

$$\int_{E^c} W(t)\chi_E(x)\,dx \leq \sqrt{\frac{t}{\pi}}\mathcal{P}(E) \tag{5.6}$$

*holds.*

Moreover, if $B$ is an Euclidean ball, he checked that

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{B^c} W(t)\chi_B(x)\,dx = \mathcal{P}(B). \tag{5.7}$$

Finally if $|E| = |B|$, then the $L^2$- inequality (5.5) is equivalent to the following

$$\int_{E^c} W(t)\chi_E(x)dx \geq \int_{B^c} W(t)\chi_B(x)dx. \tag{5.8}$$

This is easy to see; in fact,

$$\begin{aligned}
\int_{E^c} W(t)\chi_E(x)\,dx &= \int_{\mathbf{R}^n} W(t)\chi_E(x)\chi_{E^c}(x)\,dx \\
&= \int_{\mathbf{R}^n} W(t)\chi_E(x)(1-\chi_E(x))\,dx \\
&= \int_{\mathbf{R}^n} W(t)\chi_E(x)\,dx - \int_{\mathbf{R}^n} W(t)\chi_E(x)\chi_E(x)\,dx \\
&= \|W(t)\chi_E\|_{L^1(\mathbf{R}^n)} - \int_{\mathbf{R}^n} W(t/2)\chi_E(x)W(t/2)\chi_E(x)\,dx \\
&= \|\chi_E\|_{L^1(\mathbf{R}^n)} - \|W(t/2)\chi_E\|_{L^2(\mathbf{R}^n)} \\
&\geq \|\chi_B\|_{L^1(\mathbf{R}^n)} - \|W(t/2)\chi_B\|_{L^2(\mathbf{R}^n)} \\
&= \int_{B^c} W(t)\chi_B(x)\,dx,
\end{aligned}$$

whence

$$\int_{E^c} W(t)\chi_E(x)\,dx \geq \int_{B^c} W(t)\chi_B(x)\,dx.$$

Putting all these results together it is easy to prove that (5.5) implies the isoperimetric inequality. Indeed, under properties (5.6)-(5.8), for every $t > 0$

$$\mathcal{P}(E) \geq \sqrt{\frac{\pi}{t}} \int_{E^c} W(t)\chi_E(x)\,dx \geq \sqrt{\frac{\pi}{t}} \int_{B^c} W(t)\chi_B(x)\,dx$$

and as $t \to 0$, $\mathcal{P}(E) \geq \mathcal{P}(B)$.
Notice that the reverse of the Ledoux result is due to the following Riesz-Sobolev inequality (see [28, Theorem 3.7]):

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbf{R}^n \times \mathbf{R}^n} f^*(x)g^*(x-y)h^*(y)dxdy. \tag{5.9}$$

where $f^*$, $g^*$, $h^*$ denote respectively the spherical symmetrization of $f$, $g$, $h$. Now, taking $f = h = \chi_E$ and $g = g^* = G_t(\cdot)$ (where $G_t(z)$ denotes the heat kernel in $\mathbf{R}^n$) in (5.9), so that $f^* = h^* = \chi_B$, the inequality (5.5) follows immediately:

$$\begin{aligned}
\|W(t)\chi_E\|^2_{L^2(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} W(2t)\chi_E(x)\chi_E(x)dx \\
&= \int_{\mathbf{R}^n \times \mathbf{R}^n} G_{2t}(x-y)\chi_E(x)\chi_E(y)dxdy \\
&\leq \int_{\mathbf{R}^n \times \mathbf{R}^n} G_{2t}(x-y)\chi_B(x)\chi_B(y)dxdy \\
&= \int_{\mathbf{R}^n} W(2t)\chi_B(x)\chi_B(x)dx = \|W(t)\chi_B\|^2_{L^2(\mathbf{R}^n)}
\end{aligned}$$

Thus we can state the following equivalence.

**Theorem 5.1.2.** *Let $E, B$ be subset of $\mathbf{R}^n$ with $|E| = |B|$, $B$ an Euclidean ball. Then*

$$\mathcal{P}(E) \geq \mathcal{P}(B) \Longleftrightarrow \|W(t)\chi_E\|_{L^2(\mathbf{R}^n)} \leq \|W(t)\chi_B\|_{L^2(\mathbf{R}^n)} \quad \textit{for all } t \geq 0. \tag{5.10}$$

An immediate interpretation of (5.10) can be deduced by taking into account that in our assumption, (5.5) is equivalent to (5.8) and that $\int_{E^c} W(t)\chi_E(x)\,dx$ measures the amount of heat that is outside the set $E$ at time $t \geq 0$. Therefore (5.10) tells that *among all regular sets of the same volume and at the same initial temperature, the Euclidean ball (having minimum perimeter) is that which minimizes the heat outflow*.
In [33], formula (5.7) has been generalized to all sets of finite perimeter. The proof of such result is based upon the measure-theoretic properties of the reduced boundary. Moreover, in [33] it is also proved that the finiteness of the limit on the left hand side characterizes sets of finite perimeter.
Let us point out that the same characterization of finite perimeter sets is also proved, following a different approach based on the study of behavior of the difference quotient of $u$, in the papers [8], [14], [36], where convolution kernels more general than the Gauss-Weierstrass one are considered. In [33] the following theorem is proved.

**Theorem 5.1.3.** *Let $E, F \subset \mathbf{R}^n$ be sets of finite perimeter. Then the following equality holds:*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_F (\chi_E(x) - W(t)\chi_E(x))\,dx = \int_{\mathcal{F}E\cap\mathcal{F}F} \nu_E(x)\cdot\nu_F(x)d\mathcal{H}^{n-1}(x). \qquad (5.11)$$

PROOF. Since

$$W(t)\chi_E - \chi_E = \int_0^t \Delta W(s)\chi_E ds,$$

we have

$$\int_F (W(t)\chi_E - \chi_E)\,dx = \int_0^t\int_F (\Delta W(s)\chi_E)\,dx\,ds.$$

Moreover, by (4.6), integrating by parts we obtain

$$\begin{aligned}\int_F (\Delta W(s)\chi_E)\,dx &= \int_{\mathbf{R}^n}\Delta W(s)\chi_E(x)\chi_F(x)dx = -\int_{\mathbf{R}^n} D_x W(s)\chi_E(x)\cdot dD\chi_F(x)\\ &= -\int_{\mathcal{F}F} D_x W(s)\chi_E(x)\cdot\nu_F(x)d\mathcal{H}^{n-1}(x).\end{aligned}$$

Notice that, if we define for every $x\in\mathcal{F}E$ and $s>0$ the measures

$$d\mu_{s,x} = \mathcal{L}^n \llcorner \left(\frac{E-x}{\sqrt{s}}\right),$$

and set $z = \dfrac{y-x}{\sqrt{s}}$, we have

$$\begin{aligned}D_x W(s)\chi_E(x) &= \int_E D_x\left(\frac{e^{-\frac{|x-y|^2}{4s}}}{(4\pi s)^{n/2}}\right)dy = -\int_E \frac{(x-y)}{2s}\frac{e^{-\frac{|x-y|^2}{4s}}}{(4\pi s)^{n/2}}dy\\ &= \frac{1}{2\sqrt{s}}\int_{\frac{E-x}{\sqrt{s}}}\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}zdz\\ &= \frac{1}{2\sqrt{s}}\int_{\mathbf{R}^n}\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}zd\mu_{s,x}(z).\end{aligned}$$

Moreover, setting, for every $x\in\mathcal{F}E$,

$$H_{\nu_E(x)} = \{z\in\mathbf{R}^n : z\cdot\nu_E(x)\geq 0\},$$

the existence of the approximate tangent plane for $x\in\mathcal{F}E$, see (4.1.5), implies that the measures $\mu_{s,x}$ are locally weakly* convergent as $s\to 0$ to the measure

$$\mu_x = \begin{cases} 0 & \text{if } x\in E^0,\\ \mathcal{L}^n & \text{if } x\in E^1,\\ \mathcal{L}^n\llcorner H_{\nu_E(x)} & \text{if } x\in\mathcal{F}E.\end{cases}$$

Moreover for every $\varepsilon>0$ we can find a compact set $K\subset\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n\setminus K} z\cdot\nu_F(x)\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}d\mu_{s,x}(z) < \varepsilon, \quad \int_{\mathbf{R}^n\setminus K} z\cdot\nu_F(x)\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}d\mu_x(z) < \varepsilon$$

hence, since $\mu_{s,x}$ are locally weakly* convergent as $s \to 0$ to $\mu_x$

$$\lim_{s\to 0} \int_{\mathbf{R}^n} z \cdot \nu_F(x) \frac{e^{-|z|^2/4}}{(4\pi)^{n/2}} d\mu_{s,x}(z) = \int_{\mathbf{R}^n} z \cdot \nu_F(x) \frac{e^{-|z|^2/4}}{(4\pi)^{n/2}} d\mu_x. \tag{5.12}$$

Summing up, we can write

$$\sqrt{\frac{\pi}{t}} \int_F (\chi_E - W(t)\chi_E)\, dx = \frac{\sqrt{\pi}}{(4\pi)^{n/2}} \int_{\mathcal{F}F} \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2\sqrt{s}} g(x,s)\; ds\; d\mathcal{H}^{n-1}(x)\,, \tag{5.13}$$

where $g : \mathcal{F}F \times (0,t)$ is given by

$$g(x,s) = \int_{\mathbf{R}^n} e^{-|z|^2/4} z \cdot \nu_F(x) d\mu_{s,x}(z),$$

and by (5.12) we have

$$\lim_{s\to 0^+} g(x,s) = \begin{cases} \displaystyle\int_{H_{\nu_E(x)}} z \cdot \nu_F(x) e^{-|z|^2/4} dz & \text{for } x \in \mathcal{F}E \cap \mathcal{F}F \\ 0 & \text{for } x \in \left(E^0 \cup E^1\right) \cap \mathcal{F}F, \end{cases}$$

where $E^0,\ E^1$ are defined according to (4.8). This implies that for all $\varepsilon > 0$ there exists $t_0 > 0$ such that if $t < t_0$ and $x \in \left(E^0 \cup E^1\right) \cap \mathcal{F}F$, then

$$\left| \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2\sqrt{s}} g(x,s)\; ds \right| \le \frac{1}{\sqrt{t}} \int_0^t \frac{\varepsilon}{\sqrt{s}}\, ds = 2\varepsilon.$$

Now, by Theorem 4.1.7, we have that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$, then the right hand side of (5.13) reduces to the integral on $\mathcal{F}E \cap \mathcal{F}F$ and we obtain that there exists

$$\begin{aligned} &\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_F (\chi_E - W(t)\chi_E)\, dx = \frac{\sqrt{\pi}}{(4\pi)^{n/2}} \int_{\mathcal{F}E\cap\mathcal{F}F} \int_{H_{\nu_E(x)}} z \cdot \nu_F(x) e^{-|z|^2/4} dz d\mathcal{H}^{n-1}(x) \\ &= \frac{\sqrt{\pi}}{(4\pi)^{n/2}} \int_{\mathcal{F}E\cap\mathcal{F}F} \int_{H_{\nu_E(x)}} (\nu_E(x)\cdot\nu_F(x))(z\cdot\nu_E(x)) e^{-|z|^2/4} dz d\mathcal{H}^{n-1}(x) \\ &= \int_{\mathcal{F}E\cap\mathcal{F}F} \nu_E(x)\cdot\nu_F(x) d\mathcal{H}^{n-1}(x), \end{aligned}$$

because $\nu_F(x) = (\nu_E(x)\cdot\nu_F(x))\nu_E(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ and

$$\int_{H_{\nu_E(x)}} z \cdot \nu_E(x) e^{-|z|^2/4} dz = 2(4\pi)^{(n-1)/2} \qquad \forall\, x \in \mathcal{F}E.$$

□

**Remark 5.1.4.** Notice that if $|F \backslash E| = 0$ in the preceding statement, then $\nu_E(x) = \nu_F(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$, hence the equality

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_F (\chi_E - W(t)\chi_E)\, dx = \mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{F}F) \tag{5.14}$$

holds.

As a special case, we may take $E = F$ in the above theorem, and obtain the following result, which generalizes formula (5.7).

**Theorem 5.1.5.** *Let $E \subset \mathbf{R}^n$ be a set of finite perimeter; then the following equality holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{E^c} W(t)\chi_E\,dx = \mathcal{P}(E). \tag{5.15}$$

PROOF. Since $\|W(t)\chi_E\|_{L^1(\mathbf{R}^n)} = |E|$ for all $t \geq 0$, we obtain

$$\int_E (\chi_E - W(t)\chi_E)\,dx = \int_{\mathbf{R}^n}(\chi_E - W(t)\chi_E)(1-\chi_{E^c})\,dx = \int_{E^c} W(t)\chi_E\,dx$$

and the assertion follows inserting $F = E$ in (5.14). □

A sort of reverse implication is also stated.

**Theorem 5.1.6.** *Let $E \subset \mathbf{R}^n$ be a set such that either $E$ or $E^c$ has finite measure, and*

$$\liminf_{t\to 0^+}\frac{1}{\sqrt{t}}\int_{E^c} W(t)\chi_E\,dx < +\infty.$$

*Then $E$ has finite perimeter.*

PROOF. Assume that $|E| < +\infty$. We can write

$$\begin{aligned}
\frac{1}{\sqrt{t}}\langle W(t)\chi_E, \chi_{E^c}\rangle &= \frac{1}{(4\pi)^{n/2}\sqrt{t}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\chi_{E^c}(x)\chi_E(x+\sqrt{t}y)e^{-|y|^2/4}dydx \\
&= \frac{1}{(4\pi)^{n/2}\sqrt{t}}\int_{\mathbf{R}^n} e^{-|y|^2/4}\int_{\mathbf{R}^n}\Big(\chi_{E-\sqrt{t}y}(x) - \chi_E(x)\chi_{E-\sqrt{t}y}(x)\Big)\,dxdy \\
&= \frac{1}{(4\pi)^{n/2}\sqrt{t}}\int_{\mathbf{R}^n} e^{-|y|^2/4}\Big(|E| - |E\cap(E-\sqrt{t}y)|\Big)\,dy \\
&= \frac{1}{2(4\pi)^{n/2}\sqrt{t}}\int_{\mathbf{R}^n} e^{-|y|^2/4}|E\triangle(E-\sqrt{t}y)|dy \\
&= \frac{1}{2(4\pi)^{n/2}}\int_{\mathbf{R}^n}|y|e^{-|y|^2/4}\frac{|E\triangle(E-\sqrt{t}y)|}{\sqrt{t}|y|}dy,
\end{aligned}$$

where $E\triangle F = (E\cup F)\setminus(E\cap F)$ . Then, if we define

$$|D_\nu\chi_E| = \liminf_{t\to 0^+}\frac{|E\triangle(E-t\nu)|}{t},$$

from the previous estimate we get that

$$\int_{\mathbf{R}^n}|y|e^{-|y|^2/4}|D_{y/|y|}\chi_E|dy \leq \liminf_{t\to 0^+}\int_{\mathbf{R}^n}|y|e^{-|y|^2/4}\frac{|E\triangle(E-\sqrt{t}y)|}{\sqrt{t}|y|}dy < +\infty.$$

Noticing that

$$\int_{\mathbf{R}^n}|y|e^{-|y|^2/4}|D_{y/|y|}\chi_E|dy = C_n\int_{\mathbf{S}^{n-1}}|D_\nu\chi_E|d\nu,$$

we have proved that

$$\int_{\mathbf{S}^{n-1}} |D_\nu \chi_E| d\nu < +\infty.$$

This implies that the function $\nu \mapsto |D_\nu \chi_E|$ is finite for a.e. $\nu \in \mathbf{S}^{n-1}$; in particular, there exist $M > 0$ and an orthonormal system of coordinates $\nu_1, \ldots, \nu_n$ of Lebesgue points of $|D_\nu \chi_E|$ such that

$$|D_{\nu_i} \chi_E| \leq M, \quad \forall i = 1, \ldots, n.$$

Without loss of generality, we can assume that $\nu_i = e_i$; now, if $\phi \in C_c^1(\mathbf{R}^n)$, the function

$$\phi_t(x) = \frac{\phi(x + te_i) - \phi(x)}{t}$$

is uniformly convergent to $\partial_i \phi(x)$. This implies that

$$\int_{\mathbf{R}^n} \chi_E(x) \partial_i \phi(x) dx = \lim_{t \to 0^+} \int_{\mathbf{R}^n} \chi_E(x) \phi_t(x) dx.$$

But

$$\int_{\mathbf{R}^n} \chi_E(x) \phi_t(x) dx = \int_{\mathbf{R}^n} \frac{\chi_E(x - te_i) - \chi_E(x)}{t} \phi(x) dx,$$

hence

$$\left| \int_{\mathbf{R}^n} \chi_E(x) \phi_t(x) dx \right| \leq \|\phi\|_\infty \frac{|E \triangle (E + te_i)|}{t}.$$

From this it follows that

$$\begin{aligned} \left| \int_{\mathbf{R}^n} \chi_E(x) \partial_i \phi(x) dx \right| &\leq \|\phi\|_\infty \liminf_{t \to 0^+} \frac{|E \triangle (E + te_i)|}{t} \\ &= \|\phi\|_\infty |D_i \chi_E| \leq M \|\phi\|_\infty. \end{aligned}$$

In the end, we have proved that

$$\int_{\mathbf{R}^n} \chi_E(x) \operatorname{div} \phi(x) dx \leq nM \|\phi\|_\infty, \quad \forall \phi \in C_c^1(\mathbf{R}^n),$$

and then $\chi_E \in BV(\mathbf{R}^n)$. □

In connection with these results, it seems to be interesting to pursue the investigation of the relationships between the perimeter of a set in a domain and the short-time behavior of the semigroup $T(t)$ generated by a more general operator like $(A_1, D(A_1))$.

**Remark 5.1.7.** In what follows Gaussian upper and lower bounds of the fundamental solution associated with the operator $\partial_t - \mathcal{A}$ are of relevant importance. They can be found in Appendix B and are used in a form neglecting $e^{\omega t}$. This is not important for our computations since we are interested in the behavior of $T(t)$ for small $t$, see Remark B.2.1.

## 5.2 Preliminary results for problems in a domain

For every $s > 0$ and $x_0 \in \Omega$, we set

$$\Omega^{s,x_0} = \frac{\Omega - x_0}{\sqrt{s}} = \left\{ y \in \mathbf{R}^n : x_0 + \sqrt{s} y \in \Omega \right\}$$

and, given $f : \Omega \to \mathbf{R}$, $f^{s,x_0}(y) = f(x_0 + \sqrt{s}y)$. With this notation, we define the operator $\mathcal{A}^{s,x_0}$ on $\Omega^{s,x_0}$ by

$$\begin{aligned}
\mathcal{A}^{s,x_0}(y)v(y) &= \operatorname{div}(A^{s,x_0}(y)Dv(y)) + \sqrt{s}\langle B^{s,x_0}(y), Dv(y)\rangle + sc^{s,x_0}(y)v(y) \\
&= \sum_{h,k=1}^{n} a_{hk}(x_0 + \sqrt{s}y)\frac{\partial^2 v}{\partial y^h \partial y^k}(y) + \sqrt{s}\sum_{k=1}^{n}\left(\sum_{h=1}^{n} D_h a_{hk}(x_0 + \sqrt{s}y)\right)\frac{\partial v}{\partial y^k}(y) \\
&\quad + \sqrt{s}\sum_{h=1}^{n} b_h(x_0 + \sqrt{s}y)\frac{\partial v}{\partial y^h}(y) + sc(x_0 + \sqrt{s}y)v(y),
\end{aligned}$$

and the operator $\mathcal{A}^x$ on $\mathbf{R}^n$ by

$$\mathcal{A}^x v(y) = \sum_{h,k=1}^{n} a_{hk}(x)\frac{\partial^2 v}{\partial y^h \partial y^k}(y).$$

By setting $x = x_0 + \sqrt{s}y$, it is easily seen that $\mathcal{A}^{s,x_0}(y) = s\mathcal{A}(x)$. We have the following lemma.

**Lemma 5.2.1.** *Setting $u(t,x) = T(t)u_0(x)$, we can define the function $v : (0,+\infty) \times \Omega^{s,x_0} \to \mathbf{R}$ by $v(t,y) = u(ts, x_0 + \sqrt{s}y)$; then $v$ is the solution of the problem*

$$\begin{cases} \partial_t w = \mathcal{A}^{s,x_0}(y)w & \text{in } (0,+\infty) \times \Omega^{s,x_0} \\ w(0,y) = u_0^{s,x_0}(y) & \text{in } \Omega^{s,x_0} \\ \langle A^{s,x_0} Dw, \nu\rangle = 0 & \text{in } (0,+\infty) \times \partial\Omega^{s,x_0}. \end{cases} \tag{5.16}$$

Proof. By definition, we have $v(0,y) = u(0, x_0 + \sqrt{s}y) = u_0(x_0 + \sqrt{s}y) = u_0^{s,x_0}(y)$. Moreover, if we set $x = x_0 + \sqrt{s}y$, we have that $\partial/\partial y^h = \sqrt{s}\partial/\partial x^h$ and also that the unit outward normal to $\partial\Omega^{s,x_0}$ at $y$ coincides with the unit outward normal to $\partial\Omega$ at $x$; therefore,

$$\langle A^{s,x_0}(y)D_y v(t,y), \nu(y)\rangle = \sqrt{s}\langle A(x)D_x u(ts,x), \nu(x)\rangle = 0.$$

In the same way, we have

$$\partial_t v(t,y) = su'(ts, x_0 + \sqrt{s}y) = su'(ts,x) = s\mathcal{A}(x)u(ts,x) = \mathcal{A}^{s,x_0}(y)v(t,y),$$

where $u'$ denotes the derivative of $u$ with respect to its first variable, and this concludes the proof. □In order to follow the computations in Section 5.1, based on the Gauss-Weierstrass kernel $G$ we recall that the semigroup generated by $\mathcal{A}$, $\mathcal{A}^{s,x}$, $\mathcal{A}^x$, are represented through an integral kernel that will be introduced with a coherent notation (see e.g. [45]). We also denote by $(T^{s,x_0}(t))_{t\geq 0}$ the semigroup associated with problem (5.16) and by $p^{s,x_0}(t,y,z)$ its kernel. We also denote by $(T^{x_0}(t))_{t\geq 0}$ the semigroup associated with the problem

$$\begin{cases} \partial_t w(t,y) = \mathcal{A}^{x_0}(y)w(t,y) & \text{in } (0,+\infty)\times \mathbf{R}^n \\ w(0,y) = w_0(y) & \text{in } \mathbf{R}^n \end{cases}$$

and by $p^{x_0}(t,y,z)$ its kernel.

**Lemma 5.2.2.** *For the kernels the following holds*

$$p(t,x,y) = s^{-n/2} p^{s,x_0}\Big(\frac{t}{s}, \frac{x-x_0}{\sqrt{s}}, \frac{y-x_0}{\sqrt{s}}\Big). \tag{5.17}$$

Proof. The proof of Lemma 5.2.1 gives that $v(t,y) = T^{s,x_0}(t)u_0^{s,x_0}(y) = T(ts)u_0(x_0+\sqrt{s}y)$; using the kernels, we get that

$$\begin{aligned} \int_\Omega p(t,x,y)u_0(y)dy &= T(t)u_0(x) = T^{s,x_0}\left(\frac{t}{s}\right)u_0^{s,x_0}\left(\frac{x-x_0}{\sqrt{s}}\right) \\ &= \int_{\Omega^{s,x_0}} p^{s,x_0}\left(\frac{t}{s}, \frac{x-x_0}{\sqrt{s}}, z\right) u_0(x_0+\sqrt{s}z)dz \\ &= s^{-n/2}\int_\Omega p^{s,x_0}\left(\frac{t}{s}, \frac{x-x_0}{\sqrt{s}}, \frac{y-x_0}{\sqrt{s}}\right) u_0(y)dy. \end{aligned}$$

The arbitrarity of $u_0$ gives the thesis. $\square$

We have the following result.

**Proposition 5.2.3.** *For every $f\in L^1(\mathbf{R}^n)$, let $u^{s,x}(t,\xi)$ be the solution of the problem*

$$\begin{cases} \partial_t w(t,\xi) = \mathcal{A}^{s,x}(\xi)w(t,\xi) & \text{in } (0,+\infty)\times\Omega^{s,x} \\ \langle A(x+\sqrt{s}\xi)Dw(t,\xi), \nu_{\Omega^{s,x}}(\xi)\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{s,x} \\ w(0,\xi) = f(\xi) & \text{in } \Omega^{s,x} \end{cases}$$

*and let $u^x(t,\xi)$ be the solution of the problem*

$$\begin{cases} \partial_t w(t,\xi) = \mathcal{A}^{x}(\xi)w(t,\xi) & \text{in } (0,+\infty)\times\mathbf{R}^n \\ w(0,\xi) = f(\xi) & \text{in } \mathbf{R}^n \end{cases}.$$

*Then for every $t>0$ we have that $u^{s,x}(t,\cdot)$ converges to $u^x(t,\cdot)$ in $L^1_{\text{loc}}(\mathbf{R}^n)$ as $s\to 0$.*

Proof. We start by taking $f\in C_c(\mathbf{R}^n)$ and denoting by $u^{s,x}(t,\xi)$ the solution of the problem

$$\begin{cases} \partial_t w(t,\xi) = \mathcal{A}^{x}(\xi)w(t,\xi) & \text{in } (0,+\infty)\times\Omega^{s,x} \\ \langle A^{s,x}(\xi)D_\xi w(t,\xi), \nu(\xi)\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{s,x} \\ w(0,\xi) = f(\xi) & \text{in } \Omega^{s,x}. \end{cases} \tag{5.18}$$

Since $u^{s,x}$ is a classical solution, for every regular function $\varphi : [0,t_0]\times\mathbf{R}^n\to\mathbf{R}$ with $\varphi(t_0,\cdot)=0$, the following holds:

$$\begin{aligned} -\int_{\Omega^{s,x}} f(\xi)\varphi(0,\xi)d\xi \;=\; & \int_0^{t_0}\int_{\Omega^{s,x}} \Big\{u^{s,x}(t,\xi)\left(\partial_t\varphi(t,\xi) + sc^{s,x}(\xi)\right) \\ & + \frac{\partial u^{s,x}(t,\xi)}{\partial\xi^k}\Big[-a^{s,x}_{hk}(\xi)\frac{\partial\varphi(t,\xi)}{\partial\xi^h} + \sqrt{s}\varphi(t,\xi)b^{s,x}_k(\xi)\Big]\Big\}d\xi dt. \end{aligned} \tag{5.19}$$

Moreover, notice that $sc^{s,x} \to 0$, $a^{s,x}_{hk} \to a_{hk}(x)$, $\sqrt{s}b^{s,x}_k \to 0$ uniformly on compact sets as $s \to 0$.

As an auxiliary tool, let us use the $L^2$ theory, see e.g. [45, Section 5.4], recalling that there is $M > 0$ independent of $s \in [0,1]$, such that

$$\|u^{s,x}(t)\|_{L^2(\Omega^{s,x})} \le M\|f\|_{L^2(\Omega^{s,x})} \le M\|f\|_{L^2(\mathbf{R}^n)}, \tag{5.20}$$

$$\|Du^{s,x}(t)\|_{L^2(\Omega^{s,x})} \le \frac{M}{\sqrt{t}}\|f\|_{L^2(\Omega^{s,x})} \le \frac{M}{\sqrt{t}}\|f\|_{L^2(\mathbf{R}^n)}, \tag{5.21}$$

and

$$\|D^2u^{s,x}(t)\|_{L^2(\Omega^{s,x})} \le \frac{M}{t}\|f\|_{L^2(\Omega^{s,x})} \le \frac{M}{t}\|f\|_{L^2(\mathbf{R}^n)}. \tag{5.22}$$

These conditions imply that for every bounded open set $A \subset \mathbf{R}^n$, $t > 0$ fixed and $s_0$ small enough, the family $(u^{s,x}(t,\cdot))_{0<s<s_0}$ is bounded in $W^{2,2}(A)$, and then, up to subsequences, it is strongly convergent in $W^{1,2}(A)$ and also in $W^{1,1}(A)$.

We can now fix a countable dense set $D \subset [0,t_0]$ in such a way that $u^{s_h,x}(t,\cdot)$ converges to some $g(t,\cdot)$ in $W^{1,1}(A)$ for every $t \in D$ and some sequence $s_h \to 0$. By (3.2) we get that

$$\begin{aligned}\|u^{s,x}(t_2,\cdot) - u^{s,x}(t_1,\cdot)\|_{L^1(\Omega^{s,x})} &= \Big\|\int_{t_1}^{t_2} \partial_t u^{s,x}(t,\cdot)dt\Big\|_{L^1(\Omega^{s,x})} \\ \le \int_{t_1}^{t_2} \|A^{s,x}u^{s,x}(t,\cdot)\|_{L^1(\Omega^{s,x})}dt &\le c_1\|f\|_{L^1(\Omega^{s,x})}\int_{t_1}^{t_2}\frac{1}{t}dt \le c_1\|f\|_{L^1(\mathbf{R}^n)}\log\frac{t_2}{t_1},\end{aligned}$$

that is, the function $t \mapsto u^{s,x}(t,\cdot)$ is continuous from $(0,t_0)$ to $L^1(\Omega^{s,x})$; in particular, if we consider $t_1, t_2 \in D$, then the inequality

$$\begin{aligned}\|g(t_2,\cdot) - g(t_1,\cdot)\|_{L^1(A)} &\le \|g(t_2,\cdot) - u^{s_h,x}(t_2,\cdot)\|_{L^1(A)} \\ &+ \|u^{s_h,x}(t_2,\cdot) - u^{s_h,x}(t_1,\cdot)\|_{L^1(A)} + \|u^{s_h,x}(t_1,\cdot) - g(t_1,\cdot)\|_{L^1(A)}\end{aligned}$$

holds and the convergence of $u^{s,x}$ on $D$ shows that we can extend $g$ to a continuous map from $(0,t_0)$ to $L^1_{\rm loc}(\mathbf{R}^n)$; we also notice that by (3.4) we deduce also that $g(t,\cdot) \in W^{1,1}(A)$ for every $t \in (0,t_0)$. By continuity, and by the convergence of $u^{s_h,x}(t,\cdot)$ on $D$ we deduce that $u^{s_h,x}(t,\cdot) \to g(t,\cdot)$ in $L^1_{\rm loc}(\mathbf{R}^n)$ for every $t \in (0,t_0)$. In addition, conditions (3.1) allow us to apply the dominated convergence theorem, and then, taking the limit in (5.19), we get

$$-\int_A f(\xi)\varphi(0,\xi)d\xi = \int_0^{t_0}\int_A \Big(g(t,\xi)\partial_t\varphi(t,\xi) - \langle A(x)D_\xi\varphi(t,\xi), D_\xi g(t,\xi)\rangle\Big)d\xi dt$$

for all $\varphi$ as above, and then (see e.g. [40, Prop. 2.1, Ch. III]) $g(t,\cdot)$ is the solution of the problem

$$\begin{cases} \partial_t w(t,\xi) = a_{hk}(x)\frac{\partial^2 w}{\partial\xi^h\partial\xi^k}(t,\xi) & \text{in } (0,t_0)\times\mathbf{R}^n \\ w(0,\xi) = f(\xi) & \text{in } \mathbf{R}^n \end{cases}$$

for every $f \in C_c(\mathbf{R}^n)$. Then, it follows that

$$g(t,\xi) = u^x(t,\xi) = \int_{\mathbf{R}^n} p^x(t,\xi,z)f(z)dz,$$

where using the Fourier transform the kernel $p^x$ is given by

$$p^x(t,\xi,z) = \frac{1}{(4\pi t)^{n/2}|\det A^{1/2}(x)|}\exp\left(-\frac{\langle A^{-1}(x)(\xi-z),(\xi-z)\rangle}{4t}\right). \tag{5.23}$$

By the density of $C_c$ in $L^1$ we conclude. □

The following statement is an immediate consequence of Proposition 5.2.3.

**Corollary 5.2.4.** *For every $t>0$ and a.e. $\xi\in\mathbf{R}^n$, the family of measures $d\mu^{s,x} = p^{s,x}(t,\xi,\cdot)d\mathcal{L}^n \llcorner \Omega^{s,x}$ is weakly* convergent to the measure $d\mu^x = p^x(t,\xi,\cdot)d\mathcal{L}^n$ as $s\to 0$, that is, for every $\varphi\in C_c(\mathbf{R}^n)$ the following equality holds*

$$\lim_{s\to 0}\int_{\Omega^{s,x}}\varphi(z)p^{s,x}(t,\xi,z)dz = \int_{\mathbf{R}^n}\varphi(z)p^x(t,\xi,z)dz.$$

Henceforth, given the function $p(t,\xi,z)$, we shall denote by $D_1p(t,\xi,z)$ the gradient with respect to the first spatial variables $\xi$ and by $D_2p(t,\xi,z)$ the gradient with respect to the second spatial variables $z$.

**Proposition 5.2.5.** *For every $t>0$ and a.e. $\xi\in\mathbf{R}^n$, the equality*

$$\lim_{s\to 0}\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz = \int_{\mathbf{R}^n}\langle D_2p^x(t,\xi,z),\varphi(z)\rangle dz \tag{5.24}$$

*holds for every $\varphi\in L^\infty(\mathbf{R}^n,\mathbf{R}^n)$.*

Proof. We start by considering $\varphi\in C^1_c(\mathbf{R}^n,\mathbf{R}^n)$; we choose $s_0>0$ in such a way that $\operatorname{supp}\varphi\subset\Omega^{s,x}$ for all $s\le s_0$; then

$$\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz = -\int_{\Omega^{s,x}}p^{s,x}(t,\xi,z)\mathrm{div}\varphi(z)dz$$

and then, by Corollary 5.2.4

$$\begin{aligned}\lim_{s\to 0}\int_{\Omega^{s,x}}\langle D_2p^{s,x}(,\xi,z),\varphi(z)\rangle dz &= \lim_{s\to 0}-\int_{\Omega^{s,x}}p^{s,x}(t,\xi,z)\mathrm{div}\varphi(z)dz\\ &= -\int_{\mathbf{R}^n}p^x(t,\xi,z)\mathrm{div}\varphi(z)dz = \int_{\mathbf{R}^n}\langle D_2p^x(t,\xi,z),\varphi(z)\rangle dz.\end{aligned}$$

For an arbitrary $\varphi\in L^\infty(\mathbf{R}^n,\mathbf{R}^n)$ we use an approximation procedure.
First of all recall that for every $\varepsilon>0$ we can find $R>0$ and $s_0>0$ such that

$$\int_{\Omega^{s,x}\setminus B_R(0)}|D_2p^{s,x}(t,\xi,z)|\,dz\le\varepsilon,\qquad \int_{\mathbf{R}^n\setminus B_R(0)}|D_2p^x(t,\xi,z)|\,dz\le\varepsilon.$$

for all $s\le s_0$. Now, let $\eta\in C^\infty_c(\mathbf{R}^n)$, $0\le\eta\le 1$, $\eta=1$ in $B_R(0)$ and $\eta=0$ in $\mathbf{R}^n\setminus B_{2R}(0)$, and select $\varepsilon<R/2$. Then $\varphi_\varepsilon=\rho_\varepsilon*(\eta\varphi)\in C^1_c(\mathbf{R}^n,\mathbf{R}^n)$ such that $\|\varphi-\varphi_\varepsilon\|_{L^p(B_R(0))}\le\varepsilon$ for all $1\le p<\infty$ and then

$$\begin{aligned}\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz &= \int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi_\varepsilon(z)\rangle dz\\ &\quad+\int_{\Omega^{s,x}\cap B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\\ &\quad+\int_{\Omega^{s,x}\setminus B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz.\end{aligned}$$

Taking into account that $p^{s,x}(t,\xi,z) = s^{n/2}p(ts, x+\sqrt{s}\xi, x+\sqrt{s}z)$ and also that

$$\begin{aligned} D_2 p^{s,x}(t,\xi,z) &= D_z s^{n/2} p(ts, x+\sqrt{s}\xi, x+\sqrt{s}z) \\ &= s^{(n+1)/2} D_2 p(ts, x+\sqrt{s}\xi, x+\sqrt{s}z), \end{aligned}$$

by (B.2) we obtain

$$\begin{aligned} &\Big| \int_{\Omega^{s,x}\cap B_R(0)} \langle D_2 p^{s,x}(t,\xi,z), (\varphi(z)-\varphi_\varepsilon(z))\rangle dz \Big| \\ &\qquad \le s^{(n+1)/2} \|\varphi - \varphi_\varepsilon\|_{L^2(B_R)} \Big( \int_{\Omega^{s,x}} |D_2 p(ts, x+\sqrt{s}\xi, x+\sqrt{s}z)|^2 dz \Big)^{1/2} \le C\varepsilon \end{aligned}$$

with $C$ independent of $s$. Of course, the inequality

$$\Big| \int_{B_R(0)} \langle D_2 p^x(t,\xi,z), (\varphi(z)-\varphi_\varepsilon(z))\rangle dz \Big| \le C\varepsilon$$

holds as well, and then

$$\begin{aligned} &\lim_{s\to 0} \Big| \int_{\Omega^{s,x}} \langle D_2 p^{s,x}(t,\xi,z), \varphi(z)\rangle dz - \int_{\mathbf{R}^n} \langle D_2 p^x(t,\xi,z), \varphi(z)\rangle dz \Big| \\ &\le \lim_{s\to 0} \Big| \int_{\Omega^{s,x}\cap B_R(0)} \langle D_2 p^{s,x}(t,\xi,z), (\varphi(z)-\varphi_\varepsilon(z))\rangle dz \Big| \\ &\quad + \lim_{s\to 0} \Big| \int_{\Omega^{s,x}\setminus B_R(0)} \langle D_2 p^{s,x}(t,\xi,z), (\varphi(z)-\varphi_\varepsilon(z))\rangle dz \Big| \\ &\quad + \lim_{s\to 0} \Big| \int_{\Omega^{s,x}} \langle D_2 p^{s,x}(t,\xi,z), \varphi_\varepsilon(z)\rangle dz - \int_{\mathbf{R}^n} \langle D_2 p^x(t,\xi,z), \varphi_\varepsilon(z) dz\rangle \Big| \\ &\quad + \lim_{s\to 0} \Big| \int_{B_R(0)} \langle D_2 p^x(t,\xi,z), (\varphi(z)-\varphi_\varepsilon(z))\rangle dz \Big| \\ &\quad + \lim_{s\to 0} \Big| \int_{\mathbf{R}^n\setminus B_R(0)} \langle D_2 p^x(t,\xi,z), (\varphi(z)-\varphi_\varepsilon(z))\rangle dz \Big| \le C\varepsilon \end{aligned}$$

and the thesis follows from the arbitrariness of $\varepsilon$. □

## 5.3 A second characterization of $BV$ functions

The main step in the proof of (5.4) is the following result, where an asymptotic formula relating two sets of finite perimeter is shown. In the statement, we assume that $E$ has finite measure in order to give a meaning to the left hand side in (5.27) below. But, notice that, since $E$ has finite perimeter in $\Omega$, then by the relative isoperimetric inequality in the regular set $\Omega$

$$\min\{|E\cap\Omega|, |\Omega\setminus E|\} \le k\mathcal{P}(E,\Omega)^{n/n-1},$$

either $|E\cap\Omega|$ or $|\Omega\setminus E|$ is finite. Therefore, if $|E\cap\Omega|$ is infinite, then $|\Omega\setminus E|$ is finite and (5.27) applies with $\Omega\setminus E$ in place of $E$.

**Proposition 5.3.1.** *Assume that $\Omega$ be as in (2.2). Let $\mathcal{B}$ be as in (2.5), and consider $\mathcal{A}_0 = \mathrm{div}(AD)$, with $A = (a_{ij})_{ij}$ satisfying (2.107)–(2.108); let $(T_0(t))_{t\geq 0}$ be the semigroup generated by the realization of $\mathcal{A}_0$ in $L^1(\Omega)$ with homogeneous boundary condition $\mathcal{B}u = 0$; then, if $E, F \subset \mathbf{R}^n$ are sets of finite perimeter in $\Omega$, the following holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap F}(\chi_E(x) - T_0(t)\chi_E(x))dx = \int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E}\langle A(x)\nu_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x).$$

Proof. We have

$$\begin{aligned}\int_{\Omega\cap F}(T_0(t)\chi_E(x) - \chi_E(x))dx &= \int_{\Omega\cap F}\int_0^t \frac{d}{ds}T_0(s)\chi_E(x)dsdx\\ &= \int_0^t\int_{\Omega\cap F}\mathcal{A}_0T_0(s)\chi_E(x)dxds\\ &= \int_0^t\int_{\Omega\cap F}\mathrm{div}_x(A(x)D_xT_0(s)\chi_E(x))dxds\end{aligned}$$

We introduce now the kernel $p_{0,*}(t,x,y)$ of the semigroup generated by the adjoint operator $\mathcal{A}_0^*$ of $\mathcal{A}_0$; by the symmetry of the matrix $A$, the operator $\mathcal{A}_0^* = \mathcal{A}_0$. In this way we have that $p_0(t,x,y) = p_{0,*}(t,y,x)$ (see for instance [45, Theorem 5.6]) and since

$$\begin{aligned}D_{x_i}p_0(t,x,y) &= \lim_{h\to 0}\frac{p_0(t,x+he_i,y) - p_0(t,x,y)}{h} = \lim_{h\to 0}\frac{p_{0,*}(t,y,x+he_i) - p_{0,*}(t,y,x)}{h}\\ &= s^{-n/2}\lim_{h\to 0}\frac{p_{0,*}^{s,x_0}(\frac{t}{s},\frac{y-x_0}{\sqrt{s}},\frac{x-x_0}{\sqrt{s}}+\frac{he_i}{\sqrt{s}}) - p_{0,*}^{s,x_0}(\frac{t}{s},\frac{y-x_0}{\sqrt{s}},\frac{x-x_0}{\sqrt{s}})}{h}\\ &= s^{-(n+1)/2}D_2^ip_{0,*}^{s,x_0}\left(\frac{t}{s},\frac{y-x_0}{\sqrt{s}},\frac{x-x_0}{\sqrt{s}}\right)\end{aligned}$$

where $D_2^i$ denotes the $i$-th component of the gradient with respect to the second variables. Then for $t = s$ and $x = x_0$, $D_xp_0(t,x,y) = t^{-(n+1)/2}D_2p_{0,*}^{t,x}(1,\frac{y-x}{\sqrt{t}},0)$; hence integrating by parts we get

$$\begin{aligned}\int_{\Omega\cap F}\mathrm{div}(AD_xT_0(s)\chi_E(x))dx &= \int_{\Omega\cap\mathcal{F}F}\langle D_xT_0(s)\chi_E(x), A(x)\nu_F(x)\rangle d\mathcal{H}^{n-1}(x)\\ &= \int_{\Omega\cap\mathcal{F}F}\int_{\Omega\cap E}\langle D_xp_0(s,x,y), A(x)\nu_F(x)\rangle\, dy\, d\mathcal{H}^{n-1}(x)\\ &= \int_{\Omega\cap\mathcal{F}F}\int_{\Omega\cap E}s^{-(n+1)/2}\left\langle D_2p_{0,*}^{s,x}\left(1,\frac{y-x}{\sqrt{s}},0\right), A(x)\nu_F(x)\right\rangle dyd\mathcal{H}^{n-1}(x)\\ &= -\frac{1}{\sqrt{s}}\int_{\Omega\cap\mathcal{F}F}\int_{\Omega^{s,x}\cap E^{s,x}}\langle D_2p_{0,*}^{s,x}(1,z,0), A(x)\nu_F(x)\rangle dzd\mathcal{H}^{n-1}(x)\\ &= -\frac{1}{\sqrt{s}}\int_{\Omega\cap\mathcal{F}F}\int_{\mathbf{R}^n}\langle D_2p_{0,*}^{s,x}(1,z,0), A(x)\nu_F(x)\rangle d\mu^{s,x}(z)d\mathcal{H}^{n-1}(x), \qquad (5.25)\end{aligned}$$

where we have denoted by $\mu^{s,x}$ the measure

$$\mu^{s,x} = \mathcal{L}^n \llcorner (\Omega^{s,x}\cap E^{s,x})\,. \qquad (5.26)$$

These measures verify the following properties:

1. $\mu^{s,x} \xrightarrow{w^*_{\rm loc}} 0$ if $x \in E^0$:
2. $\mu^{s,x} \xrightarrow{w^*_{\rm loc}} \mathcal{L}^n$ if $x \in E^1$;
3. $\mu^{s,x} \xrightarrow{w^*_{\rm loc}} \mathcal{L}^n \llcorner H_{\nu_E(x)}$ for $x \in \mathcal{F}E$, where $H_{\nu_E(x)} = \{z \in \mathbf{R}^n : \langle z, \nu_E(x)\rangle \leq 0\}$.

These facts imply that, for $x \in E^0$, $\displaystyle\int_{\mathbf{R}^n} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle d\mu^{s,x}(z) \to 0$; indeed

$$\begin{aligned}\Big| \int_{\mathbf{R}^n} &\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\rangle \, d\mu^{s,x}(z)\Big| \\ &= s^{(n+1)/2}\Big| \int_{\mathbf{R}^n} \langle D_x p_0(s,x,z\sqrt{s}+x), A(x)\nu_F(x)\rangle \, d\mu^{s,x}(z)\Big| \\ &\leq c_1\|A\|_\infty \int_{\mathbf{R}^n} e^{-b|z|^2}\, d\mu^{s,x}(z).\end{aligned}$$

Now, let $\varepsilon > 0$ be given, we consider $\eta \in C^\infty_c(\mathbf{R}^n)$ such that $\displaystyle\int_{\mathbf{R}^n} (1-\eta)\, e^{-b|z|^2}\, d\mu^{s,x}(z) \leq \varepsilon$, then there exists $s_0 > 0$ such that if $|s| < s_0$

$$\int_{\mathbf{R}^n} e^{-b|z|^2}\, d\mu^{s,x}(z) = \int_{\mathbf{R}^n} \eta\, e^{-b|z|^2}\, d\mu^{s,x}(z) + \int_{\mathbf{R}^n} (1-\eta)\, e^{-b|z|^2}\, d\mu^{s,x}(z) < \varepsilon.$$

Moreover, for $x \in E^1$

$$\begin{aligned}\int_{\Omega^{s,x}} \chi_{E^{s,x}}(z)\big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle\, dz &= \int_{\Omega^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle\, dz \\ &+ \int_{\Omega^{s,x}\setminus E^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle\, dz\end{aligned}$$

Now,

$$\int_{\Omega^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle\, dz \to A(x)\nu_F(x)\cdot \int_{\mathbf{R}^n} D_2 p^{x}_{0,*}(1,z,0) dz = 0$$

and for every $\varepsilon > 0$ there exists $t_0$ small enough, such that for $|s| < t_0$, by (B.2)

$$\int_{\Omega^{s,x}\setminus E^{s,x}} |D_2 p^{s,x}_{0,*}(1,z,0)|\, dz < \varepsilon$$

therefore if $x \in E^1$ and $s$ is sufficiently small,

$$\frac{1}{\sqrt{t}}\int_0^t \frac{1}{\sqrt{s}} \int_{\Omega^{s,x}\cap E^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle\, dz < 2\varepsilon.$$

Now, taking into account that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$, we can consider only points $x \in \mathcal{F}F \cap \mathcal{F}E$; in this case we obtain that

$$\int_{\mathbf{R}^n} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle d\mu^{s,x}(z) \longrightarrow \int_{H_{\nu_E(x)}} \big\langle D_2 p^{x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle dz.$$

Taking into account (5.23) and the symmetry of $A$, we get that

$$D_2 p^x_{0,*}(1,z,0) = -\frac{1}{2(4\pi)^{n/2}|\det A^{1/2}(x)|}\exp\left(-\left\langle A^{-1}(x)z,z\right\rangle/4\right)A^{-1}(x)z,$$

and then, since for $x \in \mathcal{F}F \cap \mathcal{F}E$ we have $\nu_F(x) = \langle \nu_E(x), \nu_F(x)\rangle \nu_E(x)$

$$\int_{H_{\nu_E(x)}} \left\langle D_2 p^x_{0,*}(1,z,0), A(x)\nu_F(x)\right\rangle dz =$$

$$= -\frac{\langle \nu_E(x), \nu_F(x)\rangle}{2(4\pi)^{n/2}|\det A^{1/2}(x)|}\int_{H_{\nu_E(x)}} \exp\left(-\left\langle A^{-1}(x)z,z\right\rangle/4\right)\langle z, \nu_E(x)\rangle dz$$

$$= -\frac{\langle \nu_E(x), \nu_F(x)\rangle}{\pi^{n/2}}\int_{H_{A^{1/2}(x)\nu_E(x)}} e^{-|z|^2}\left\langle z, A^{1/2}(x)\nu_E(x)\right\rangle dz.$$

For the computation of this last integral, we consider an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ with

$$e_n = \frac{A^{1/2}(x)\nu_E(x)}{|A^{1/2}(x)\nu_E(x)|};$$

we then obtain

$$\int_{H_{A^{1/2}(x)\nu_E(x)}} \left\langle z, A^{1/2}(x)\nu_E(x)\right\rangle e^{-|z|^2} dz = |A^{1/2}(x)\nu_E(x)|\int_{H_{A^{1/2}(x)\nu_E(x)}} z_n e^{-|z|^2} dz$$

$$= \pi^{(n-1)/2}|A^{1/2}(x)\nu_E(x)|\int_{-\infty}^{0} z_n e^{-z_n^2} dz_n = -\frac{\pi^{(n-1)/2}}{2}|A^{1/2}(x)\nu_E(x)|.$$

At the end, we have obtained that

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap F}(T_0(t)\chi_E - \chi_E)dx = -\int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E}\langle \nu_E, \nu_F\rangle |A^{1/2}\nu_E| d\mathcal{H}^{n-1}.$$

□

With a perturbation argument we establish the result stated in Proposition 5.3.1 for the semigroup $T(t)$ generated by the complete operator $(A_1, D(A_1))$ in $L^1(\Omega)$.

**Theorem 5.3.2.** *Assume $\Omega$, $\mathcal{B}$ be as in Proposition 5.3.1 and let $\mathcal{A}$ be as in* (5.1) *with coefficients satisfying* (5.2). *Denote by $T(t)$ the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$, then, if $E, F \subset \mathbf{R}^n$ are sets of finite perimeter in $\Omega$, the following holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap F}(\chi_E(x) - T(t)\chi_E(x))dx = \int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E}\langle A(x)\nu_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x). \tag{5.27}$$

Proof. By using Proposition 5.3.1 and the notations fixed there, it suffices to prove that

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\cap F}(T_0(t)\chi_E(x) - T(t)\chi_E(x))dx = 0. \tag{5.28}$$

In order to get the claim, we set $u(t,x) = (T(t)\chi_E)(x)$ and $v(t,x) = (T_0(t)\chi_E)(x)$, so that the function $z = u - v$ solves the problem

$$\begin{cases} \partial_t z - \mathcal{A}z = \langle B, Dv\rangle + cv & t > 0,\ x \in \Omega \\ z(0) = 0 & x \in \Omega \\ \langle ADz, \nu\rangle = 0 & t > 0,\ x \in \partial\Omega \end{cases}$$

and can be written as follows

$$u - v = \int_0^t T(t-s)\big(\langle B, Dv(s)\rangle + cv(s)\big)\, ds. \tag{5.29}$$

Using (3.1) we have

$$\|u - v\|_{L^1(\Omega)} \le c_0 \int_0^t \Big(\|\langle B, Dv(s)\rangle\|_{L^1(\Omega)} + \|cv(s)\|_{L^1(\Omega)}\Big) ds. \tag{5.30}$$

If we prove that

$$\|u - v\|_{L^1(\Omega)} = o(\sqrt{t}) \quad \text{as } t \to 0 \tag{5.31}$$

we conclude. For the last term in (5.30) we have that

$$\int_\Omega |c(x)T_0(s)\chi_E(x)|\, dx \le c_0 \|c\|_\infty |\Omega \cap E|$$

and then

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_0^t \int_\Omega |c(x)T_0(s)\chi_E(x)|\, dxds = 0.$$

For the first term in (5.30), we notice that

$$\left| \int_\Omega \int_{\Omega\cap E} \langle B(x), D_x p_0(s,x,y)\rangle dydx \right| \le \|B\|_\infty |\Omega \cap E| \int_\Omega |D_x p_0(s,x,y)| dx$$

and using Gaussian estimates (B.2) we get

$$\int_\Omega |D_x p_0(s,x,y)| dx \le \frac{C}{\sqrt{s}}$$

for some constant $C$ depending only on the operator $\mathcal{A}$ and the dimension $n$. However we can write

$$\begin{aligned}
\int_\Omega \langle B, D_x T_0(s)\chi_E\rangle dx &= \int_\Omega dx \int_{\Omega\cap E} \langle B(x), D_x p_0(s,x,y)\rangle dy \\
&= s^{-(n+1)/2} \int_\Omega dx \int_{\Omega\cap E} \left\langle B(x), D_2 p_{0,*}^{s,x}\left(1, \frac{y-x}{\sqrt{s}}, 0\right)\right\rangle dy \\
&= \frac{1}{\sqrt{s}} \int_\Omega dx \int_{\Omega^{s,x}\cap E^{s,x}} \big\langle B(x), D_2 p_{0,*}^{s,x}(1,z,0)\big\rangle dz \\
&= \frac{1}{\sqrt{s}} \int_\Omega dx \int_{\mathbf{R}^n} \big\langle B(x), D_2 p_{0,*}^{s,x}(1,z,0)\big\rangle d\mu^{s,x}(z).
\end{aligned}$$

where $\mu^{s,x}$ is defined in (5.26) and satisfies 1., 2. and 3. of Proposition 5.3.1. With the same argument previously used, we can deduce that for $x \in E^0 \cup E^1$, the limit of the above integral as $t \to 0$ vanishes; then, taking into account that $|\Omega \setminus (E^0 \cup E^1)| = 0$, we have then obtained that

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_0^t \int_\Omega \int_{\Omega\cap E} |\langle B(x), D_x p_0(s,x,y)\rangle|\, dydxds = 0$$

for $\mathcal{H}^{n-1}$-a.a. $x \in E^0 \cup E^1$. Therefore (5.31) is proved and the proof is complete □

Specializing the above result for $F = E^c$ we get the following

**Corollary 5.3.3.** *Under assumption of Theorem 5.3.1, let $(T(t))_{t\geq 0}$ be the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$; then, if $E \subset \mathbf{R}^n$ is a set with finite perimeter in $\Omega$, the following equality holds:*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap E^c} T(t)\chi_E dx = \int_{\Omega\cap\mathcal{F}E} |A^{1/2}(x)\nu_E(x)| d\mathcal{H}^{n-1}(x). \tag{5.32}$$

Using an argument similar to the one used in [33, Theorem 3.4] and the lower bound for the kernel $p(t,x,y)$, it is possible to prove the converse of the statement in Corollary 5.3.3.

**Proposition 5.3.4.** *Let $E \subset \mathbf{R}^n$ be a set such that either $E$ or $E^c$ has finite measure in $\Omega$, and such that*

$$\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{E^c\cap\Omega} T(t)\chi_E(x)dx < +\infty,$$

*then $E$ has finite perimeter in $\Omega$, that is $\chi_E \in BV(\Omega)$.*

Proof. Define $E_\Omega := E\cap\Omega$ and assume $|E_\Omega| < \infty$. From (B.17) we have

$$\begin{aligned}
\frac{1}{\sqrt{t}}\int_{E^c} T(t)\chi_E(x)\,dx &= \int_\Omega\int_\Omega p(t,x,y)\chi_E(y)\chi_{E^c}(x)\,dy\,dx \\
&\geq \frac{C_1}{t^{(n+1)/2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-c_1\frac{|x-y|^2}{t}}\chi_{E_\Omega}(y)\big(\chi_\Omega(x)-\chi_E(x)\big)\,dy\,dx \\
&= \frac{C_1}{\sqrt{t}}\int_{\mathbf{R}^n} e^{-c_1|z|^2}\int_{\mathbf{R}^n}\chi_\Omega(x)\chi_{E_\Omega}(z\sqrt{t}+x)(1-\chi_E(x))\,dx\,dz \\
&= \frac{C_1}{\sqrt{t}}\int_{\mathbf{R}^n} e^{-c_1|z|^2}\int_{\mathbf{R}^n}\chi_\Omega(x)\big(\chi_{E_\Omega - z\sqrt{t}}(x) - \chi_{E_\Omega - z\sqrt{t}}(x)\chi_E(x)\big)\,dx\,dz \\
&= C_1\int_{\mathbf{R}^n} e^{-c_1|z|^2}|z|\frac{|(E_\Omega\Delta(E_\Omega - z\sqrt{t}))\cap\Omega|}{\sqrt{t}|z|}\,dz
\end{aligned}$$

In fact, denoting by

$$|D_\nu\chi_E|(\Omega) = \liminf_{t\to 0}\frac{|(E\Delta(E-t\nu))\cap\Omega|}{t},$$

by assumption we get that

$$\begin{aligned}
&\int_{\mathbf{R}^n}|z|e^{-c_1|z|^2}|D_{\frac{z}{|z|}}\chi_{E_\Omega}|(\Omega)\,dz \\
&\qquad\leq \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}\chi_E(y)\chi_{E^c}(x)p(t,x,y)dxdy < +\infty.
\end{aligned}$$

This implies, using an argument similar to that used in Theorem 5.1.6, that there exist $M > 0$ such that $|D_{e_i}\chi_{E_\Omega}|(\Omega) \leq M$ for $i = 1,\ldots,n$. Finally, let $\varphi \in C_c^1(\Omega,\mathbf{R}^n)$; then

$$\int_\Omega \chi_E(x)D_i\varphi(x)\,dx = \lim_{t\to 0^+}\int_\Omega\chi_E(x)\frac{\varphi(x+te_i)-\varphi(x)}{t}\,dx$$

But

$$\Big|\int_\Omega \chi_E(x)\frac{\varphi(x+te_i)-\varphi(x)}{t}\Big| = \Big|\int_\Omega \frac{\chi_{E_\Omega+te_i}(x)-\chi_{E_\Omega}(x)}{t}\varphi(x)\,dx\Big|$$
$$\leq \|\varphi\|_{L^\infty(\Omega)}\frac{|(E_\Omega\Delta(E_\Omega+te_i))\cap\Omega|}{t}$$

Thus

$$\Big|\int_\Omega \chi_E(x)D_i\varphi(x)\,dx\Big| \leq \|\varphi\|_{L^\infty(\Omega)}\liminf_{t\to0^+}\frac{|(E_\Omega\Delta(E_\Omega+te_i))\cap\Omega|}{t}$$
$$= \|\varphi\|_{L^\infty(\Omega)}|D_{e_i}\chi_{E_\Omega}|(\Omega) \leq M\|\varphi\|_{L^\infty(\Omega)}$$

and

$$\int_\Omega \chi_E(x)\mathrm{div}\varphi(x)\,dx \leq nM\|\varphi\|_{L^\infty(\Omega)}$$

that is $|D\chi_E|(\Omega) < +\infty$. □

We are now in a position to prove the main result of this section, namely, the announced characterization of $BV$ functions (5.4). The strategy is the same as for $\mathbf{R}^n$ and is based on (4.13).

**Theorem 5.3.5.** *Let $\Omega$, $\mathcal{A}$, $\mathcal{B}$ be as in Theorem 5.3.2, let $(T(t))_{t\geq0}$ be the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$ and let $u\in L^1(\Omega)$; then $u\in BV(\Omega)$ if and only if*

$$\liminf_{t\to0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty;$$

*moreover, in this case the following equality holds*

$$|Du|_A(\Omega) = \lim_{t\to0}\frac{\sqrt{\pi}}{2\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy. \tag{5.33}$$

Proof. The "if" part. We start by considering $u\in L^1(\Omega)$; for $\tau\in\mathbf{R}$ we denote by $E_\tau = \{u>\tau\}$ and, since the semigroup is positive and contractive, we obtain that

$$\begin{aligned} 0 &\leq \int_{\mathbf{R}}\liminf_{t\to0}\frac{1}{\sqrt{t}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau \leq \liminf_{t\to0}\frac{1}{\sqrt{t}}\int_{\mathbf{R}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau \\ &\leq \liminf_{t\to0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}\int_{\mathbf{R}}|\chi_{E_\tau}(x)-\chi_{E_\tau}(y)|p(t,x,y)dxdyd\tau \\ &= \liminf_{t\to0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty \end{aligned}$$

and then, thanks to Proposition 5.3.4, almost every level $E_\tau$ has finite perimeter and equation (5.32) holds. Then, using coarea formula (4.13), we get

$$\begin{aligned} |Du|_A(\Omega) &= \int_{\mathbf{R}}\mathcal{P}_A(E_\tau,\Omega)d\tau = \int_{\mathbf{R}}\lim_{t\to0}\sqrt{\frac{\pi}{t}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau \\ &\leq \liminf_{t\to0}\sqrt{\frac{\pi}{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty \end{aligned}$$

that is $u \in BV_A(\Omega)$.
The other implication follows from (5.33). To prove (5.33), we define the function

$$g_t(\tau) = \sqrt{\frac{\pi}{t}} \int_{E_\tau^c \cap \Omega} T(t)\chi_{E_\tau}(x)dx.$$

For this function we have the following estimate

$$\begin{aligned}
|g_t(\tau)| =& \sqrt{\frac{\pi}{t}} \Big| \int_0^t \int_{E_\tau^c \cap \Omega} \mathcal{A}T(s)\chi_{E_\tau} dxds \Big| = \sqrt{\frac{\pi}{t}} \Big| \int_0^t \Big( \int_{\mathcal{F}E_\tau \cap \Omega} \langle ADT(s)\chi_{E_\tau}, \nu_{E_\tau} \rangle d\mathcal{H}^{n-1} \\
&+ \int_{E_\tau^c \cap \Omega} \langle B, DT(s)\chi_{E_\tau} \rangle dx + \int_{E_\tau^c \cap \Omega} cT(s)\chi_{E_\tau})dx \Big) ds \Big| \\
\leq & \sqrt{\frac{\pi}{t}} \int_0^t \Big( \|A\|_\infty \int_{\mathcal{F}E_\tau} |DT(s)\chi_{E_\tau}| d\mathcal{H}^{n-1} \\
&+ \|B\|_\infty \int_{E_\tau^c \cap \Omega} \int_{E_\tau \cap \Omega} |D_x p(s,x,y)| dxdy + \|c\|_\infty \int_{E_\tau^c \cap \Omega} \int_{E_\tau \cap \Omega} |p(s,x,y)| dxdy \Big) ds \\
\leq & cM_0(\mathcal{P}(E_\tau, \Omega) + \min\{|E_\tau \cap \Omega|, |E_\tau^c \cap \Omega|\}) = h(\tau)
\end{aligned}$$

where the last inequality follows from the estimates (B.2) on the kernel $p(s,x,y)$. We have that $h \in L^1(\mathbf{R})$ since

$$\int_{\mathbf{R}} \mathcal{P}(E_\tau, \Omega) d\tau = |Du|(\Omega)$$

and, denoted by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$,

$$\begin{aligned}
\int_{\mathbf{R}} \min\{|E_\tau \cap \Omega|, |E_\tau^c \cap \Omega|\} d\tau \leq & \int_0^\infty |E_\tau \cap \Omega| d\tau + \int_{-\infty}^0 |E_\tau^c \cap \Omega| d\tau \\
= & \int_0^\infty \int_\Omega \chi_{E_\tau} dxd\tau + \int_{-\infty}^0 \int_\Omega \chi_{E_\tau^c} dxd\tau \\
= & \int_\Omega \int_0^\infty \chi_{\{u>\tau\}} d\tau dx + \int_\Omega \int_0^\infty \chi_{\{-u \geq \tau\}} d\tau dx \\
= & \int_\Omega u^+ dx + \int_\Omega u^- dx = \int_\Omega |u| dx.
\end{aligned}$$

Then we can apply Corollary 5.3.3 and Lebesgue dominated convergence theorem to the functions $g_t$ in order to obtain

$$\begin{aligned}
|Du|_A(\Omega) &= \int_{\mathbf{R}} \mathcal{P}_A(E_\tau, \Omega) d\tau = \int_{\mathbf{R}} \lim_{t \to 0} \sqrt{\frac{\pi}{t}} \int_{E_\tau^c \cap \Omega} T(t)\chi_{E_\tau} dx \\
&= \lim_{t \to 0} \sqrt{\frac{\pi}{t}} \int_{\mathbf{R}} \int_{\Omega \times \Omega} (\chi_{E_\tau}(y) - \chi_{E_\tau}(y)\chi_{E_\tau}(x)) p(t,x,y) dxdyd\tau \\
&= \lim_{t \to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega \times \Omega} (u(y) - \min\{u(y), u(x)\}) p(t,x,y) dxdy
\end{aligned}$$

since $\chi_{E_\tau}(y)\chi_{E_\tau}(x) \neq 0$ if and only if $\tau < \min\{u(x), u(y)\}$; finally, the assertion follows by noticing that $\min\{u(y), u(x)\} = \frac{1}{2}(u(x) + u(y) - |u(x) - u(y)|)$. □